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ON AN INFINITE-DIMENSIONAL LIMIT OF THE STEINBERG REPRESENTATIONS

Abstract. We present a construction of the Steinberg representation that allows for automatically passing to an infinite-dimensional limit.

Recall that the representation theory of infinite-dimensional classical groups and infinite symmetric groups is a relatively old well-developed topic. For infinite-dimensional groups over finite fields, progress appeared comparatively recently, in [3, 9, 10] and [6, 7]. This note contains a construction intermediate between these works.

Notation. Denote by $\mathbb{F}_q$ the field with q elements, and let $\mathbb{F}_q^n$ be the coordinate n-dimensional linear space with the standard basis e_j. Denote by $\mathrm{GL}(n)$ the group of all invertible matrices of order n. It acts on $\mathbb{F}_q^n$ by the multiplication $x \mapsto xg$ of a row $x \in \mathbb{F}_q^n$ by a matrix $g \in \mathrm{GL}(n)$.

Denote by S_n the symmetric group. It is generated by the transpositions $\tau_j = (j, j+1)$, the relations being $\tau_j^2 = 1$, $(\tau_j\tau_{j+1})^3 = 1$, and $\tau_k\tau_j = \tau_j\tau_k$ for $|k - j| \geqslant 2$.

By $\ell_2(Z)$ we denote the space of complex functions on a finite set Z equipped with the ℓ_2 inner product.

1. Schubert cells. Denote by $\mathrm{Fl}(n)$ the space of complete flags $\mathcal{V}$ in $\mathbb{F}_q^n$,

$$\mathcal{V} : 0 = V_0 \subset V_1 \subset \cdots \subset V_n = \mathbb{F}_q^n, \quad \dim V_j = j.$$

Fix the standard flag

$$\mathcal{E} :\ E_0 \subset E_1 \subset \cdots \subset E_n, \quad E_j = \sum_{i \leqslant j} \mathbb{F}_q e_i.$$

Denote by $\mathrm{B}(n) \subset \mathrm{GL}(n)$ the stabilizer of $\mathcal{E}$. It consists of the lower triangular matrices. For each $\sigma \in S_n$ we consider the flag

$$\mathcal{E}^\sigma :\ E_0^\sigma \subset E_1^\sigma \subset \cdots \subset E_n^\sigma \quad \text{where} \quad E_j^\sigma = \mathbb{F}_q e_{\sigma(1)} \oplus \cdots \oplus \mathbb{F}_q e_{\sigma(j)}.$$

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Orbits of $\mathrm{B}(n)$ on $\mathrm{Fl}(n)$ are called *Schubert cells*; for details, see [2, 10.2]. They are enumerated by the elements $\sigma \in S_n$: each orbit contains a unique flag $\mathcal{E}^\sigma$, we denote this orbit by X^σ.

2. The Steinberg representation. Denote by $\mathrm{Fl}_j(n)$ the space of incomplete flags containing subspaces of all dimensions except j. Denote by $\pi_j : \mathrm{Fl}(n) \to \mathrm{Fl}_j(n)$ the map forgetting V_j. There is a natural map $\Pi_j : \ell_2(\mathrm{Fl}(n)) \to \ell_2(\mathrm{Fl}_j(n))$, defined by

$$\Pi_j f(\mathcal{W}) = \frac{1}{q+1} \sum_{\mathcal{V}:\pi_j(\mathcal{V})=\mathcal{W}} f(\mathcal{W}).$$

In fact, the summation is taken over all flags

$$W_0 \subset \cdots \subset W_{j-1} \subset Y \subset W_{j+1} \subset \cdots \subset W_n,$$

such flags are enumerated by the subspaces Y satisfying $W_{j-1} \subset Y \subset W_{j+1}$; or, equivalently, over the set of lines in the two-dimensional space W_{j+1}/W_{j-1}.

Theorem 1. *There exists a unique irreducible representation of* $\mathrm{GL}(n)$ *that is contained in* $\ell_2(\mathrm{Fl}(n))$ *and is not contained in the spaces* $\ell_2(\mathrm{Fl}_j(n))$.

This representation is called the *Steinberg representation*; for its fascinating properties, see the surveys [4, 8].

3. Definition of the Steinberg representation via reproducing kernels. We define a function $k(\mathcal{V},\mathcal{W})$ on $\mathrm{Fl}(n)\times\mathrm{Fl}(n)$ as the number of all pairs (i,j), where i, j range in $\{0,1,\dots,n-1\}$, such that

$$\dim V_i \cap W_j = \dim V_{i+1} \cap W_{j+1}. \tag{1}$$

Define a kernel $K(\cdot,\cdot)$ on $\mathrm{Fl}(n)$ by

$$K(\mathcal{V},\mathcal{W}) = (-q)^{-k(\mathcal{V},\mathcal{W})}.$$

By definition, the kernel $K(\cdot,\cdot)$ is $\mathrm{GL}(n)$-invariant.

Proposition 1. *The function*

$$\kappa(\mathcal{W}) := k(\mathcal{E},\mathcal{W})$$

is constant on Schubert cells X^σ. The value of κ on X^σ coincides with the number $I(\sigma)$ of inversions of σ. The number of points of X^σ coincides with $q^{k(\mathcal{E},\mathcal{W})}$.

Note that $I(\sigma)$ coincides with the length of a shortest decomposition of σ into a product of the generators τ_j.

Proof. Let us evaluate the number of points of X^σ. Consider an example. Let $n = 6$,

$$\sigma = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}. \tag{2}$$

The vectors e_σ are the rows of this matrix. A $\mathrm{B}(n)$-orbit of the collection $\{e_\sigma\}$ consists of arbitrary collections of the type

$$\begin{array}{c} (\ * \ * \ * \ \circ \ 0 \ 0 \ ), \\ (\ * \ \circ \ 0 \ 0 \ 0 \ 0 \ ), \\ (\ * \ * \ * \ * \ \circ \ 0 \ ), \\ (\ \circ \ 0 \ 0 \ 0 \ 0 \ 0 \ ), \\ (\ * \ * \ * \ * \ * \ \circ \ ), \\ (\ * \ * \ \circ \ 0 \ 0 \ 0 \ ), \end{array}$$

where $*$ denotes arbitrary elements of $\mathbb{F}_q$ and $\circ$ are nonzero elements. We have $\circ$'s on the former positions of units and $*$'s on the positions to the left of units. Elements of flags (subspaces) are linear combinations $\sum\limits_{j \leqslant k} c_j e_{\sigma(j)}$. Replacements of the form

$$e_{\sigma(j)} \to \lambda e_{\sigma(j)} + \sum_{i<j} a_i e_{\sigma(i)}, \quad \lambda \neq 0,$$

do not change the flag. Therefore, we can get 1 on the positions of $\circ$'s and 0 under all units. Thus we see that any flag in the $\mathrm{B}(n)$-orbit of $\mathcal{E}^\sigma$ is generated by a collection of vectors

$$\begin{array}{c} (\ * \ * \ * \ 1 \ 0 \ 0 \ ), \\ (\ * \ 1 \ 0 \ 0 \ 0 \ 0 \ ), \\ (\ * \ 0 \ * \ 0 \ 1 \ 0 \ ), \\ (\ 1 \ 0 \ 0 \ 0 \ 0 \ 0 \ ), \\ (\ 0 \ 0 \ * \ 0 \ 0 \ 1 \ ), \\ (\ 0 \ 0 \ 1 \ 0 \ 0 \ 0 \ ). \end{array} \tag{3}$$

Now for each star we have a unit under this star and a unit to the right of the star. This pair of units corresponds to an inversion in σ.

Next, let us evaluate the number (i, j) of pairs satisfying (1). The dimension of $E_i \cap F_j$ is the number of units in the left upper $i \times j$ corner of the matrix σ. Condition (1) means that the $i \times j$ and $(i+1) \times (j+1)$ corners contain the same units. Therefore, units in the $(i+1)$th row and $(j+1)$th column are outside the $(i+1) \times (j+1)$ corner. Hence we have $*$ on the $(i+1)(j+1)$th place in (3). □

Lemma 1. *The kernel $K(\cdot,\cdot)$ is positive definite.*[1]

Consider the Euclidean space H_n determined by the reproducing kernel[2] $K(\mathcal{V}, \mathcal{W})$.

Lemma 2. *The representation of* $\mathrm{GL}(n)$ *in H_n coincides with the Steinberg representation.*

Proofs of the lemmas. By the Frobenius reciprocity, any subrepresentation in $\ell^2(\mathrm{Fl}(n))$ contains a $\mathrm{B}(n)$-invariant vector. Denote by η a $\mathrm{B}(n)$-invariant function in the Steinberg subrepresentation St in ℓ_2. Denote by $\eta[\sigma]$ its value on a Schubert cell X^σ. By definition, η satisfies the equations $\Pi_j \eta = 0$. It is easy to see that these equations have the form

$$q\eta[\tau_j \sigma] + \eta[\sigma] = 0 \quad \text{if} \quad I(\tau_j \sigma) > I(\sigma).$$

These recurrence relations have a unique (up to a constant factor) solution, namely, $\eta[\sigma] = (-q)^{-I(\sigma)}$. This also proves Theorem 1.

Denote by $M(\cdot,\cdot)$ the reproducing kernel determining the subspace St. This means that the functions

$$\delta_{\mathcal{V}}(\mathcal{W}) = M(\mathcal{V}, \mathcal{W})$$

are contained in St and for any function f on $\mathrm{Fl}(n)$ we have

$$f(\mathcal{V}) = \langle f, \delta_{\mathcal{V}} \rangle_{\ell_2(\mathrm{Fl}(n))}.$$

Since St is $\mathrm{GL}(n)$-invariant, the kernel M is $\mathrm{GL}(n)$-invariant, $M(g\mathcal{V}, g\mathcal{W}) = M(\mathcal{V}, \mathcal{W})$. Since the action of $\mathrm{GL}(n)$ on $\mathrm{Fl}(n)$ is transitive, the kernel is determined by its values for $\mathcal{V} = \mathcal{E}$, i.e., by the function $\delta_{\mathcal{E}}$. Moreover, $\delta_{\mathcal{E}}(\mathcal{W})$ is $\mathrm{B}(n)$-invariant, and therefore $\delta_{\mathcal{E}}(\mathcal{W}) = s \cdot \eta(\mathcal{W})$. □

Remark. This construction of the Steinberg representation is a rephrasing of [1, Theorem 10.2].

[1] I.e., for any collection of points $\mathcal{V}_i \in \mathrm{Fl}(n)$, we have $\det_{i,j}\{K(\mathcal{V}_i, \mathcal{V}_j)\} \geqslant 0$.

[2] See, e.g., [5, Sec. 7.1].

4. The infinite-dimensional limit. Preliminaries. Consider the linear space L whose vectors are two-sided sequences

$$x = (\dots, x_{-1}, x_0, x_1, \dots)$$

such that $x_k = 0$ for sufficiently large k. We represent operators in L as infinite matrices $g = g_{ij}$, where $-\infty < i, j < \infty$. Denote by $\mathrm{B}(2\infty)$ the group of all infinite matrices g such that $g_{ij} = 0$ for $i < j$ and g_{ii} are invertible (i.e., we consider all invertible lower triangular matrices). Denote by $\mathrm{GL}(2\infty)$ the group of *finitary*[3] invertible matrices. Denote by $\mathrm{GLB}(2\infty)$ the group of matrices generated by $\mathrm{GL}(2\infty)$ and $\mathrm{B}(2\infty)$. The group $\mathrm{GLB}(2\infty)$ acts in L by the transformations $x \mapsto xg$.

Denote by E_j, where $-\infty < j < \infty$, the subspace in L consisting of the vectors

$$(\dots, x_{j-1}, x_j, 0, 0, \dots).$$

Thus we get the *standard flag* $\mathcal{E}$:

$$\dots \subset E_{-1} \subset E_0 \subset E_1 \subset \cdots .$$

We define the flag space $\mathrm{Fl}(2\infty)$ as the space of complete flags coinciding with the standard flag in all but a finite number of terms. More precisely, consider flags $\mathcal{V}$ having the following form. Fix $M \leqslant N$. Set $V_j = E_j$ if $j \leqslant M$ and $j \geqslant N$. Consider the finite-dimensional space E_N/E_M and a complete flag in E_N/E_M,

$$0 = F_0 \subset F_1 \subset \dots \subset F_{N-M} = E_N/E_M.$$

For $0 \leqslant \alpha \leqslant N - M$, we set $V_{M+\alpha}$ equal to the preimage of F_α under the projection $E_N \to L/E_M$.

Setting $M = -n$, $N = n$, we see that the space $\mathrm{Fl}(2\infty)$ is an inductive limit of the chain

$$\dots \longrightarrow \mathrm{Fl}(2n+1) \longrightarrow \mathrm{Fl}(2n+3) \longrightarrow \dots .$$

The group $\mathrm{GLB}(2\infty)$ acts on the space $\mathrm{Fl}(2\infty)$.

[3]This means that $g - 1$ has finitely many nonzero matrix elements.

5. The infinite-dimensional limit of the Steinberg representations. We define a function $K(\mathcal{V},\mathcal{W})$ on $\mathrm{Fl}(2\infty)\times\mathrm{Fl}(2\infty)$ as the number of pairs $(i,j)\in\mathbb{Z}\times\mathbb{Z}$ such that

$$V_{i+1}\cap W_{j+1}=V_i\cap W_j.$$

The kernel $K(\mathcal{V},\mathcal{W})=(-p)^{-k(\mathcal{V},\mathcal{W})}$ is positive definite on each space $\mathrm{Fl}(2n+1)$ and, therefore, on the inductive limit $\mathrm{Fl}(2\infty)$. We consider the Hilbert space determined by the reproducing kernel K and the unitary representation of $\mathrm{GL}(2\infty)$ in this space.

6. Comparison with earlier papers. (a) Consider the space L_+ consisting of sequences $(x_0,x_1,\dots)$ such that $x_j=0$ for all but a finite number of j. The same construction gives the Steinberg representation obtained in [3].

(b) Grassmannians and flags in the space L were considered in [6]. However, the topic of [6] is the group of all continuous transformations of (the locally compact Abelian group) L; this group is larger than $\mathrm{GLB}(2\infty)$. Also, [6] treats another space of flags, which has empty intersection with $\mathrm{Fl}(2\infty)$.

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