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DESCRIBING ORBIT SPACE OF GLOBAL UNITARY ACTIONS FOR MIXED QUDIT STATES

ABSTRACT. The unitary $U(d)$-equivalence relation between elements
of the space $\mathfrak{P}_+$ of mixed states of d-dimensional quantum sys-
tem defines the orbit space $\mathfrak{P}_+/U(d)$ and provides its description in
terms the ring $\mathbb{R}[\mathfrak{P}_+]^{U(d)}$ of $U(d)$-invariant polynomials. We prove
that the semi-algebraic structure of $\mathfrak{P}_+/U(d)$ is determined com-
pletely by two basic properties of density matrices, their semi-posi-
tivity and Hermicity. Particularly, it is shown that the Processi–
Schwarz inequalities in elements of integrity basis for $\mathbb{R}[\mathfrak{P}_+]^{U(d)}$
defining the orbit space, are identically satisfied for all elements
of $\mathfrak{P}_+$.

§1. INTRODUCTION

The basic symmetry of isolated quantum systems is the unitary invari-
ance. It sets the equivalence relations between the states and defines the
physically relevant factor space. For composite systems implementation
of this symmetry has very specific features leading to a such non-trivial
phenomenon as the entanglement of quantum states.

The space of mixed states, $\mathfrak{P}_+$, of d-dimensional binary quantum sys-
tem is locus in quo for two unitary groups action: the group $U(d)$ and the
tensor product group $U(d_1) \otimes U(d_2)$, where d_1, d_2 stand for dimensions
of subsystems, $d = d_1 d_2$. Both groups act on a state $\varrho \in \mathfrak{P}_+$ in adjoint
manner

$$(\mathrm{Ad}\, g)\varrho = g\, \varrho\, g^{-1}. \tag{1}$$

As a result of this action one can consider two equivalent classes of ϱ;
the *global* $U(d)$-orbit and the *local* $U(d_1) \otimes U(d_2)$-orbit. The collection

of all U(d)-orbits, together with the quotient topology and differentiable structure defines the "global orbit space", $\mathfrak{P}_+/\mathrm{U}(d)$, while the orbit space $\mathfrak{P}_+/\mathrm{U}(d_1) \otimes \mathrm{U}(d_2)$ represents the "local orbit space", or the so-called *entanglement space* $\mathcal{E}_{d_1 \times d_2}$. The latter space is proscenium for manifestations of the intriguing effects occurring in quantum information processing and communications.

Both orbit spaces admit representations in terms of the elements of integrity basis for the corresponding ring of G-invariant polynomials, where G is either $\mathrm{G} = \mathrm{U}(d)$ or $\mathrm{G} = \mathrm{U}(d_1) \otimes \mathrm{U}(d_2)$. This can be done implementing the Processi and Schwarz method, introduced in 80th of last century for description of the orbit space of a compact Lie group action on a linear space [1,2]. According to the Processi and Schwarz the orbit space is identified with the semi-algebraic variety, defined by the syzygy ideal for the integrity basis and the semi-positivity condition of a special, so-called "gradient matrix", $\mathrm{Grad}(z) \geqslant 0$, that is constructed from the integrity basis elements. In the present note we address the question of application of this generic approach to the construction of $\mathfrak{P}_+/\mathrm{U}(d)$ and $\mathfrak{P}_+/\mathrm{U}(d_1) \otimes \mathrm{U}(d_2)$. Namely, we study whether the semi-positivity of Grad-matrix introduces new conditions on the elements of the integrity basis for the corresponding ring $\mathbb{R}[\mathfrak{P}_+]^{\mathrm{G}}$. Below it will be shown that for the global unitary invariance, $\mathrm{G} = \mathrm{U}(d)$, the semi-algebraic structure of the orbit space is determined solely from the physical conditions on density matrices, their semi-positivity and Hermicity. The conditions $\mathrm{Grad}(z) \geqslant 0$ do not bring new restrictions on the elements of integrity basis for $\mathbb{R}[\mathfrak{P}_+]^{\mathrm{U}(d)}$. Opposite to this case, for the local symmetries the Processi and Schwarz inequalities impact on the algebraic and geometric properties of the entanglement space.

Our presentation is organized as follows. In section 2 the Processi and Schwarz method is briefly stated in the form applicable to analysis of adjoint unitary action on the space of states. In section 3 the semi-algebraic structure of $\mathfrak{P}_+/\mathrm{U}(d)$ is discussed. The final section is devoted to a detailed consideration of two examples, the orbit space of qutrit (d=3) and the global orbit space of four-level quantum system (d=4).

§2. The Processi–Schwarz method

Here we briefly state the above mentioned method for the orbit space construction elaborated by Processi and Schwartz for the case of compact Lie group action on a linear space [1, 2].

Consider a compact Lie group G acting linearly on the real d-dimensional vector space V. Let $\mathbb{R}[V]^{\mathrm{G}}$ is the corresponding ring of the G-invariant polynomials on V. Assume $\mathcal{P} = (p_1, p_2, \ldots, p_q)$ is a set of homogeneous polynomials that form the integrity basis,

$$\mathbb{R}[x_1, x_2, \ldots, x_d]^{\mathrm{G}} = \mathbb{R}[p_1, p_2, \ldots, p_q].$$

Elements of the integrity basis define the polynomial mapping:

$$p: \quad V \to \mathbb{R}^q; \quad (x_1, x_2, \ldots, x_d) \to (p_1, p_2, \ldots, p_q). \tag{2}$$

Since p is constant on the orbits of G it induces a homeomorphism of the orbit space V/G and the image X of p-mapping; $V/G \simeq X$ [1, 2]. In order to describe X in terms of $\mathcal{P}$ uniquely, it is necessary to take into account the *syzygy ideal* of $\mathcal{P}$, i.e.,

$$I_{\mathcal{P}} = \{h \in \mathbb{R}[y_1, y_2, \ldots, y_q] \; : \; h(p_1, p_2, \ldots, p_q) = 0\} \subseteq \mathbb{R}[V].$$

Let $Z \subseteq \mathbb{R}^q$ denote the locus of common zeros of all elements of $I_{\mathcal{P}}$, then Z is affine variety in $\mathbb{R}^q$ such that $X \subseteq Z$. Denote by $\mathbb{R}[Z]$ the *coordinate ring* of Z, that is, the ring of polynomial functions on Z. Then the following isomorphism takes place [3]

$$\mathbb{R}[Z] \simeq \mathbb{R}[y_1, y_2, \ldots, y_q]/I_{\mathcal{P}} \simeq \mathbb{R}[V]^{\mathrm{G}}.$$

Therefore, the subset Z essentially is determined by $\mathbb{R}[V]^{\mathrm{G}}$, but to describe X the further steps are required. According to [1, 2] the necessary information on X is encoded in the semi-positivity of $q \times q$ matrix with elements given by the inner products of gradients, $\mathrm{grad}(p_i)$:

$$\|\mathrm{Grad}\|_{ij} = (\mathrm{grad}\,(p_i), \mathrm{grad}\,(p_j)).$$

Briefly summarizing all above, the G-orbit space can be identified with the semi-algebraic variety, defined as points, satisfying two conditions:

  a) $z \in Z$, where Z is the surface defined by the syzygy ideal for the integrity basis of $\mathbb{R}[V]^{\mathrm{G}}$;
  b) $\mathrm{Grad}(z) \geqslant 0$.

Having in mind these basic facts one can pass to the construction of the orbit space $\mathfrak{P}_+/\mathrm{U}(d)$. At first we describe the generic semi-algebraic structure and further exemplify it considering two simple, three and four level quantum systems.

§3. Semi-algebraic structure of $\mathfrak{P}_+/\mathrm{U}(d)$

The first step making the Procesi–Schwarz method applicable to the case we are interested in consists in the linearization of the adjoint $\mathrm{U}(d)$-action (1). For the unitary action one can achieve this as follows. Consider the space $\mathcal{H}_{d \times d}$ of $d \times d$ Hermitian matrices and define the mapping

$$\mathcal{H}_{d \times d} \to \mathbb{R}^{d^2};$$

$$\varrho_{11} = v_1, \quad \varrho_{12} = v_2, \ldots, \varrho_{1d} = v_d, \quad \varrho_{21} = v_{d+1} \ldots, \quad \varrho_{dd} = v_{d^2}.$$

Then it can be easily verified that the linear representation on $\mathbb{R}^{d^2}$

$$\boldsymbol{v}' = L\boldsymbol{v}, \quad L \in \mathrm{U}(d) \otimes \overline{\mathrm{U}(d)},$$

where a line over expression means the complex conjugation, is isomorphic to the initial adjoint $\mathrm{U}(d)$ action (1).

Now the corresponding integrity basis $\mathcal{P} = (p_1, p_2, \ldots, p_q)$ for the ring of invariant polynomials is required for the mapping (2). For its construction the following observation is in order. Starting from the center $\mathcal{Z}(\mathfrak{su}(\mathrm{d}))$ of the universal enveloping algebra $\mathfrak{U}(\mathfrak{su}(\mathrm{d}))$, according to the well-known Gelfand's theorem, one can define an isomorphic commutative symmetrized algebra of invariants $S(\mathfrak{su}(\mathrm{d}))$, which by turn is isomorphic to the algebra of invariant polynomials over $\mathfrak{su}(\mathrm{d})$ [4]. The later provides the required resource for coordinates that can be used to parameterize the orbit space $\mathfrak{P}_+/\mathrm{U}(d)$. For our purpose it is convenient to choose the integrity basis that is formed by the so-called trace invariants. Namely, we use below the polynomial ring $\mathbb{R}[v_1, v_2, \ldots, v_{d^2}]^{\mathrm{U}(d)} = \mathbb{R}[t_1, t_2, \ldots, t_d]$, with n basis elements

$$t_k = \mathrm{tr}\left(\varrho^k\right), \quad k = 1, 2 \ldots, d. \tag{3}$$

In terms of the integrity basis (3) the Grad−matrix reads

$$\mathrm{Grad}(t_1, t_2, \ldots, t_d) = \begin{pmatrix} d & 2t_1 & 3t_2 & \cdots & dt_{d-1} \\ 2t_1 & 2^2 t_2 & 2 \cdot 3t_3 & \cdots & 2 \cdot dt_d \\ 3t_2 & 2 \cdot 3t_3 & 3^2 t_4 & \cdots & 3 \cdot dt_{d+1} \\ \vdots & \vdots & \vdots & \vdots\vdots\vdots & \vdots \\ dt_{d-1} & 2 \cdot dt_d & 3 \cdot dt_{d+1} & \cdots & d^2 t_{2d-2} \end{pmatrix}. \tag{4}$$

In (4), polynomials t_k with $k > d$ are expressed as polynomials in $(t_1, t_2, \ldots, t_d)$. From (4) one can easily obtain that

$$\mathrm{Grad}(t_1, t_2, \ldots, t_d) = \chi \mathrm{Disc}\,(t_1, t_2, \ldots, t_d)\,\chi^T, \tag{5}$$

where $\chi = (1, 2, \ldots, d)$ and Disc $(t_1, t_2, \ldots, t_d)$ denotes the matrix

$$
\text{Disc}\,(t_1, t_2, \ldots, t_d) = \begin{pmatrix}
d & t_1 & t_2 & \cdots & t_{d-1} \\
t_1 & t_2 & t_3 & \cdots & t_d \\
t_2 & t_3 & t_4 & \cdots & t_{d+1} \\
\vdots & \vdots & \vdots & \vdots\vdots\vdots \\
t_{d-1} & t_d & t_{d+1} & \cdots & t_{2d-2}
\end{pmatrix}. \tag{6}
$$

In one's turn the matrix (6) can be written as "square" of the Vandermonde matrix, Disc $(t_1, t_2, \ldots, t_d)\ = \Delta\Delta^T$,

$$
\Delta(x_1, \ldots, x_d) = \begin{pmatrix}
1 & x_1 & x_1^2 & \ldots & x_1^{d-1} \\
1 & x_2 & x_2^2 & \ldots & x_2^{d-1} \\
1 & x_3 & x_3^2 & \ldots & x_3^{d-1} \\
\vdots & \vdots & \vdots & \vdots\vdots\vdots \\
1 & x_d & x_d^2 & \ldots & x_d^{d-1}
\end{pmatrix}, \tag{7}
$$

whose columns are determined by powers of roots $(x_1, x_2, \ldots x_d)$ of the characteristic equation:

$$
\det \| x - \varrho \| = x^d - S_1 x^{d-1} + S_2 x^{d-2} - \cdots + (-1)^d S_d = 0\,. \tag{8}
$$

The semi-positivity condition of the matrix (6) guaranties reality of the roots of (8). Thus, semi-positivity of the Grad-matrix is equivalent to the reality condition of eigenvalues of the density matrix ϱ written in terms of the U(d) polynomial scalars. Finally, noting that the density matrices by construction are Hermitian, we convinced that the Procesi–Schwarz inequalities are satisfied identically on $\mathfrak{P}_+$.

Summarizing, the algebraic structure of the orbit space $\mathfrak{P}_+/\mathrm{U}(d)$ is completely determined by the inequalities in elements of the integrity basis for polynomial ring $\mathbb{R}[t_1, t_2, \ldots, t_d]$ originating from the Hermicity and semi-positivity requirements on density matrices.

§4. Two examples

Algebraic structure of the orbit space of quantum systems is highly intricate. The examples of d=3 (qutrit) and d=4, considered below, demonstrate the degree of its complexity even for the low dimensional systems.

4.1. Orbit space of qutrit. Qutrit is 3-dimensional quantum system and the integrity basis for U(3)-invariant polynomial consist from first, second and third order trace polynomials; t_1, t_2, t_3. For a visibility below we consider the case of normalized density matrices, supposing $t_1 = 1$.[1]

The condition of the eigenvalues reality is

$$0 \leqslant \frac{1}{6} \left(3t_2^3 - 21t_2^2 + 36t_3 t_2 + 9t_2 - 18t_3^2 - 8t_3 - 1 \right) , \qquad (9)$$

while the semi-positivity of density matrices formulated as non-negativity of coefficients of characteristic equation (8) reads

$$0 \leqslant \frac{1}{2}(1 - t_2) \leqslant \frac{1}{3} \, ,$$

$$0 \leqslant \frac{1}{6}(1 - 3t_2 + 2t_3) \leqslant \frac{1}{9} \, .$$

Resolving the inequalities

$$\frac{1}{3} \leqslant t_2 \leqslant 1$$

$$\frac{3}{2}t_2 - \frac{1}{2} \leqslant t_3 \leqslant \frac{3}{2}t_2 - \frac{1}{6}$$

$$-4 + 18t_2 - \sqrt{2}(3t_2 - 1)^{3/2} \leqslant 18t_3 \leqslant -4 + 18t_2 + \sqrt{2}(3t_2 - 1)^{3/2}$$

we get the intersection domain shown on Fig. 1. The triangle domain A-B-C, bounded by the lines:

$$\text{A-B} \quad t_3 = \frac{1}{18}(-4 + 18t_2 + \sqrt{2}(3t_2 - 1)^{3/2})$$

$$\text{A-C} \quad t_3 = \frac{1}{18}(-4 + 18t_2 - \sqrt{2}(3t_2 - 1)^{3/2})$$

$$\text{B-C} \quad t_3 = \frac{3}{2}t_2 - \frac{1}{2}$$

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[1]It is worth to note that description of the qutrit orbits is similar to the studies of the flavor symmetries of hadrons, performed more than forty ears ago by by Michel and Radicati [5] (cf. the method adaptation to the analysis of space of quantum states [6–8]).

with vertexes [2]: $A\left(\frac{1}{3}, \frac{1}{9}\right)$, $B(1, 1)$ and $C\left(\frac{1}{2}, \frac{1}{4}\right)$, represents the orbit space of qutrit in parametrization of trace polynomial coordinates.
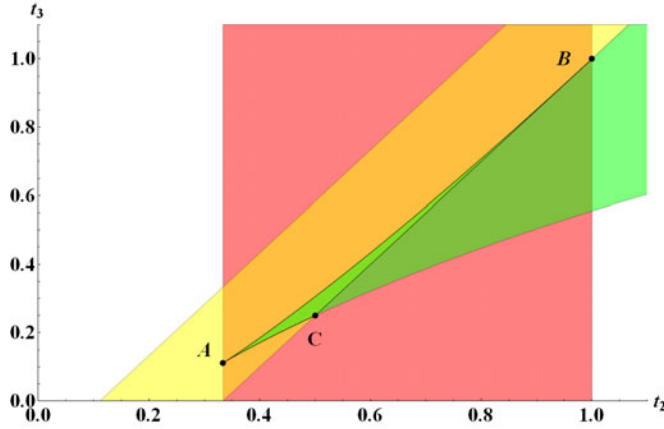

Fig. 1. Triangle domain A-B-C as the orbit space of qutrit.

Now it is in order to discuss correspondence between the above algebraic results and known classification of orbits with respect to their stability group. Having in mind this issue consider the Bloch parametrization for qutrit

$$\rho = \frac{1}{3}\left(\mathbb{I}_3 + \sqrt{3}\,\boldsymbol{\xi}\cdot\boldsymbol{\lambda}\right), \tag{10}$$

where $\boldsymbol{\xi} = (\xi_1, \xi_2, \cdots, \xi_8) \in \mathbb{R}^8$ denote the Bloch vector and $\boldsymbol{\lambda}$ is the vector, whose components are elements $(\lambda_1, \lambda_2, \cdots, \lambda_8)$ of $\mathfrak{su}(3)$ algebra

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[2]Note that the straight line B–C is tangent to the curve A-B at the point B

$$\frac{dt_3}{dt_2} = 1 + \frac{1}{\sqrt{2}}(3t_2 - 1)^{3/2}, \quad \frac{dt_3}{dt_2}\big|_{t_2=1} = \frac{3}{2}\,.$$

basis, say the Gell–Mann matrices,

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} \quad \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \tag{11}$$

$$\lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \quad \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix},$$

obeying

$$[\lambda_i, \lambda_j] = 2\imath f_{ijk}\lambda_k, \quad \mathrm{tr}\,(\lambda_i \lambda_j) = 2\delta_{ij}, \tag{12}$$

with non-vanishing structure constants

$$f_{123} = 2f_{147} = 2f_{246} = 2f_{257} = 2f_{345} = -2f_{156}$$
$$= -2f_{367} = \frac{2}{\sqrt{3}} f_{458} = \frac{2}{\sqrt{3}} f_{678} = 1. \tag{13}$$

To analyse the adjoint orbit $\mathcal{O}_\varrho$ that passes through the point ϱ, we define the set of tangent vectors:

$$l_i = \lim_{\theta_1, \theta_2, \ldots \theta_8 \to 0} \frac{\partial}{\partial \theta_i} \left[U\left(\theta_1, \theta_2, \ldots \theta_8\right) \, \varrho \, U\left(\theta_1, \theta_2, \ldots \theta_8\right) \right] = \imath[\lambda_i, \varrho]. \tag{14}$$

By definition, the dimension of orbit $\dim(\mathcal{O}_\varrho)$ is given by the dimension of the tangent space to the orbit $T_{\mathcal{O}_\varrho}$ and therefore equals to the the number of linearly independent vectors among eight tangent vectors $l_1, l_2, \ldots, l_8$. This number depends on the point ϱ and according to the well-known theorem from linear algebra is given by the rank of the so-called Gram matrix

$$A_{ij} = \frac{1}{2} \left\| \mathrm{tr}(l_i l_j) \right\|. \tag{15}$$

In the Bloch parameterization (10) we easily find that

$$A_{ij} = \frac{4}{3} f_{ims} f_{jns} \xi_m \xi_n. \tag{16}$$

To estimate the rank of matrix (15) it is convenient to pass to the diagonal representative of the matrix ϱ:

$$\varrho = W \begin{pmatrix} x_1 & 0 & 0 \\ 0 & x_2 & 0 \\ 0 & 0 & x_3 \end{pmatrix} W^+ , \tag{17}$$

where $W \in \mathrm{SU}(3)/S_3$ and the descending order for ϱ matrix eigenvalues

$$1 \geqslant x_1 \geqslant x_2 \geqslant x_3 \geqslant 0 ,$$

is chosen. The later constraints allow to avoid a double counting due to the $S_3 \subset U(3)$ symmetry of permutation of the density matrix eigenvalues. Using the principal axis transformation (17) and taking into account the adjoint properties of Gell–Mann matrices $W^+ \lambda_i W = O_{ij}\lambda_j$ with $O \in SO(8)$, the matrix A_{ij} can be written as

$$A_{ij} = O_{ik} A_{kl}^{\mathrm{diag}} O_{lj}^T . \tag{18}$$

Matrix A_{kl}^{diag} in (18) is the matrix (15) constructed from vectors $l_i^{\mathrm{diag}} = i[\lambda_i, \varrho_{\mathrm{diag}}]$ tangent to the orbit of the diagonal matrix $\varrho_{\mathrm{diag}} := \mathrm{diag}(x_1, x_2, x_3)$.

Since we are interesting in determination of rank$|A|$, the relation (18) allows to reduce this question to the evaluation of the rank of the diagonal representative ϱ_{diag}. For diagonal matrices the Bloch vector is $\boldsymbol{\xi}^{\mathrm{diag}} = (0, 0, 0, \xi_3, 0, 0, 0, \xi_8)$. Taking into account the values for structure constants from (13), the expression for $|A^{\mathrm{diag}}|$ reads

$$A^{\mathrm{diag}} = \frac{1}{3} \mathrm{diag}\Big(4\xi_3^2, 4\xi_3^2, 0, (\xi_3 + \sqrt{3}\xi_8)^2, (\xi_3 + \sqrt{3}\xi_8)^2,$$
$$(\xi_3 - \sqrt{3}\xi_8)^2, (\xi_3 - \sqrt{3}\xi_8)^2, 0\Big) . \tag{19}$$

From (19), we conclude that there are orbits of three different dimensions:

- the orbits of maximal dimension, $\dim(\mathcal{O}_\varrho) = 6$,
- the orbits of dimension, $\dim(\mathcal{O}_\varrho) = 4$,
- zero dimensional orbit, one point $\boldsymbol{\xi} = 0$.

These orbits are in agreement with their classification based on the group of transformations G_ϱ – the isotropy group (or stability group), which stabilize point $\varrho \in \mathcal{O}_\varrho$. The orbits of different dimensions have a different stability groups; for the points lying on the orbit of maximal dimension the stability group is the Cartan subgroup $\mathrm{U}(1) \otimes \mathrm{U}(1) \otimes \mathrm{U}(1)$,

while the stability group of points with diagonal representative λ_8 is $U(2) \otimes U(1)$. The dimensions of listed orbits agrees with the general formula

$$\dim \mathcal{O}_\varrho = \dim G - \dim G_\varrho \,. \tag{20}$$

Since the isotropy group of any two points on the orbit are the same up to conjugation, the orbits can be partitioned into sets with equivalent isotropy groups [3]. This set is known as *"strata"*.

Concluding we refer to the relations between the triangle A-B-C, depicted on the Figure 1, and the corresponding strata. The domain inside the triangle ABC corresponds to the principal strata with the stability group $U(1) \times U(1) \times U(1)$. The discriminant is positive $|\text{Disc}| > 0$ and the density matrix has three different real eigenvalues, the representative matrix reads $\frac{1}{3}(\mathbb{I}_3 + \sqrt{3}\,(\xi_3 \lambda_3 + \xi_8 \lambda_8))$, with ξ_3 and ξ_8 subject to the following constraints

$$0 < 1 - \xi_3^2 - \xi_8^2 < 1,$$
$$0 < (2\xi_8 - 1)(1 - \sqrt{3}\xi_3 + \xi_8)(1 + \sqrt{3}\xi_3 + \xi_8) < 1 \,.$$

The S_3 coefficient vanishes at line B-C. The boundary line B-C, excluding vertices B and C also belongs to the principal stratum, while points B and C belong to the stratum of lower dimension. On the sides A-B and A-C the discriminant is zero $|\text{Disc}| = 0$, hence, the density matrix has three real eigenvalues and two of them are equal. At point B two eigenvalues of ϱ are zero. The lines (A-B)/{A} and (A-C)/{A} represent the degenerate 4-dimensional orbits whose stability group is $U(2) \otimes U(1)$. Finally, the point A is the zero dimensional stratum corresponding to the maximally mixed state $\varrho = \frac{1}{3}\mathbb{I}_3$. The details of the orbit types are collected in the nable below.

4.2. Orbit space of a four-level quantum system. The density matrix ϱ of a 4-level quantum system in the Bloch form reads

$$\rho = \frac{1}{4}\left(\mathbb{I}_4 + \sqrt{6}\,\vec{\xi}\cdot\vec{\lambda}\right)\,, \tag{21}$$

where the traceless part of ϱ is given by scalar product of 15-dimensional Bloch vector $\vec{\xi} = \{\xi_1, \ldots, \xi_{15}\} \in \mathbb{R}^{15}$ with λ-vector whose components are

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[3]The isotropy group of a point ϱ depends only on the algebraic multiplicity of the eigenvalues of the matrix ϱ.

dim $\mathcal{O}$	Strata	Stability group	Representative matrix	Constraints
6	Interior of triangle ABC	$U(1)\otimes U(1)\otimes U(1)$	$\frac{1}{3}(\mathbb{I}_3+\sqrt{3}(\xi_3\lambda_3+\xi_8\lambda_8))$	Disc $>0, S_2>0, S_3>0$
	Boundary: $(B-C)/\{B,C\}$	$U(1)\otimes U(1)\otimes U(1)$	$\frac{1}{3}(\mathbb{I}_3+\sqrt{3}(\xi_3\lambda_3+\frac{1}{2}\lambda_8))$	Disc $>0, S_2>0, S_3=0$
4	Boundary: $(A-B)/\{A\}$ $(A-C)/\{A\}$	$U(2)\otimes U(1)$	$\frac{1}{3}(\mathbb{I}_3+\sqrt{3}\,\xi_8\lambda_8)$ $\frac{1}{3}(\mathbb{I}_3+\sqrt{3}\xi_8(\sqrt{3}\lambda_3+\lambda_8))$ $\frac{1}{3}(\mathbb{I}_3+\sqrt{3}\xi_8(-\sqrt{3}\lambda_3+\lambda_8))$	Disc $=0, S_2>0, S_3\geqslant 0$
0	Point: $\{A\}$	$U(3)$	$\frac{1}{3}\mathbb{I}_3$	Disc $=S_2=S_3=0$

Table 1. The stratum decomposition for the orbit space of qutrit.

elements of the Hermitian basis of the Lie algebra $\mathfrak{su}(4)$

$$\lambda_i\lambda_j = \frac{1}{2}\,\delta_{ij}\mathbb{I}_4 + (d_{ijk}+i\,f_{ijk})\lambda_k, \quad i,j,k=1,\ldots,15.$$

The corresponding integrity basis for for polynomial ring $\mathbb{R}[\mathfrak{P}_+]^{\mathrm{U}(4)}$ can be fixed by three U(4)-invariant polynomials, the Casimir scalars $\mathfrak{C}_2,\mathfrak{C}_3,\mathfrak{C}_4$

$$\mathfrak{C}_2=\vec{\xi}\cdot\vec{\xi}\quad \mathfrak{C}_3=\sqrt{\frac{3}{2}}\,d_{ijk}\xi_i\xi_j\xi_k\,,\quad \mathfrak{C}_4=\frac{3}{2}\,d_{ijk}d_{lmk}\xi_i\xi_j\xi_l\xi_m\,, \quad (22)$$

The semi-positivity of (21) formulated as non-negativity of coefficients S_2,S_3 and S_4 of the characteristic polynomial (8) [4]

$$S_2=\tfrac{3}{8}(1-\mathfrak{C}_2)\geqslant 0 \tag{23}$$

$$S_3=\tfrac{1}{16}(1-3\mathfrak{C}_2+2\mathfrak{C}_3)\geqslant 0, \tag{24}$$

$$S_4=\det\rho=\tfrac{1}{256}((1-3\mathfrak{C}_2)^2+8\mathfrak{C}_3-12\mathfrak{C}_4)\geqslant 0 \tag{25}$$

Now we are in position to compute the Grad-matrix in terms of the SU(4) Casimir scalars:

$$\mathrm{Grad}=\begin{pmatrix}4\mathfrak{C}_2 & 6\mathfrak{C}_3 & 8\mathfrak{C}_4\\ 6\mathfrak{C}_3 & 9\mathfrak{C}_4 & 12\mathfrak{C}_2\mathfrak{C}_3\\ 8\mathfrak{C}_4 & 12\mathfrak{C}_2\mathfrak{C}_3 & 4(\mathfrak{C}_3^2+3\mathfrak{C}_2\mathfrak{C}_4)\end{pmatrix}. \tag{26}$$

[4]For details we refer to [9].

Passing to the equivalent matrix $Q \operatorname{Grad} Q^T$, with $Q = \operatorname{diag}(2, 3, 2)$, we arrive at the following form for the Processi–Schwarz inequalities

$$\mathfrak{C}_2 + \mathfrak{C}_3^2 + 3\mathfrak{C}_2\mathfrak{C}_4 + \mathfrak{C}_4 \geqslant 0, \tag{27}$$

$$\mathfrak{C}_3^2 \left(-4\mathfrak{C}_2^2 + \mathfrak{C}_2 + \mathfrak{C}_4 - 1\right) + \mathfrak{C}_4 \left(3\mathfrak{C}_2^2 + 3\mathfrak{C}_2\mathfrak{C}_4 + \mathfrak{C}_2 - 4\mathfrak{C}_4\right) \geqslant 0, \tag{28}$$

$$-4\mathfrak{C}_2^3\mathfrak{C}_3^2 + 3\mathfrak{C}_2^2\mathfrak{C}_4^2 + 6\mathfrak{C}_2\mathfrak{C}_3^2\mathfrak{C}_4 - \mathfrak{C}_3^4 - 4\mathfrak{C}_4^3 \geqslant 0. \tag{29}$$

The domains describing the semi-positivity of ρ (23)–(25), and its residually part after imposing condition of the semi-positivity of Grad-matrix (27)–(29) are depicted on Fig. 2.
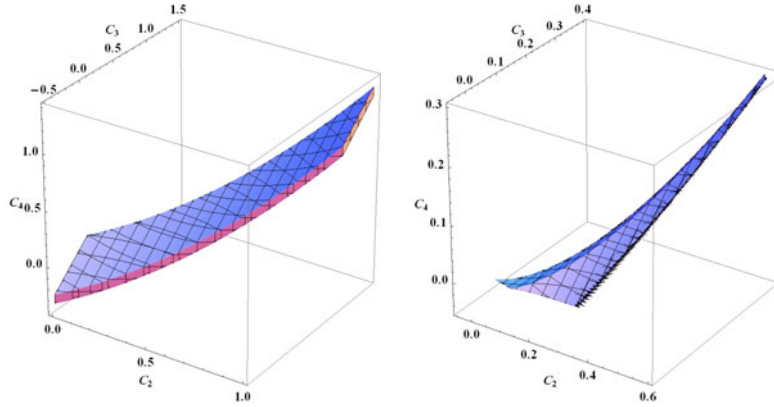

Fig. 2. On the left side: $\rho \geqslant 0$. On the right side: $\rho \geqslant 0 \cap \operatorname{Grad} \geqslant 0$.

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