Записки научных
семинаров ПОМИ
Том 421, 2014 г.

I. A. Krepkiy

THE SANDPILE GROUPS OF CHAIN-CYCLIC GRAPHS

Abstract. Firstly, we consider the graphs obtained by gluing a family of arbitrary finite graphs to the edges of a cyclic graph and prove that the sandpile group of the resulting graph does not depend on a specific way of doing that. Then, we define the class of chain-cyclic graphs, which are the graphs obtained by connecting a finite family of cyclic graphs along a line. Two kinds of formulas for calculating the sandpile groups of chain-cyclic graphs are proved.

§1. Notations

Let M be an arbitrary square integer matrix. Denote the multiset of numbers arranged on the diagonal of the Smith normal form of M (see [3]) by $\overline{M}$.

We will use (as in [2]) the definition of the sandpile group of a graph in terms of the Smith normal form of the Laplacian matrix of this graph:

Definition 1. *The sandpile group of a graph G is the group $S(G) \cong \bigoplus_{a\in(\overline{M}\setminus\{0\})} C_a$, where M is a Laplacian matrix of G. (Here and below C_n is a cyclic group of order n.)*

(More detailed information about the two classical definitions of the sandpile groups can be found in [1].)

One can also write $S(G) \cong \bigoplus_{a\in\overline{M'}} C_a$, where M' is obtained from M by removing the row and column containing an arbitrary diagonal element (due to the properties of the Laplacian matrix).

Also, $\overline{A} = \overline{B}$, where the matrix B is obtained from the matrix A by a single operation of addition/subtraction of one column/row to another (due to the properties of the Smith normal form). Hence, the permutation of rows or columns of a matrix, as well as their multiplication by -1 does not affect the Smith normal form of this matrix.

Key words and phrases: sandpile group, Smith normal form, chain-cyclic graph.
The work on this paper was supported by RFBR grant 13-01-00383.

§2. Main theorem

Let F and G be some graphs. Let f_1, f_2 be non-negative integer-valued functions on the set of vertices of F, and g_1, g_2 be non-negative integer-valued functions on the set of vertices of G. Denote by T a cyclic graph consisting of n vertices. We can label the vertices of these graphs by natural numbers as follows:

1. r vertices of F are numbered from 1 to r;
2. n vertices of T are numbered from $r+1$ to $r+n$ in the order they appear in the loop;
3. s vertices of G are numbered from $r+n+1$ to $r+n+s$.

Now, for $i \in \mathbb{Z}, 0 \leqslant i \leqslant n-2$ we construct a new graph H_i as follows:

1. connect each vertex v of G with $r+n-1$th vertex of T by $g_1(v)$ edges;
2. connect each vertex v of G with $r+n$th vertex of T by $g_2(v)$ edges;
3. connect each vertex v of F with $r+i$th vertex of T by $f_1(v)$ edges (if $1 \leqslant i \leqslant n-2$); or connect each vertex v of F with $r+n$th vertex of T by $f_1(v)$ edges (if $i = 0$);
4. connect each vertex v of F with $r+i+1$th vertex of T by $f_2(v)$ edges.

For example, if the graphs F, T, G are as shown in Fig. $1--3$ and functions f_1, f_2, g_1, g_2 are such that

$f_1(1) = 1, \quad f_1(2) = 0,\ f_1(3) = 0,$
$f_2(1) = 0, \quad f_2(2) = 0,\ f_2(3) = 2,$
$g_1(10) = 1,\ g_1(11) = 1,$
$g_2(10) = 1,\ g_2(11) = 1$

then the resulting graphs H_1 and H_2 are shown in Figs. 4 and 5.

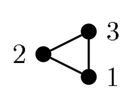

Fig. 1. F

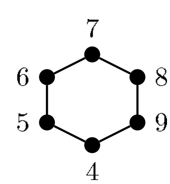

Fig. 2. T

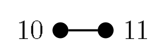

Fig. 3. G

Theorem 1. *The structure of the sandpile group of H_i does not depend on the choice of i.*

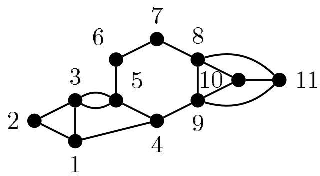

Fig. 4. H_1

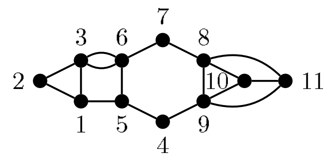

Fig. 5. H_2

Proof. It suffices to show that $S(H_k) \cong S(H_{k+1})$ for any $k \in \mathbb{Z}$, $0 \leqslant k \leqslant n-3$.

Here we only consider the case $1 \leqslant k \leqslant n-4$. (For the cases $k=0$ and $k=n-3$ the proof does not change, but the matrices are somewhat different from those described below.)

We denote the Laplacian matrices of H_k and H_{k+1} by A and B, respectively:

$$A = \begin{pmatrix} A_{1,1} & A_{1,2} & A_{1,3} \\ A_{2,1} & A_{2,2} & A_{2,3} \\ A_{3,1} & A_{3,2} & A_{3,3} \end{pmatrix}, \qquad B = \begin{pmatrix} B_{1,1} & B_{1,2} & B_{1,3} \\ B_{2,1} & B_{2,2} & B_{2,3} \\ B_{3,1} & B_{3,2} & B_{3,3} \end{pmatrix}.$$

Both matrices are divided into 9 blocks.

Block $(1,1)$ is the Laplacian matrix of F, which is modified by subtraction of p_i from the diagonal elements.

$$A_{1,1} = B_{1,1} = \begin{pmatrix} -p_1 & & & & \\ & -p_2 & & & \\ & & \ddots & & \\ & & & -p_{r-1} & \\ & & & & -p_r \end{pmatrix}.$$

Block $(3,3)$ is the Laplacian matrix of G, which is modified by subtraction of q_i from the diagonal elements.

(In these formulas we only show the numbers that should be subtracted from the diagonal elements of the corresponding matrices.)

Block $(2,2)$ is the Laplacian matrix of T, which is modified by subtraction of w', x', y', z' from the diagonal elements corresponding to the

$$A_{3,3} = B_{3,3} = \begin{pmatrix} -q_1 & & & & \\ & -q_2 & & & \\ & & \ddots & & \\ & & & -q_{s-1} & \\ & & & & -q_s \end{pmatrix}.$$

vertices connected to the vertices of F and G.

$$A_{2,2} = \begin{pmatrix} -2 & 1 & & & & & & & & & & 1 \\ 1 & -2 & 1 & & & & & & & & & \\ & 1 & \ddots & \ddots & & & & & & & & \\ & & \ddots & -2 & 1 & & & & & & & \\ & & & 1 & -2-w' & 1 & & & & & & \\ & & & & 1 & -2-x' & 1 & & & & & \\ & & & & & 1 & -2 & 1 & & & & \\ & & & & & & 1 & -2 & 1 & & & \\ & & & & & & & 1 & \ddots & \ddots & & \\ & & & & & & & & \ddots & -2 & 1 & \\ & & & & & & & & & 1 & -2-y' & 1 \\ 1 & & & & & & & & & & 1 & -2-z' \end{pmatrix}$$

$$B_{2,2} = \begin{pmatrix} -2 & 1 & & & & & & & & & & 1 \\ 1 & -2 & 1 & & & & & & & & & \\ & 1 & \ddots & \ddots & & & & & & & & \\ & & \ddots & -2 & 1 & & & & & & & \\ & & & 1 & -2 & 1 & & & & & & \\ & & & & 1 & -2-w' & 1 & & & & & \\ & & & & & 1 & -2-x' & 1 & & & & \\ & & & & & & 1 & -2 & 1 & & & \\ & & & & & & & 1 & \ddots & \ddots & & \\ & & & & & & & & \ddots & -2 & 1 & \\ & & & & & & & & & 1 & -2-y' & 1 \\ 1 & & & & & & & & & & 1 & -2-z' \end{pmatrix}$$

Here the numbers x', y', z', w' are equal to the numbers of edges that connect the graphs F and G with the respective vertices of the cycle T. Note that

$$w' = \sum_{1 \leqslant i \leqslant r} w_i, \quad x' = \sum_{1 \leqslant i \leqslant r} x_i,$$
$$y' = \sum_{1 \leqslant i \leqslant s} y_i, \quad z' = \sum_{1 \leqslant i \leqslant s} z_i,$$

$$p_i = x_i + w_i, \quad q_i = y_i + z_i.$$

Blocks $(1,2)$ and $(2,1)$ contain numbers x_i and w_i, which correspond to the numbers of edges connecting different vertices of F with two vertices of T.

Block $(1,2)$:

$$\begin{pmatrix} 0 & \cdots & 0 & w_1 & x_1 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & w_r & x_r & 0 & \cdots & 0 \end{pmatrix}.$$

The nonzero columns of $A_{1,2}$ are numbered as k and $k+1$, and the nonzero columns of $B_{1,2}$ are numbered as $k+1$ and $k+2$. $A_{2,1} = A_{1,2}^{\intercal}$, $B_{2,1} = B_{1,2}^{\intercal}$.

Blocks $(2,3)$ and $(3,2)$ contain numbers y_i and z_i, which correspond to the number of edges connecting different vertices of G with two vertices of T.

$$A_{3,2} = B_{3,2} = A_{2,3}^{\intercal} = B_{2,3}^{\intercal} = \begin{pmatrix} 0 & \cdots & 0 & y_1 & z_1 \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & y_s & z_s \end{pmatrix}.$$

Blocks $(1,3)$ and $(3,1)$ are empty:

$$A_{1,3} = B_{1,3} = A_{3,1}^{\intercal} = B_{3,1}^{\intercal} = \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix}.$$

The main difference between these matrices (matrices A and B) is the difference between numbers of rows and columns that contain numbers w_i, x_i, y_i, z_i.

Let us denote by A' the matrix obtained from A by removing the $(n+r)$th row and column:

$$A' = \begin{pmatrix} A'_{1,1} & A'_{1,2} & A'_{1,3} \\ A'_{2,1} & A'_{2,2} & A'_{2,3} \\ A'_{3,1} & A'_{3,2} & A'_{3,3} \end{pmatrix}.$$

Here $A'_{1,1} = A_{1,1}$, $A'_{1,3} = A_{1,3}$, $A'_{3,1} = A_{3,1}$, $A'_{3,3} = A_{3,3}$.

Block $(1,2)$:

$$A'_{1,2} = A'^{\mathsf{T}}_{2,1} = \begin{pmatrix} 0 & \cdots & 0 & w_1 & x_1 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & w_r & x_r & 0 & \cdots & 0 \end{pmatrix}.$$

The nonzero columns of $A'_{1,2}$ are numbered as k and $k+1$.
Block $(3,2)$:

$$A'_(3,2) = A'^{\mathsf{T}}_{2,3} = \begin{pmatrix} 0 & \cdots & 0 & y_1 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & y_s \end{pmatrix}.$$

Block $(2,2)$:

$$A'_{2,2} = \begin{pmatrix} -2 & 1 & & & & & & & & & & \\ 1 & -2 & 1 & & & & & & & & & \\ & 1 & \ddots & \ddots & & & & & & & & \\ & & \ddots & -2 & 1 & & & & & & & \\ & & & 1 & -2-w' & 1 & & & & & & \\ & & & & 1 & -2-x' & 1 & & & & & \\ & & & & & 1 & -2 & 1 & & & & \\ & & & & & & 1 & -2 & 1 & & & \\ & & & & & & & 1 & \ddots & \ddots & & \\ & & & & & & & & \ddots & -2 & 1 \\ & & & & & & & & & 1 & -2-y' \end{pmatrix}$$

Let us denote by B' the matrix obtained from B by removing the $(r+1)$th row and column:

$$B' = \begin{pmatrix} B'_{1,1} & B'_{1,2} & B'_{1,3} \\ B'_{2,1} & B'_{2,2} & B'_{2,3} \\ B'_{3,1} & B'_{3,2} & B'_{3,3} \end{pmatrix}.$$

Here $B'_{1,1} = B_{1,1}$, $B'_{1,3} = B_{1,3}$, $B'_{3,1} = B_{3,1}$, $B'_{3,3} = B_{3,3}$.
Blocks $(1,2)$ and $(2,1)$ coincide with the respective blocks of the matrix A':
$B'_{1,2} = A'_{1,2}$, $B'_{2,1} = B'^{\mathsf{T}}_{1,2}$.
Block $(3,2)$:

$$B'_{3,2} = B'^{\mathsf{T}}_{2,3} = \begin{pmatrix} 0 & \cdots & 0 & y_1 & z_1 \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & y_s & z_s \end{pmatrix}.$$

Block $(2,2)$:

$$
B'_{2,2}=\begin{pmatrix}
-2 & 1 & & & & & & & & & & & \\
1 & \ddots & \ddots & & & & & & & & & & \\
& \ddots & -2 & 1 & & & & & & & & & \\
& & 1 & -2 & 1 & & & & & & & & \\
& & & 1 & -2-w' & 1 & & & & & & & \\
& & & & 1 & -2-x' & 1 & & & & & & \\
& & & & & 1 & -2 & 1 & & & & & \\
& & & & & & 1 & \ddots & \ddots & & & & \\
& & & & & & & \ddots & -2 & 1 & & & \\
& & & & & & & & 1 & -2-y' & 1 \\
& & & & & & & & & 1 & -2-z'
\end{pmatrix}
$$

We need to show that $\overline{A'}=\overline{B'}$. To do this, we perform a series of actions with columns and rows of B'.

We add the last $s+1$ rows to the $(r+n-2)$th row. Only the blocks $(2,2)$ and $(2,3)$ get changed:

$$
(2,2)=\begin{pmatrix}
-2 & 1 & & & & & & & & & & & \\
1 & \ddots & \ddots & & & & & & & & & & \\
& \ddots & -2 & 1 & & & & & & & & & \\
& & 1 & -2 & 1 & & & & & & & & \\
& & & 1 & -2-w' & 1 & & & & & & & \\
& & & & 1 & -2-x' & 1 & & & & & & \\
& & & & & 1 & -2 & 1 & & & & & \\
& & & & & & 1 & \ddots & \ddots & & & & \\
& & & & & & & \ddots & -2 & 1 & & & \\
& & & & & & & & 1 & -1 & -1 \\
& & & & & & & & & 1 & -2-z'
\end{pmatrix}
$$

$$
(2,3)=\begin{pmatrix}
0 & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & 0 \\
z_1 & \cdots & z_s
\end{pmatrix}.
$$

We add the last $s+1$ columns to the $(r+n-2)$th column. Only the blocks $(2,2)$ and $(3,2)$ get changed:

$$(2,2)=\begin{pmatrix}
-2 & 1 & & & & & & & & & \\
1 & \ddots & \ddots & & & & & & & & \\
 & \ddots & -2 & 1 & & & & & & & \\
 & & 1 & -2 & 1 & & & & & & \\
 & & & 1 & -2-w' & 1 & & & & & \\
 & & & & 1 & -2-x' & 1 & & & & \\
 & & & & & 1 & -2 & 1 & & & \\
 & & & & & & 1 & \ddots & \ddots & & \\
 & & & & & & & \ddots & -2 & 1 & \\
 & & & & & & & & 1 & -2 & -1 \\
 & & & & & & & & & -1 & -2-z'
\end{pmatrix}$$

$$(3,2)=\begin{pmatrix}
0 & \cdots & 0 & z_1 \\
\vdots & \ddots & \vdots & \vdots \\
0 & \cdots & 0 & z_s
\end{pmatrix}.$$

We add the last s rows to the $(r+n-1)$th row. Only the blocks $(2,2)$ and $(2,3)$ get changed:

$$(2,2)=\begin{pmatrix}
-2 & 1 & & & & & & & & & \\
1 & \ddots & \ddots & & & & & & & & \\
 & \ddots & -2 & 1 & & & & & & & \\
 & & 1 & -2 & 1 & & & & & & \\
 & & & 1 & -2-w' & 1 & & & & & \\
 & & & & 1 & -2-x' & 1 & & & & \\
 & & & & & 1 & -2 & 1 & & & \\
 & & & & & & 1 & \ddots & \ddots & & \\
 & & & & & & & \ddots & -2 & 1 & \\
 & & & & & & & & 1 & -2 & -1 \\
 & & & & & & & & & -1 & -2
\end{pmatrix}$$

$$(2,3)=\begin{pmatrix}
0 & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & 0 \\
-y_1 & \cdots & -y_s
\end{pmatrix}.$$

We add the last s columns to the $(r+n-1)$th column. Only the blocks $(2,2)$ and $(3,2)$ get changed:

$$(2,2)=\begin{pmatrix}
-2 & 1 & & & & & & & & & & \\
1 & \ddots & \ddots & & & & & & & & & \\
 & \ddots & -2 & 1 & & & & & & & & \\
 & & 1 & -2 & 1 & & & & & & & \\
 & & & 1 & -2-w' & 1 & & & & & & \\
 & & & & 1 & -2-x' & 1 & & & & & \\
 & & & & & 1 & -2 & 1 & & & & \\
 & & & & & & 1 & \ddots & \ddots & & & \\
 & & & & & & & \ddots & -2 & 1 & & \\
 & & & & & & & & 1 & -2 & -1 \\
 & & & & & & & & & -1 & -2-y'
\end{pmatrix}$$

$$(3,2)=\begin{pmatrix} 0 & \cdots & 0 & -y_1 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & -y_s \end{pmatrix}.$$

Finally, we change signs of $(r+n-1)$th row and column to obtain the matrix A'. Only the blocks $(2,2)$, $(2,3)$, and $(3,2)$ get changed:

$$(2,2)=\begin{pmatrix}
-2 & 1 & & & & & & & & & & \\
1 & \ddots & \ddots & & & & & & & & & \\
 & \ddots & -2 & 1 & & & & & & & & \\
 & & 1 & -2 & 1 & & & & & & & \\
 & & & 1 & -2-w' & 1 & & & & & & \\
 & & & & 1 & -2-x' & 1 & & & & & \\
 & & & & & 1 & -2 & 1 & & & & \\
 & & & & & & 1 & \ddots & \ddots & & & \\
 & & & & & & & \ddots & -2 & 1 & & \\
 & & & & & & & & 1 & -2 & 1 \\
 & & & & & & & & & 1 & -2-y'
\end{pmatrix}=A'_{2,2},$$

$$(2,3)=\begin{pmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \\ y_1 & \cdots & y_s \end{pmatrix}=A'_{2,3}; \quad (3,2)=\begin{pmatrix} 0 & \cdots & 0 & y_1 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & y_s \end{pmatrix}=A'_{3,2}.$$

The blocks $(1,1)$, $(1,2)$, $(2,1)$, $(1,3)$, $(3,1)$, $(3,3)$ have not changed and coincide with the corresponding blocks of the matrix A. So $\overline{A'}=\overline{B'}$ and the theorem is proved. □

There is also a weaker version of this theorem that is applicable to a smaller class of graphs. The proof of the second theorem can be done without cumbersome manipulations with matrices.

Let F be an arbitrary planar graph and f is one of its edges that is adjacent to the outer face of F. Let G be an arbitrary planar graph and g is one of its edges that is adjacent to the outer face of G. Let T be a cyclic planar graph on n vertices. The edges of T are marked as $t_0, t_1, \ldots, t_{n-1}$ in the order they appear on the loop.

For $i \in \mathbb{Z}$, $1 \leqslant i \leqslant n-1$ we construct a planar graph J_i as follows:

1. We merge the edges t_0 and g in such a way that all vertices of G are placed on the outer face of T.

2. We merge the edges t_i and f in such a way that all vertices of F are placed on the outer face of T.

We assume that the structure of F and G (as planar graphs) is preserved and notice that the cycle of T now has exactly $n-2$ edges that are adjacent to the outer face of J_i.

Theorem 2. *The structure of the sandpile group of J_i does not depend on the choice of i.*

Proof. It suffices to show that for every $2 \leqslant i \leqslant n-1$ the sandpile groups of J_1 and J_i are isomorphic. There is a theorem which states that the sandpile groups of a planar graph and its dual graph are isomorphic ([4]). It is obvious that the graphs J_1' and J_i' (that are duals, respectively, of J_1 and J_i) are isomorphic, which implies that $S(J_1) \cong S(J_1') \cong S(J_i') \cong S(J_i)$. $\square$

§3. Chain-cyclic graphs

The theorem proved above can be used to calculate the sandpile groups for a class of graphs that we call chain-cyclic graphs. This is the union of a countable family of finite sets of graphs denoted by $\mathrm{CH}(a_1, a_2, \ldots, a_n)$, where $a_i \in \mathbb{Z}$, $a_i \geqslant 2$, $i \in [1 \ldots n]$. In order to give the exact definition, we assume that every graph G belonging to this class has a distinguished cycle $L(G) = [v_1, \ldots, v_k]$.

Starting from $n = 1$, we declare that the class $\mathrm{CH}(a)$ contains only one graph, namely, the cycle T_a on a vertices $v_1, \ldots, v_a$ (numbered along the cycle), and we set $L(T_a) = [v_1, \ldots, v_a]$.

Then we proceed by induction. Every graph $G \in \mathrm{CH}(a_1, a_2, \ldots, a_{n+1})$ is constructed from some graph $H \in \mathrm{CH}(a_1, a_2, \ldots, a_n)$. Let $L(H) = [w_1, w_2, \ldots, w_k]$. Fix an arbitrary integer i, such that $1 \leqslant i < k$. Add to the graph H a linear chain of $a_{n+1} - 2$ vertices $v_1, \dots, v_{a_{n+1}-2}$, connected by edges in accordance with the order of indices. Connect vertices v_1 and w_i by one edge and connect vertices $v_{a_{n+1}-2}$ and w_{i+1} by another edge. Now let $L(G) = [w_i, v_1, \dots, v_{a_{n+1}-2}, w_{i+1}]$. (If $a_{n+1} = 2$, we just need to add one more edge between vertices w_i and w_{i+1} and suppose that $L(G) = [w_i, w_{i+1}]$.) At this point the construction of G is complete. For example, Fig. 6 shows a graph of class $\mathrm{CH}(3, 6, 4, 6)$ (here straight lines denote the edges of the subgraph of class $\mathrm{CH}(3, 6, 4)$).

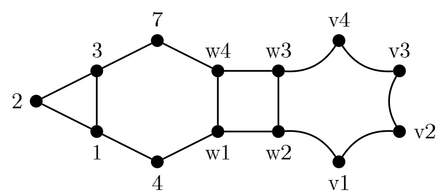

Fig. 6. $G \in \mathrm{CH}(3, 6, 4, 6)$

Less formally, any graph $G \in \mathrm{CH}(a_1, a_2, \dots, a_n)$ consists of a “chain”, obtained through a series of connection (by edges) of an ordered set of cyclic graphs with lengths $a_1, a_2, \dots, a_n$. For example Fig. 7–9 show three different graphs belonging to the class $\mathrm{CH}(3, 6, 4, 6)$.

It is obvious (by main theorem) that for any $G, H \in \mathrm{CH}(a_1, a_2, \dots, a_n)$ we have $S(G) \cong S(H)$. For example, the sandpile group of each of three graphs shown in Fig. 7–9 has the structure of C_{373}.

For each class $\mathrm{CH}(a_1, a_2, \dots, a_n)$ we choose a canonical representative of this class – a graph that is arranged in such a way that all of its n main cycles have a common vertex. We will denote this graph by $\mathrm{Ch}(a_1, a_2, \dots, a_n)$. For example, the canonical representative of the class $\mathrm{CH}(3, 6, 4, 6)$ is shown in Fig. 10.

Now our task is to calculate the sandpile group of each of these canonical representatives.

What is the Laplacian matrix of the graph $\mathrm{Ch}(a_1, a_2, \dots, a_n)$? Suppose that it has $k + 1$ vertices. Let us enumerate them as follows. The only common vertex of all cycles is assigned the number $k+1$. All other vertices are given the numbers from 1 to k to match the order they appear in the “outer” cycle of the whole graph. An example of numbering of the vertices in $\mathrm{Ch}(3, 6, 4, 6)$ is shown in Fig. 10.

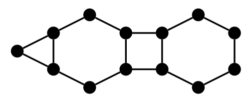

Fig. 7. $G \in \mathrm{CH}(3, 6, 4, 6)$

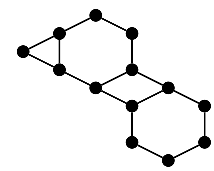

Fig. 8. $G \in \mathrm{CH}(3, 6, 4, 6)$

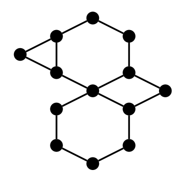

Fig. 9. $G \in \mathrm{CH}(3, 6, 4, 6)$

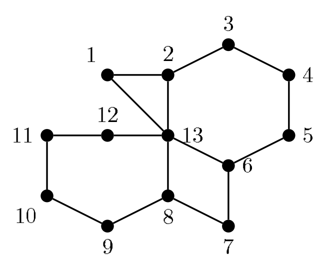

Fig. 10. $\mathrm{Ch}(3, 6, 4, 6)$

Obviously, in such a numbering the Laplacian matrix of the graph $\mathrm{Ch}(a_1, a_2, \ldots, a_n)$ looks as follows:

$$\left(\begin{array}{c|c|c|c|c|c}
h_1 & 1 & & & & z_1 \\ \hline
1 & h_2 & 1 & & & z_2 \\ \hline
 & 1 & \ddots & \ddots & & \vdots \\ \hline
 & & \ddots & h_{k-1} & 1 & z_{k-1} \\ \hline
 & & & 1 & h_k & z_k \\ \hline
z_1 & z_2 & \cdots & z_{k-1} & z_k & h_{k+1}
\end{array}\right)$$

We remove the last column and the last row of the matrix and denote the resulting matrix by M:

$$\left(\begin{array}{c|c|c|c|c} h_1 & 1 & & & \\ \hline 1 & h_2 & 1 & & \\ \hline & 1 & \ddots & \ddots & \\ \hline & & \ddots & h_{k-1} & 1 \\ \hline & & & 1 & h_k \end{array}\right).$$

To calculate the sandpile group of the graph $\mathrm{Ch}(a_1, a_2, \ldots, a_n)$ it is sufficient to compute the Smith normal form of the matrix M.

We transform M as follows:

1. From the second row we substract the first row h_2 times. The second row takes the form $(r_2, 0, 1, 0, \ldots, 0)$, where, $r_2 = 1 - r_1 h_2$, $r_1 = h_1$.

2. For i from 3 to k we repeat the following procedure:

From the the ith row we substract the i_1th row h_i times, then once from the ith row we subtract the $(i-2)$th row.

As a result, the ith row becomes $(r_i, 0, \ldots, 0, 1, 0, \ldots, 0)$, where the number 1 is in the $(i+1)$th position. $r_i = -r_{i-1}h_i - r_{i-2}$.

3. Finally we move the last line to the first position.

Now, the matrix takes the form:

$$\left(\begin{array}{c|c|c|c|c} r_k & & & & \\ \hline r_1 & 1 & & & \\ \hline \vdots & & \ddots & & \\ \hline r_{k-2} & & & 1 & \\ \hline r_{k-1} & & & & 1 \end{array}\right).$$

It is clear that the "unwanted" elements of the first column can be removed by manipulation with other columns. Therefore, the Smith normal form of the matrix M looks like

$$\left(\begin{array}{c|c|c|c|c} 1 & & & & \\ \hline & 1 & & & \\ \hline & & \ddots & & \\ \hline & & & 1 & \\ \hline & & & & r_k \end{array}\right).$$

So the group of the graph $\mathrm{Ch}(a_1, a_2, \ldots, a_n)$ is cyclic and we need only to determine its cardinality.

We use the well-known statement about the cardinality of the sandpile group of a graph. Namely, the cardinality of the sandpile group of a graph equals the number of spanning trees of this graph ([1]).

Let us count the number of spanning trees of a graph of class $\mathrm{CH}(a_1, a_2, \dots, a_n, a_{n+1})$.

We define a series of functions $F_i(x_1, x_2, \dots, x_i)$, $i \in \mathbb{N}$:

$$F_n(a_1, \dots, a_n) = |\mathrm{Ch}(a_1, \dots, a_n)|.$$

Next, recall that our graph is obtained from the graph of class $\mathrm{CH}(a_1, a_2, \dots, a_n)$ by adding of $a_{n+1} - 2$ vertices and $a_{n+1} - 1$ edges. These additional edges together with edge u (which previously belonged to the graph of class $\mathrm{CH}(a_1, a_2, \dots, a_n)$) constitute a cycle of length a_{n+1}. (For example, Fig. 11 shows a graph of class $\mathrm{CH}(3, 6, 4, 6)$ with its edge u.)

To obtain a spanning tree, we need to remove some edges belonging to this cycle. Here we have two options:

1. Remove exactly one of $a_{n+1} - 1$ edges (which differ from u) in the cycle of length a_n (edge that did not belong to the graph of class $\mathrm{CH}(a_1, a_2, \dots, a_n)$). (Example of the result of such an operation on the graph of class $\mathrm{CH}(3, 6, 4, 6)$ is shown in Fig. 12.) It is clear that we can not remove more edges from this set, if we are going to get a spanning tree. We get a graph with number of spanning trees equal to the number of spanning trees of $\mathrm{Ch}(a_1, a_2, \dots, a_n)$, which means that here we have $(a_{n+1} - 1) \cdot F_n(a_1, a_2, \dots, a_n)$ opportunities to construct a spanning tree.

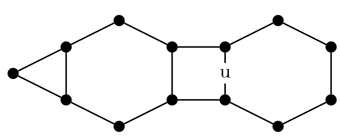

Fig. 11. The graph of class CH(3, 6, 4, 6)

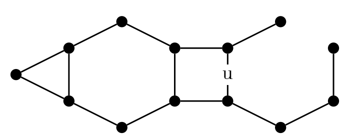

Fig. 12. $\mathrm{CH}(3, 6, 4, 6) \to \mathrm{CH}(3, 6, 4)$

2. Do not delete any of $a_{n+1} - 1$ edges (which differ from u) of cycle of length n (edges that did not belong to the graph of class $\mathrm{CH}(a_1, a_2, \dots, a_n)$). In this case, the only way to get rid of the cycle is to remove the edge u. After removing it, we get a graph of class $\mathrm{CH}(a_1, a_2, \dots, a_{n-1}, a_n + a_{n+1} - 2)$. (Fig.13 shows a graph obtained from the graph of class $\mathrm{CH}(3, 6, 4, 6)$ by removing the edge u. The edges, which we agreed not to remove, are marked

by curved lines. Fig.14 shows the graph, obtained from the graph of class CH$(3,6,8)$ by "contraction" of fixed edges.) But since we have agreed not to touch the $a_{n+1}-1$ edges, then it is clear that the number of spanning trees that we can get is the same as the number of spanning trees of the graph Ch$(a_1,a_2,\ldots,a_{n-1},a_n-1)$, which means that here we have $F_n(a_1,a_2,\ldots,a_n-1)$ opportunities to construct a spanning tree. Generally speaking, there is an inaccuracy. There is no description of how to act in the case $a_n=2$. This inaccuracy will be corrected later.

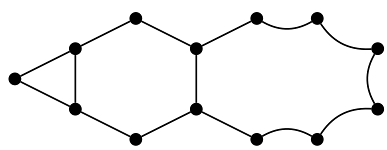

Fig. 13. A graph of class CH$(3,6,8)$

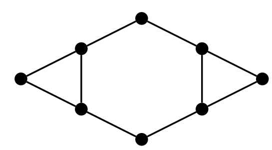

Fig. 14. A graph of class CH$(3,6,3)$

It is also clear that the cyclic graph Ch(a) has exactly a spanning trees. So now the functions F_i are defined as follows:

$$F_1(x_1)=x_1, F_{i+1}(x_1,x_2,\ldots,x_i,x_{i+1}) = F_i(x_1,x_2,\ldots,x_i-1)+(x_{i+1}-1)\cdot F_i(x_1,x_2,\ldots,x_i). \quad (1)$$

Generally speaking, such a definition of the functions F_n is "redundant" in the sense that their values are determined by including the cases where some of the arguments are equal to 1 (despite the fact that in the description of Ch$(a_1,\ldots,a_n)$ all the arguments are larger than 1). This feature is used to eliminate the inaccuracies that arose in counting the spanning trees of a graph. Suppose we have a graph of the class CH$(a_1,a_2,\ldots,a_k,2,2,\ldots,2,2,m)$, where $a_k>2$. We can calculate the corresponding value of F_n, using the new recursive definition:

$$\begin{aligned}
F_n(a_1, a_2, \dots, a_k, 2, 2, \dots, 2, 2, m) \\
&= (m-1) \cdot F_{n-1}(a_1, a_2, \dots, a_k, 2, 2, \dots, 2, 2) \\
&+ F_{n-1}(a_1, a_2, \dots, a_k, 2, 2, \dots, 2, 1) \\
&= (m-1) \cdot F_{n-1}(a_1, a_2, \dots, a_k, 2, 2, \dots, 2, 2) \\
&+ (1-1) \cdot F_{n-2}(a_1, a_2, \dots, a_k, 2, 2, \dots, 2) \\
&+ F_{n-2}(a_1, a_2, \dots, a_k, 2, 2, \dots, 1) \\
&= \dots = (m-1) \cdot F_{n-1}(a_1, a_2, \dots, a_k, 2, 2, \dots, 2, 2) \\
&+ F_k(a_1, a_2, \dots, a_k - 1).
\end{aligned}$$

This result is consistent with the process of counting the spanning trees:

1. The number $(m-1) \cdot F_{n-1}(a_1, a_2, \dots, a_k, 2, 2, \dots, 2, 2)$ corresponds to the case when we remove one of $m-1$ edges (which differ from u) in a cycle of length m.

2. When we intend to save the $m-1$ edges, we are obliged to remove not only the u, but all other edges connecting vertices that are connected by the edge u. When we do that, we get a graph of class $\mathrm{CH}(a_1, a_2, \dots, a_k+m-2)$. Since we have agreed not to touch $m-1$ of its edges, then the number of spanning trees that we can get equals the number of spanning trees of the graph $\mathrm{Ch}(a_1, a_2, \dots, a_k-1)$. This number is precisely $F_k(a_1, a_2, \dots, a_k-1)$.

Hence the functions F_n are well-defined.

Now we find a non-recursive representation of functions F_n. To do this, we need some auxiliary objects.

For non-negative integers i, j define the set $C_{i,j}$.

$I \in C_{i,j}$ if and only if all of the following conditions are true:

1. $I \subset \mathbb{N}$.
2. $|I| = i$.
3. $x \in I \Rightarrow 1 \leqslant x \leqslant j$.
4. $(a, b \in I, a < b \mid (\forall t \in (a, b) \Rightarrow t \notin I)) \Rightarrow (b - a \equiv 1 \pmod 2)$.
5. $j - \max(I) \equiv 0 \pmod 2$.

We also define the sets $A_{i,j}$ and $B_{i,j}$ as follows:

1. $C_{i,j} = A_{i,j} \bigsqcup B_{i,j}$.
2. $I \in B_{i,j} \Leftrightarrow j \in I$.

We define the functions $\alpha_{i,j}, \beta_{i,j}, \gamma_{i,j}$ and $\beta'_{i,j}$:

$\alpha_{i,j}(x_1, \dots, x_j) = \sum\limits_{I \in A_{i,j}} 1 * \prod\limits_{i \in I} x_i$.

$\beta_{i,j}(x_1,\dots,x_j) = \sum_{I\in B_{i,j}} 1 * \prod_{i\in I} x_i$.
$\gamma_{i,j}(x_1,\dots,x_j) = \sum_{I\in C_{i,j}} 1 * \prod_{i\in I} x_i$.
$\beta'_{i,j}(x_1,\dots,x_{j-1}) = \beta_{i,j}(x_1,\dots,x_{j-1},1)$.
Thus, for example, with $i=3$, $j=7$ these functions are:
$\alpha_{3,7}(x_1,\dots,x_7) = x_1x_2x_3 + x_1x_2x_5 + x_1x_4x_5 + x_3x_4x_5$
$\beta_{3,7}(x_1,\dots,x_7) = x_1x_2x_7 + x_1x_4x_7 + x_1x_6x_7 + x_3x_4x_7 + x_3x_6x_7 + x_5x_6x_7$
$\gamma_{3,7}(x_1,\dots,x_7) = x_1x_2x_3 + x_1x_2x_5 + x_1x_2x_7 + x_1x_4x_5 + x_1x_4x_7 + x_1x_6x_7 + x_3x_4x_5 + x_3x_4x_7 + x_3x_6x_7 + x_5x_6x_7$
$\beta'_{3,7}(x_1,\dots,x_6) = x_1x_2 + x_1x_4 + x_1x_6 + x_3x_4 + x_3x_6 + x_5x_6$

It is clear that $\gamma_{i,j} = \alpha_{i,j} + \beta_{i,j}$.

It is also easy to check that for any positive i,j the following equation is true:

$$\gamma_{i,j} = x_j \cdot \gamma_{i-1,j-1} + \beta'_{i+1,j-1}. \tag{2}$$

For $n\in\mathbb{N}$ we define a function G_n:

$$G_n(x_1,\dots,x_n) = \gamma_{n,n} - \gamma_{n-2,n} + \gamma_{n-4,n} - \gamma_{n-6,n} + \cdots , \tag{3}$$

where the sum extends while the first subscript is non-negative.

To be more specific:
1. $G_{4k} = \gamma_{4k,4k} - \gamma_{4k-2,4k} + \cdots + \gamma_{0,4k}$;
2. $G_{4k+1} = \gamma_{4k+1,4k+1} - \gamma_{4k-1,4k+1} + \cdots + \gamma_{1,4k+1}$;
3. $G_{4k+2} = \gamma_{4k+2,4k+2} - \gamma_{4k,4k+2} + \cdots - \gamma_{0,4k+2}$;
4. $G_{4k+3} = \gamma_{4k+3,4k+3} - \gamma_{4k+1,4k+3} + \cdots - \gamma_{1,4k+3}$.

Theorem 3. $G_n(x_1,\dots,x_n) = F_n(x_1,\dots,x_n)$

Proof. $G_1(x_1) = \gamma_{1,1} = x_1 = F_1(x_1)$.

Suppose that $G_i = F_i$. We must show that $G_{i+1} = F_{i+1}$. To do this, according to (1), it suffices to show that

$$G_{i+1}(x_1,x_2,\dots,x_i,x_{i+1}) = F_i(x_1,x_2,\dots,x_i-1)$$

$$+(x_{i+1}-1)\cdot F_i(x_1,x_2,\dots,x_i) = \text{[by the induction hypothesis]}$$

$$= G_i(x_1,x_2,\dots,x_i-1) + (x_{i+1}-1)\cdot G_i(x_1,x_2,\dots,x_i).$$

Indeed, consider, for example, the case $i = 4k$ (for the remaining cases the chain of equalities is built in a similar way):

$$
\begin{aligned}
&G_i(x_1, x_2, \dots, x_i - 1) + (x_{i+1} - 1)\cdot G_i(x_1, x_2, \dots, x_i)\\
&= G_{4k}(x_1, x_2, \dots, x_{4k} - 1) + (x_{4k+1} - 1)\cdot G_{4k}(x_1, x_2, \dots, x_{4k})\\
&= (\gamma_{4k,4k} - \gamma_{4k-2,4k} + \dots + \gamma_{0,4k})(x_1, x_2, \dots, x_{4k} - 1)\\
&+ (x_{4k+1}\cdot\gamma_{4k,4k} - x_{4k+1}\cdot\gamma_{4k-2,4k} + \dots - x_{4k+1}\cdot\gamma_{2,4k}\\
&+ x_{4k+1}\cdot\gamma_{0,4k}) - (\gamma_{4k,4k} - \gamma_{4k-2,4k} + \dots - \gamma_{2,4k} + \gamma_{0,4k})\\
&= (\alpha_{4k,4k} - \alpha_{4k-2,4k} + \dots + \alpha_{0,4k})(x_1, x_2, \dots, x_{4k} - 1)\\
&+ (\beta_{4k,4k} - \beta_{4k-2,4k} + \dots + \beta_{0,4k})(x_1, x_2, \dots, x_{4k} - 1)\\
&+ (x_{4k+1}\cdot\gamma_{4k,4k} - x_{4k+1}\cdot\gamma_{4k-2,4k} + \dots - x_{4k+1}\cdot\gamma_{2,4k} + x_{4k+1}\cdot\gamma_{0,4k})\\
&- (\gamma_{4k,4k} - \gamma_{4k-2,4k} + \dots - \gamma_{2,4k} + \gamma_{0,4k})\\
&= (\alpha_{4k,4k} - \alpha_{4k-2,4k} + \dots + \alpha_{0,4k}) + (\beta_{4k,4k} - \beta_{4k-2,4k} + \dots + \beta_{0,4k})\\
&- (\beta'_{4k,4k} - \beta'_{4k-2,4k} + \dots + \beta'_{0,4k})\\
&+ (x_{4k+1}\cdot\gamma_{4k,4k} - x_{4k+1}\cdot\gamma_{4k-2,4k} + \dots - x_{4k+1}\cdot\gamma_{2,4k}\\
&+ x_{4k+1}\cdot\gamma_{0,4k}) - (\gamma_{4k,4k} - \gamma_{4k-2,4k} + \dots - \gamma_{2,4k} + \gamma_{0,4k})\\
&= (\gamma_{4k,4k} - \gamma_{4k-2,4k} + \dots - \gamma_{2,4k} + \gamma_{0,4k})\\
&+ (\beta'_{4k,4k} - \beta'_{4k-2,4k} + \dots + \beta'_{0,4k})\\
&+ (x_{4k+1}\cdot\gamma_{4k,4k} - x_{4k+1}\cdot\gamma_{4k-2,4k} + \dots - x_{4k+1}\cdot\gamma_{2,4k} + x_{4k+1}\cdot\gamma_{0,4k})\\
&- (\gamma_{4k,4k} - \gamma_{4k-2,4k} + \dots - \gamma_{2,4k} + \gamma_{0,4k})\\
&= (\beta'_{4k,4k} - \beta'_{4k-2,4k} + \dots + \beta'_{0,4k})\\
&+ (x_{4k+1}\cdot\gamma_{4k,4k} - x_{4k+1}\cdot\gamma_{4k-2,4k} + \dots - x_{4k+1}\cdot\gamma_{2,4k} + x_{4k+1}\cdot\gamma_{0,4k})\\
&= x_{4k+1}\cdot\gamma_{4k,4k} - (\beta'_{4k,4k} + x_{4k+1}\cdot\gamma_{4k-2,4k}) + \dots + (\beta'_{2,4k}\\
&+ x_{4k+1}\cdot\gamma_{0,4k}) - \beta'_{0,4k} = [\text{by } (2)] = x_1x_2\dots x_{4k}x_{4k+1} - \gamma_{4k-1,4k+1}\\
&+ \dots + \gamma_{1,4k+1} - 0\\
&= \gamma_{4k+1,4k+1} - \gamma_{4k+1,4k+1} + \dots + \gamma_{1,4k+1} = G_{4k+1} = G_{i+1}. \qquad \square
\end{aligned}
$$

Thus, we can formulate the last theorem.

Theorem 4. $S(\mathrm{Ch}(a_1, \dots, a_n)) \cong C_{F_n(a_1,\dots,a_n)} \cong C_{G_n(a_1,\dots,a_n)}$, *where* F_n, G_n *are defined by* (1), (3).

§4. Acknowledgments

The author wishes first to thank his supervisor, Dr. Sergei V. Duzhin for valuable advices and guidance. Secondly, the author is grateful to Vladimir O. Zolotov for some interesting conversations related to the subject of this article.

References

1. A. E. Holroyd, L. Levine, K. Meszaros, Y. Peres, J. Propp, D.B. Wilson, *Chip-firing and rotor-routing on directed graphs.* http://arxiv.org/abs/0801.3306.
2. I. A. Krepkiy, *Sandpile groups of triangular binary trees.* http://www.pdmi.ras.ru/preprint/2012/12-21.html.
3. K. R. Matthews, *Smith normal form.* — Linear Algebra, Lect. Notes, Univ. Queensland, http://www.numbertheory.org/courses/MP274/smith.pdf.
4. R. Cori, D. Rossin, *On the Sandpile group of dual graphs.* http://www.sciencedirect.com/science/article/pii/S0195669899903669.

St.Petersburg State University,
Universitetsky pr. 28,
Peterhof, 198504 St.Petersburg, Russia
E-mail: feb418@gmail.com

Поступило 26 декабря 2013 г.