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A NOTE ON THE TANGENT BUNDLE AND GAUSS FUNCTOR OF POSETS AND MANIFOLDS

Abstract. We introduce a notion of the tangent bundle of a poset. In the case where the poset is the poset of simplices of a combinatorial manifold, the construction produces the best possible combinatorial model for the geometric compactified tangent bundle.

1. Introduction. To any poset P we can associate the "tangent bundle" of P – a poset map $\mathrm{E}\mathsf{P} \xrightarrow{\mathrm{T}\mathsf{P}} \mathsf{P}$ supplemented by two sections $\mathsf{P} \xrightarrow{s_0, s_\infty} \mathrm{E}\mathsf{P}$ and a functor $\mathsf{P} \xrightarrow{\mathbf{G}} \mathbf{Posets}$. The functor $\mathbf{G}$ is related to the map $\mathrm{T}\mathsf{P}$ in such a way that $\mathrm{E}\mathsf{P} \xrightarrow{\mathrm{T}\mathsf{P}} \mathsf{P}$ coincides with $\operatorname{Hocolim} \mathbf{G} \xrightarrow{\Pi} \mathsf{P}$. Curiously, this pure abstract nonsense construction can be geometrically and topologically justified. In [13] it is shown that in the case where P is a "strict abstract manifold" (examples are combinatorial manifolds and the boundary complexes of arbitrary convex polytopes) the following happens. The order complex of $\mathrm{E}\mathsf{P}$ is a combinatorial manifold. The map $B\mathrm{E}\mathsf{P} \xrightarrow{B\mathrm{T}\mathsf{P}} B\mathsf{P}$ together with two sections Bs_0, Bs_∞ is a Kuiper–Lashoff $(S^n, 0, \infty)$ model for the tangent PL bundle of $B\mathsf{P}$. The image of the functor $\mathbf{G}$ naturally lives in a nice category $\mathfrak{R}_n$. The discrete category $\mathfrak{R}_n$ can be viewed as a discrete replacement of the structure group for the discrete replacements of PL fiber bundles with fiber $\mathbb{R}^n$. The main statement it that this discretization is exact: $B\mathfrak{R}_n \approx B\mathrm{PL}_n$ and $B\mathsf{P} \xrightarrow{B\mathbf{G}} B\mathfrak{R}_n$ is a homotopy model of the PL Gauss map for the PL manifold $B\mathsf{P}$. Here $B*$ is the classifying space functor, which in the case of posets coincides with the geometric realization of the order complex; in the case of categories, with Milnor's geometric realization of the nerve; and in the case of a simplicial group, with the classifying space for principal bundles. These results can be viewed as a purely combinatorial variation of some constructions

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from [4, 8, 15]. This also proves the PL double for MacPherson's conjecture on modeling the real Grassmanian $BO(n)$ by the poset of oriented matroids [9, 10] (see [14]). In this note, we formulate an abstract notion of the tangent bundle of a poset and briefly discuss related results and their proofs from [13].

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2. The tangent bundle and Gauss map of a PL manifold. Milnor in [12] defined the notion of an n-dimensional PL microbundle, the simplicial structure group PL_n of microbundles, and the classifying space $B\mathrm{PL}_n$. This theory creates a canonical one-to-one correspondence between the isomorphism classes of n-dimensional PL microbundles on a polyhedron K and the homotopy classes of maps from K to $B\mathrm{PL}_n$. Milnor also defined the notion of the tangent microbundle of a PL manifold M^n. A map $M^n \xrightarrow{G} B\mathrm{PL}_n$ representing the tangent microbundle of M^n is called the *Gauss map* of M^n, and BPL_n is a PL *Grassmanian*. The space $B\mathrm{PL}_n$ and the Gauss map are defined up to homotopy. In [6] and [7], Kuiper and Lashof developed the theory of models for piecewise linear $\mathbb{R}^n$-bundles. In particular, they established a canonical one-to-one correspondence between the isomorphism classes of n-dimensional PL microbundles and the $(S^n, \infty)[(S^n, 0, \infty)]$ fiber bundles. A piecewise linear (S^n, ∞) fiber bundle is a PL fiber bundle with fiber S^n and a section labeled by ∞. A piecewise linear $(S^n, 0, \infty)$ fiber bundle is a PL fiber bundle with fiber S^n and two sections labeled by 0 and ∞. These sections should have no common points.

3. The tangent bundle and Gauss functor of a poset. Here we introduce a very general and simple abstract construction.

Let P be a poset, $p \in \mathsf{P}$. We denote by $\mathrm{Star}\, p$ the subposet of P that is the union of all principal ideals containing p. Formally,

$$\mathrm{Star}\, p = \{x \in \mathsf{P} | \text{there exists } y : p \leqslant y, x \leqslant y\}.$$

We also denote

$$\mathrm{Link}\, p = \{x \in \mathrm{Star}\, p | p \nleqslant x\}.$$

Define a new poset EP as follows. Denote by $D\mathsf{P}$ the subset of $\mathsf{P} \times \mathsf{P}$ formed by all pairs (x, y) such that there exists $z \in \mathsf{P}$ with $z \geqslant x, z \geqslant y$. Pick a new element $\infty \notin \mathsf{P}$. Set

$$\mathsf{EP} = D\mathsf{P} \cup \mathsf{P} \times \{\infty\} \subset \mathsf{P} \times \mathsf{P} \cup \{\infty\}.$$

Now we define a partial order on $\mathrm{E}\mathsf{P}$. Set

$$(x_1, y_1) \leqslant (x_2, y_2) \Leftrightarrow (x_1 \underset{\mathsf{P}}{\leqslant} x_2) \wedge \begin{cases} y_1 \underset{\mathsf{P}}{\leqslant} y_2 & \text{if } y_1, y_2 \in \mathsf{P}, \\ y_1 \in \mathrm{Link}_{\mathsf{P}}\, x_2 & \text{if } y_1 \in \mathsf{P}, y_2 = \infty, \\ y_1 = \infty, y_2 = \infty. \end{cases}$$

Denote by $\mathrm{E}\mathsf{P} \xrightarrow{\mathrm{T}\mathsf{P}} \mathsf{P}$ the projection to the first argument. We fix two sections of the poset map $\mathrm{T}\mathsf{P}$: the diagonal

$$\mathsf{P} \xrightarrow{s_0} \mathrm{E}\mathsf{P}, \quad s_0(x) = (x, x) \in D\mathsf{P} \subset \mathrm{E}\mathsf{P},$$

and the "section at infinity":

$$\mathsf{P} \xrightarrow{s_\infty} \mathrm{E}\mathsf{P}, \quad s_\infty(x) = (x, \infty).$$

We call the set of data $\langle \mathrm{T}\mathsf{P}, s_0, s_\infty \rangle$ defined above the *tangent bundle of the poset* P.

Any element $x \in \mathsf{P}$ gives rise to the subposet $\mathbf{G}_x = (\mathrm{T}\mathsf{P})^{-1}(x) \subset \mathrm{E}\mathsf{P}$ with the induced order. We can describe the structure of $\mathbf{G}_x$:

$$\mathbf{G}_x = \{(x, y) | y \in \mathrm{Star}_{\mathsf{P}}\, x\} \cup \{(x, \infty)\},$$

$$(x, y_1) \underset{\mathbf{G}_x}{\leqslant} (x, y_2) \Leftrightarrow \begin{cases} y_1 \underset{\mathsf{P}}{\leqslant} y_2 & \text{if } y_1, y_2 \in \mathsf{P}, \\ y_1 \in \mathrm{Link}_{\mathsf{P}}\, x & \text{if } y_2 = \infty, \\ y_1 = \infty, y_2 = \infty. \end{cases}$$

We will call the special diagonal element $(x, x) \in \mathbf{G}_x$ the 0-element.

To any pair $x_1 \leqslant x_2$ of comparable elements in P we associate a poset map

$$\mathbf{G}_{x_1} \xrightarrow{\mathbf{G}_{x_1 \leqslant x_2}} \mathbf{G}_{x_2}$$

defined as follows:

$$\mathbf{G}_{x_1 \leqslant x_2}(x_1, y) = \begin{cases} (x_2, y) & \text{if } y \in \mathrm{Star}\, x_2, \\ (x_2, \infty) & \text{if } y \notin \mathrm{Star}\, x_2, \\ (x_2, \infty) & \text{if } y = \infty. \end{cases}$$

Consider the category $\mathbf{Posets}^{0,\infty}$ of all posets with two elements specially labelled, one by 0 and the other one by ∞. The morphisms of $\mathbf{Posets}^{0,\infty}$ are the poset maps preserving the labelled elements. The poset maps $\mathbf{G}_{x_1 \leqslant x_2}$ preserve the elements labelled by 0 (diagonal) and by ∞. So we can regard $\mathbf{G}$ as a functor $\mathsf{P} \xrightarrow{\mathbf{G}} \mathbf{Posets}^{0,\infty}$. We will call $\mathbf{G}$ *the Gauss functor of the poset* P.

The tangent bundle $\langle \mathsf{TP}, s_0, s_\infty \rangle$ can be identified with a certain classical construction associated with $\mathbf{G}$. The construction is known as "categorial homotopy colimit," or "Grothendieck construction" [3], or "double bar construction" [11]. Probably, its first indication is in Whitehead's construction of the cone of simplicial map [16]. In our situation, the construction looks as follows. Let P be a poset, and let $\mathbf{F}$ be any functor $\mathsf{P} \xrightarrow{\mathbf{F}} \mathbf{Posets}$. With the functor $\mathbf{F}$ we associate a new poset $\operatorname{Hocolim} \mathbf{F}$ and a poset map $\operatorname{Hocolim} \mathbf{F} \xrightarrow{\Pi} \mathsf{P}$. Put

$$\operatorname{Hocolim} \mathbf{F} = \{(x, y) | x \in \mathsf{P}, y \in \mathbf{F}_x\}$$

and $(x_0, y_0) \leqslant (x_1, y_1)$ if and only if $x_0 \underset{\mathsf{P}}{\leqslant} x_1$ and $\mathbf{F}_{x_0 \underset{\mathsf{P}}{\leqslant} x_1}(y_0) \underset{\mathbf{F}_{x_1}}{\leqslant} y_1$. The projection $\operatorname{Hocolim} \mathbf{F} \xrightarrow{\Pi} \mathsf{P}$ is the projection to the first argument. With a functor $\mathsf{P} \xrightarrow{\mathbf{H}} \mathbf{Posets}^{0,\infty}$ we associate three functors $\mathsf{P} \to \mathbf{Posets}$. One is the composite of $\mathbf{H}$ and the forgetful functor (we denote it by $\widetilde{\mathbf{H}}$). The two others are the constant functors $\mathbf{0}, \boldsymbol{\infty}$ sending the whole P to 0 and ∞, respectively. The triad

$$(\operatorname{Hocolim} \widetilde{\mathbf{H}}, \operatorname{Hocolim} \mathbf{0}, \operatorname{Hocolim} \boldsymbol{\infty})$$

is exactly $\operatorname{Hocolim} \widetilde{\mathbf{H}}$ together with the graphs of two sections s_0 and s_∞ of the projection Π. Thus Hocolim of a functor with values in $\mathbf{Posets}^{0,\infty}$ is naturally equipped with 0 and ∞ sections of the projection Π.

The essential property of our general construction of the tangent bundle and Gauss functor of a poset P is that the tangent bundle $\langle \mathsf{EP} \xrightarrow{\mathsf{TP}} \mathsf{P}, s_0, s_\infty \rangle$ coincides with $\langle \operatorname{Hocolim} \widetilde{\mathbf{G}} \xrightarrow{\Pi} \mathsf{P}, s_0, s_\infty \rangle$.

4. Abstract manifolds. Here we define a slight generalization of Alexander's combinatorial manifold. As a reference on the combinatorial topology of posets and ball complexes, one may use [2].

A PL *ball complex* is a pair (X, U) where X is a compact Euclidean polyhedron and U is a covering of X by closed PL balls such that the following axioms are satisfied:

plbc1: the relative interiors of the balls from U form a partition of X;

plbc2: the boundary of each ball from U is a union of balls from U.

A PL ball complex is defined up to a PL homeomorphism only by the combinatorics of adjunctions of its balls. Let $\mathcal{D}$ be a PL ball complex. Consider the poset $\mathsf{P} = \mathbf{P}(\mathcal{D})$ of all its balls ordered by inclusion. Then $\mathcal{D}$ is a cellular complex PL homeomorphic to the complex $(B\mathsf{P}, \{B\mathsf{P}_{\leqslant p}\}_{p \in \mathsf{P}})$.

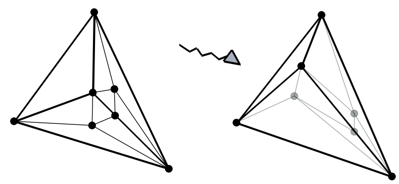

Fig. 1. An abstract aggregation.

Here $\mathsf{P}_{\leqslant p}$ is a principal ideal. For a poset Q, we denote by $B\mathsf{Q}$ the geometric realization of the order complex of Q.

This makes it possible to define an abstract PL ball complex: a finite poset P is called an *abstract* PL *ball complex* if for any $p \in \mathsf{P}$ the polyhedron $B\mathsf{P}_{<p}$ is a PL sphere. If P is an abstract ball complex, then $(B\mathsf{P}, \{B\mathsf{P}_{\leqslant p}\}_{p\in\mathsf{P}})$ is a PL ball complex [1].[1] We will say that principal ideals of an abstract ball complex P are its *balls*. A poset P is *pure* if all maximal chains have the same length. A pure poset P is an *abstract manifold* if both P and P^{op} are abstract PL ball complexes. Here P^{op} is P with the opposite order. If P is an abstract manifold, then the simplicial complex $\operatorname{Ord}\mathsf{P}$ is a combinatorial manifold in the classical sense of Alexander.

Let P_0, P_1 be abstract manifolds. We call a poset map $\mathsf{P}_0 \xrightarrow{\xi} \mathsf{P}_1$ an *aggregation morphism* if for any rank k ball O of P, the polyhedron $B\xi^{-1}(O)$ is a k-dimensional PL ball.

An aggregation morphism ξ can be realized up to an isomorphism as a geometric aggregation of PL ball complex structures on $B\mathsf{P}_0$. Consider the ball complex structure $(B\mathsf{P}_0, \{B\mathsf{P}_{0\leqslant p}\}_{p\in\mathsf{P}_0})$ on $B\mathsf{P}_0$. Then after gluing together all the balls that are sent by the morphism ξ to the same ball of P_1, we will obtain a geometric representation of P_1 up to an isomorphism. Figure 1 illustrates an abstract aggregation, and Fig. 2, a geometric aggregation. The composition of aggregation morphisms is an aggregation morphism.

We define an *abstract n-sphere* as an abstract manifold P such that the polyhedron $B\mathsf{P}$ is PL homeomorphic to S^n. Consider the category $\mathfrak{R}_n$ where the objects are the abstract n-spheres with one element of maximal

[1] The classical combinatorial characterization of ball complexes is in the topological category, not in the PL category, but for the PL case the theory works without any changes.

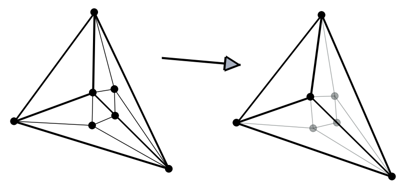

Fig. 2. A geometric aggregation.

rank specially labelled by "∞." Morphisms of $\mathfrak{R}_n$ are aggregations sending the ∞-element to the ∞-element. Thus $\mathfrak{R}_n$ is a subcategory of the category $\mathbf{Posets}^\infty$ of all posets with fixed element labelled by "∞."

We need one more definition. An abstract n-dimensional manifold P is *strict* if for any $x \in \mathsf{P}$ the pair of polyhedra $(B\,\mathrm{Star}\,x, B\,\mathrm{Link}\,x)$ is PL homeomorphic to the pair (D^n, S^{n-1}). Examples of strict abstract manifolds are the simplex posets of combinatorial manifolds and the face posets of convex polytopes.

5. The tangent bundle and Gauss functor of a strict abstract manifold. Now we can formulate our theorem on the tangent bundle of a poset in the case of a strict abstract manifold. The general message is that in the case of strict abstract manifolds (Sec. 4), the abstract construction (Sec. 3) represents the classical geometric construction (Sec. 2).

Let M^n be a strict abstract n-dimensional manifold. Denote by $\mathsf{M}^n \xrightarrow{\mathbf{G}^\infty} \mathbf{Posets}^\infty$ the composition of the Gauss functor $\mathbf{G}$ and the forgetful functor $\mathbf{Posets}^{0,\infty} \to \mathbf{Posets}^\infty$. We will also call $\mathbf{G}^\infty$ the "Gauss functor."

Theorem 1.
1. *The image of the Gauss functor* $\mathsf{M}^n \xrightarrow{\mathbf{G}^\infty} \mathbf{Posets}^\infty$ *belongs to* $\mathfrak{R}_n$.
2. *The order complex of* $\mathrm{E}\mathsf{M}^n$ *is a combinatorial manifold.*
3. *The* PL *map* $B\mathrm{E}\mathsf{M}^n \xrightarrow{B\mathrm{T}\mathsf{M}^n} B\mathsf{M}^n$ *together with two sections* Bs_0, Bs_∞ *is a Kuiper–Lashof* $(S^n, 0, \infty)$ *model of the Milnor tangent bundle of* $B\mathsf{M}^n$.
4. *The space* $B\mathfrak{R}_n$ *is homotopy equivalent to the space* $B\mathrm{PL}_n$.
5. *The map* $B\mathsf{M}^n \xrightarrow{B\mathbf{G}^\infty} B\mathfrak{R}_n$ *is a homotopy model of the* PL *Gauss map of* $B\mathsf{M}^n$.

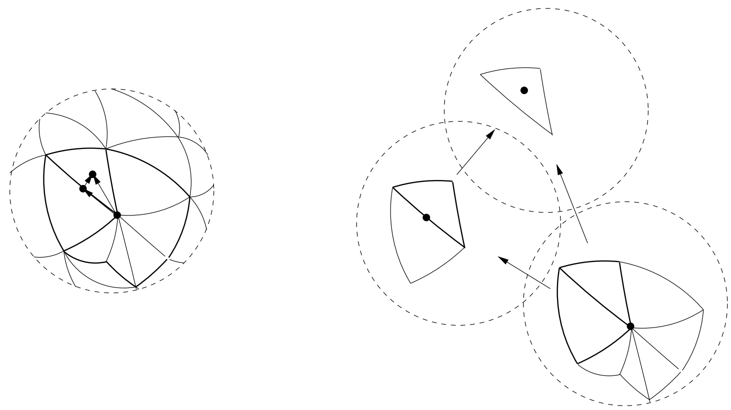

Fig. 3. The Gauss functor of a combinatorial sphere.

If one verifies Claim 1, then Claims 2 and 3 follow from the Kuiper–Lashof theory, simple properties of the Hocolim construction, and Alexander's trick. Claim 5 is a summary of 1–4 modulo standard abstract nonsense. The real thing to prove is Claim 4 ([13, Theorem C]).

To verify Claim 1 and the overall naturalness of $\mathbf{G}^\infty$, it is useful to consider the case where M^n is a combinatorial sphere S^n (see Fig. 3). Let s be a simplex of S^n. Then one can imagine $\mathbf{G}^\infty(s) \in \mathfrak{R}_n$ as follows: glue together all the simplices of S^n that do not contain s. This will be our new ball of $\mathbf{G}^\infty$ marked by ∞. The ball is naturally attached by Link s to Star s, and together they form the sphere $\mathbf{G}^\infty(s)$ with a marked ∞-ball. We should mention that while S^n is a combinatorial manifold, the "tangent sphere" $\mathbf{G}^\infty(s)$ is an abstract manifold, since the ∞-ball is usually nonsimplicial. Let $s_0 \subset s_1$ be a pair of simplices of S^n. Then $\operatorname{Star} s_0 \supset \operatorname{Star} s_1$. When we pass from $\mathbf{G}^\infty(s_0)$ to $\mathbf{G}^\infty(s_1)$, the simplices from $\operatorname{Star} s_0 \setminus \operatorname{Star} s_1$ dissolve in the ∞-ball of $\mathbf{G}^\infty(s_1)$. This operation is exactly the morphism $\mathbf{G}^\infty_{s_0 \subset s_1}$, and, as we see, this is exactly an aggregation morphism from $\mathfrak{R}_n$. In Fig. 4 we show the cellular $(S^1, 0, \infty)$ model of the tangent bundle of a triangle obtained by our recipe.

6. $B\mathfrak{R}_n \approx B\mathrm{PL}_n$**.** Now we will discuss the problem of proving Claim 4 of Theorem 1. Let X be a compact PL manifold. Consider the category $\mathfrak{R}(X)$ of all abstract manifolds M such that the polyhedron $B\mathsf{M}$ is PL homeomorphic to X. The morphisms of $\mathfrak{R}(X)$ are the aggregation morphisms.

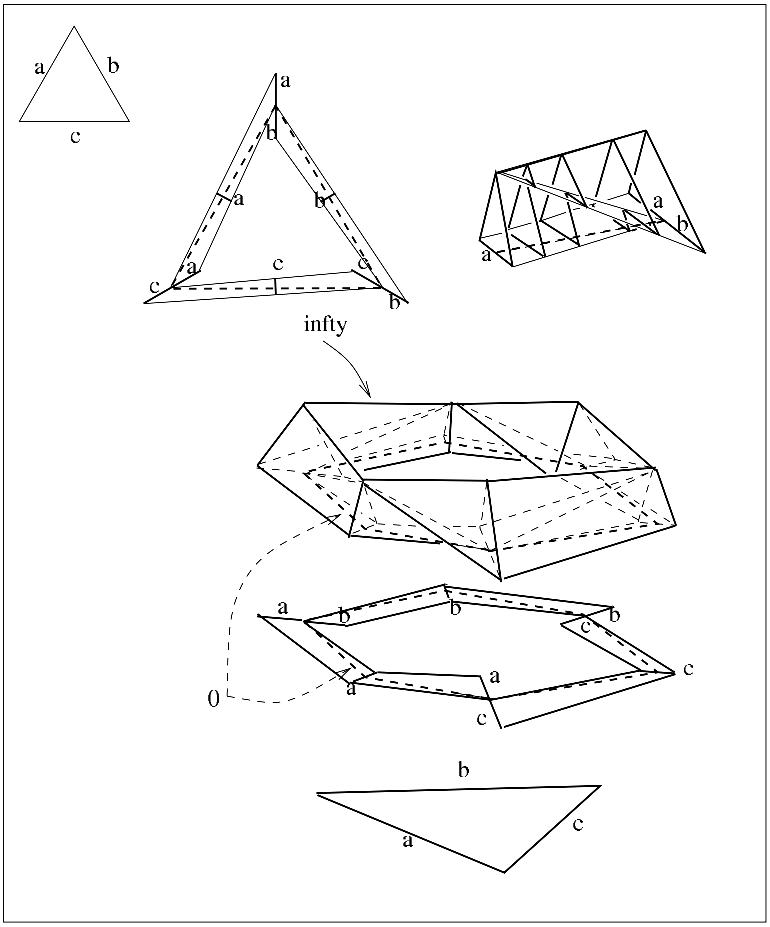

Fig. 4. The cellular $(S^1, 0, \infty)$ model of the tangent bundle of a triangle.

Theorem A in [13] states that

$$B\mathfrak{R}(X) \approx B\mathrm{PL}(X) \tag{1}$$

where PL(X) is the simplicial group of PL homeomorphisms of X. The proof that $B\mathfrak{R}_n \approx B\mathrm{PL}_n$ is a cosmetic variation of the general scheme developed for (1).

Let L be a PL polyhedron. We say that an $\mathfrak{R}(X)$-*coloring* of L is the following object: a linear triangulation K of L, $|\mathsf{K}| = L$, and an assignment to each vertex of K of an abstract manifold from $\mathfrak{R}(X)$, and to each 1-simplex of K, an aggregation morphism in such a way that all 2-simplices of K become commutative triangles in $\mathfrak{R}(X)$. Thus an $\mathfrak{R}(X)$-coloring of L is just a commutative diagram in $\mathfrak{R}(X)$ drawn on the 2-skeleton of some triangulation of L. The concordance of two $\mathfrak{R}(X)$-colorings ξ_0, ξ_1 of L is a coloring of the polyhedron $L \times [0,1]$ that induces the coloring ξ_i on the ith side. By abstract nonsense, to prove (1) is the same as to establish a functorial one-to-one correspondence between the isomorphism classes of PL fiber bundles on L with fiber X and the concordance classes of $\mathfrak{R}(X)$-colorings of L.

We mention how we would like, but cannot, establish such a correspondence. This speculation is borrowed from [15]. To any $\mathfrak{R}(X)$-coloring of L we can apply the construction Hocolim, and its geometric realization will produce a triangulated fiber bundle with fiber X. On the other hand, one can triangulate any fiber bundle with base L. To any triangulated fiber bundle J with base L and fiber X one can canonically associate ([4, 15]) some $\mathfrak{R}(X)$-coloring of the first barycentric subdivision of the base of J. We denote this construction by $\mathrm{Hocolim}^{-1}$. The composition $\mathrm{Hocolim} \circ \mathrm{Hocolim}^{-1}$ applied to a bundle produces an isomorphic bundle. We would prove (1) in a nice and short way if we could establish some canonical concordance between any $\mathfrak{R}(X)$-coloring ξ of L and the coloring $\mathrm{Hocolim}^{-1}\mathrm{Hocolm}\,\xi$. This would magically eliminate geometry. Unfortunately, there is no way to see such a canonical concordance. This is the cause of some published and many unpublished mistakes. From the theory developed in [13] it follows that ξ and $\mathrm{Hocolim}^{-1}\mathrm{Hocolm}\,\xi$ are concordant, but the concordance is transcendental.

Instead of using the Hocolim construction, we construct a bundle on L from an $\mathfrak{R}(X)$-coloring of L using the traditional construction of trivializations and structure homeomorphisms. Let K be an $\mathfrak{R}(X)$-colored simplicial complex, $|K| = L$. The coloring induces a coloring of the k-simplex of K by the chain

$$\mathsf{Q}_0 \rightsquigarrow \mathsf{Q}_1 \rightsquigarrow \cdots \rightsquigarrow \mathsf{Q}_k$$

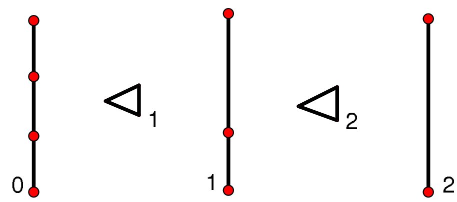

Fig. 5.

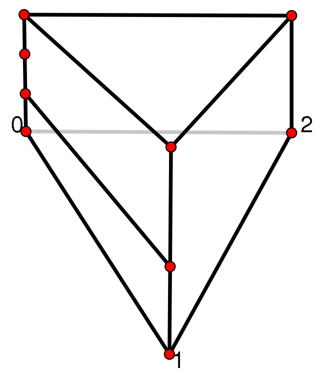

Fig. 6.

of abstract aggregations. According to the speculations in Sec. 4, one can realize this chain by the chain

$$Q = (Q_0 \trianglelefteq Q_1 \trianglelefteq \cdots \trianglelefteq Q_k)$$

of geometric aggregations of geometric PL ball complexes. With the chain Q one can associate a ball decomposition of the trivial bundle $X \times \Delta^k \xrightarrow{\pi} \Delta^k$ into the horizontal *"prisms"* which are the trivial subbundles with a ball as a fiber. Figures 5 and 6 illustrate the construction of the prismatic decomposition on π by the chain of geometric aggregations. The combinatorics of the coloring associates to any pair of simplices $s_0 \subset s_1$ in K a combinatorial isomorphism of two prismatic structures on the trivial bundle over s. By Alexander's trick, one can represent all these combinatorial isomorphisms by fiberwise structure PL homeomorphisms of the fiber bundle with base L and fiber X. All these structure homeomorphisms map prisms to prisms. As a result, we obtain from $\mathfrak{R}(X)$-colorings the class of fiber bundles with unusual structure homeomorphisms – "prismatic" ones.

In this setup, the inverse problem is to learn how to deform the structure homeomorphisms of an arbitrary PL fiber bundle into a "prismatic" form and present a consistent coloring in a controllable way.

At this point it is useful to recall the proof of the lemma on fragmentation of isotopy. This lemma was proved by Hudson [5] in the PL case. It states that for any covering $U = \{U_i\}_i$ of a manifold X by open balls and for any PL homeomorphism $X \xrightarrow{f} X$ that is isotopic to the identity there exists a finite decomposition $f = f_1 \circ \ldots \circ f_m$ such that for every i there exists j such that $\operatorname{supp} f_i \subset U_j$. The proof of the fragmentation lemma contains more information than the statement. In the proof, we pick an arbitrary PL isotopy F connecting f and the identity. Then we deform F in the class of isotopies with fixed ends to an isotopy F' of a special form. The isotopy F' corresponds to a chain of isotopies that are fixed on the complements to open balls from U.

The isotopy F is the same thing as a fiberwise homeomorphism

$$X \times [0,1] \xrightarrow{F} X \times [0,1]$$

commuting with the projection to $[0,1]$ and such that $F_0 = \mathrm{id}$ and $F_1 = f$. The homeomorphism F is the same thing as a one-dimensional foliation $\mathcal{F}$ on $X \times [0,1]$ transversal to the fibers of the projection (Fig. 7). The homeomorphism F' corresponds to a foliation $\mathcal{F}'$ with the following property: for any point $b \in [0,1]$, all the points $x \in X$ such that the leaf of $\mathcal{F}$ "is not horizontal" at (x,b) are contained in an element of U (Fig. 8). Inspecting the drawing of $\mathcal{F}'$, we see that one can subdivide the base $[0,1]$ into intervals $u_1, \ldots, u_m$ and introduce a prismatic structure on all subbundles $X \times u_i \xrightarrow{\pi_2} u_i$ such that the induced homeomorphisms $F'\lfloor_{u_i}$ are prismatic. Thus the construction of the fragmentation lemma allows us to deform a fiberwise homeomorphism of the trivial bundle over an interval into a system of prismatic homeomorphisms over a subdivision of the interval. The deformation $F \rightsquigarrow F'$ has a canonical form and possesses a coordinate generalization to homeomorphisms of the trivial bundle over a cube. So, realizing the program of such a generalization together with the development of appropriate surgery for fiberwise homeomorphisms takes about 100 pages in [13].

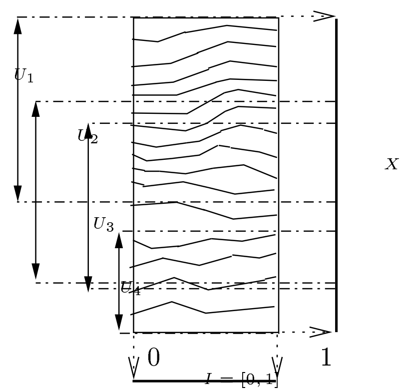

Fig. 7.

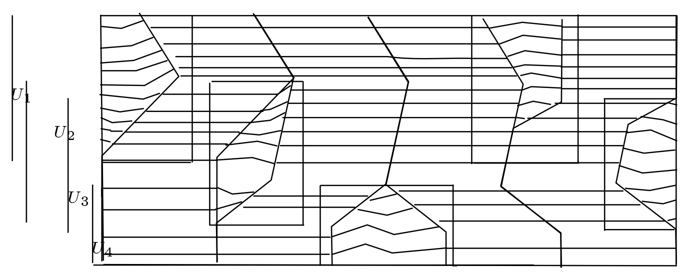

Fig. 8.

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