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A COMBINATORIAL INTERPRETATION OF THE SCALAR PRODUCTS OF STATE VECTORS OF INTEGRABLE MODELS

Abstract. The representation of Bethe wave functions of certain integrable models via Schur functions allows one to apply the well-developed theory of symmetric functions to the calculation of thermal correlation functions. The algebraic relations arising in the calculation of scalar products and correlation functions are based on the Binet–Cauchy formula for the Schur functions. We provide a combinatorial interpretation of the formula for the scalar products of Bethe state vectors in terms of nests of self-avoiding lattice paths constituting so-called watermelon configurations. The proposed interpretation is, in turn, related to the enumeration of boxed plane partitions.

Dedicated to Anatoly Moiseevich Vershik

§1. Introduction

Symmetric functions, Young diagrams, boxed plane partitions, and vicious walkers [1–4] play an important role in the contemporary theoretical physics [5–9]. The N-particle wave functions of a certain class of integrable models on a chain are expressed in terms of Schur functions [10–16]. The Schur functions are defined by the Jacobi–Trudi relation:

$$S_{\boldsymbol{\lambda}}(\mathbf{x}) \equiv S_{\boldsymbol{\lambda}}(x_1, x_2, \dots, x_N) \equiv \frac{\det(x_j^{\lambda_k+N-k})_{1\leqslant j,k\leqslant N}}{\mathcal{V}_N(\mathbf{x})}\,, \tag{1}$$

where $\mathcal{V}_N(\mathbf{x})$ is the Vandermonde determinant

$$\mathcal{V}_N(\mathbf{x}) \equiv \det(x_j^{N-k})_{1\leqslant j,k\leqslant N} = \prod_{1\leqslant m<l\leqslant N} (x_l - x_m)\,. \tag{2}$$

A partition $\boldsymbol{\lambda} \equiv (\lambda_1, \lambda_2, \dots, \lambda_N)$ is a nonincreasing sequence $M \geqslant \lambda_1 \geqslant \lambda_2 \geqslant \dots \geqslant \lambda_N \geqslant 0$ of nonnegative integers, called the parts of $\boldsymbol{\lambda}$. A

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partition $\boldsymbol{\lambda}$ can be represented by the corresponding Young diagram, the arrangement of squares with coordinates (i, j) such that $1 \leqslant j \leqslant \lambda_i$, see [1].

For bosonic models defined on the chain of $M+1$ sites, there is a one-to-one correspondence between the sets of occupation numbers $\{n_M, n_{M-1}, \dots, n_1, n_0\}$ and the partitions $\boldsymbol{\lambda} = (M^{n_M}, (M-1)^{n_{M-1}}, \dots, 1^{n_1}, 0^{n_0})$, where each number S appears n_S times in $\boldsymbol{\lambda}$. For the Heisenberg spin-$\frac{1}{2}$ chains of $M+N$ sites, the coordinates of the spin "down" states ("particles") constitute a strict decreasing partition $\boldsymbol{\mu} = (\mu_1, \mu_2, \dots, \mu_N)$, where $M+N-1 \geqslant \mu_1 > \mu_2 > \dots > \mu_N \geqslant 0$. The parts of $\boldsymbol{\mu}$ and $\boldsymbol{\lambda}$ are related: $\mu_j = \lambda_j + N - j$.

The bijection between the coordinates of the particles and the Young diagram of the partition $\boldsymbol{\lambda}$ is demonstrated in Fig. 1.

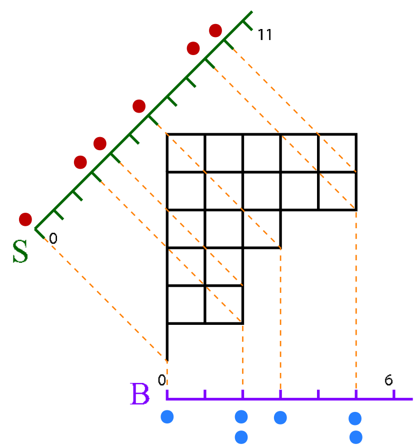

Fig. 1. Configurations of spins and bosons on chains and the corresponding Young diagram of the related partition $\boldsymbol{\lambda} = (5, 5, 3, 2, 2, 0)$.

The calculation of the correlation functions of integrable models of a special type, [10–13, 15, 16], is based on the Binet–Cauchy formula:

$$\sum_{\boldsymbol{\lambda} \subseteq M^N} S_{\boldsymbol{\lambda}}(\mathbf{x}) S_{\boldsymbol{\lambda}}(\mathbf{y}) = \frac{\det(M_{kj})_{1 \leqslant k, j \leqslant N}}{\mathcal{V}_N(\mathbf{x}) \mathcal{V}_N(\mathbf{y})}, \tag{3}$$

where the summation is over all partitions $\boldsymbol{\lambda}$ with at most N parts each of which is at most M, and $\mathcal{V}_N(\mathbf{x})$ is the Vandermonde determinant (2).

The entries M_{kj} in (3) are

$$M_{kj} = \frac{1 - (x_k y_j)^{M+N}}{1 - x_k y_j} \,. \tag{4}$$

The calculation of the scalar products of N-particle Bethe state vectors [15, 16] is based on Eq. (3).

To obtain the q-parameterized Binet–Cauchy relation, we put $\mathbf{y} = \mathbf{q} \equiv (q, q^2, \ldots, q^N)$, $\mathbf{x} = \mathbf{q}/q \equiv (1, q, \ldots, q^{N-1})$ in (3) and obtain

$$\sum_{\boldsymbol{\lambda} \subseteq M^N} S_{\boldsymbol{\lambda}}(\mathbf{q}) S_{\boldsymbol{\lambda}}(\mathbf{q}/q) = \mathcal{V}_N^{-1}(\mathbf{q}) \mathcal{V}_N^{-1}(\mathbf{q}/q) \det \left(\frac{1 - q^{(M+N)(j+k-1)}}{1 - q^{j+k-1}} \right)_{1 \leqslant j,k \leqslant N} . \tag{5}$$

Relation (5) is used in the calculation of the amplitudes of the low temperature asymptotics of the correlation functions in the limit when the total number of sites is large enough, $M \gg 1$, while the number of particles N is moderate: $1 \ll N \ll M$, see [15]. The connection of this equation to the enumeration of plane partitions in the $N \times N \times M$ box in the framework of quantum inverse scattering method [17, 18] was established in [10]. We will represent the box of size $L \times N \times P$ as the set of integer lattice points

$$\mathcal{B}(L, N, P) = \left\{ (i, j, k) \in \mathbb{N}^3 \,\middle|\, 0 \leqslant i \leqslant L,\ 0 \leqslant j \leqslant N,\ 0 \leqslant k \leqslant P \right\}.$$

In [10], the determinant in the right-hand side of Eq. (5) was expressed as the Kuperberg determinant [19], which led to the answer:

$$\mathcal{V}_N^{-1}(\mathbf{q}) \mathcal{V}_N^{-1}(\mathbf{q}/q) \det \left(\frac{1 - q^{(M+N)(j+k-1)}}{1 - q^{j+k-1}} \right)_{1 \leqslant j,k \leqslant N} = \prod_{k=1}^{N} \prod_{j=1}^{N} \frac{1 - q^{M+j+k-1}}{1 - q^{j+k-1}} . \tag{6}$$

This formula is MacMahon's generating function for the boxed plane partitions [3].

As follows from [16], the sum of Schur functions in the left-hand side of (5) can be expressed in terms of the *q-binomial determinant*:

$$\sum_{\boldsymbol{\lambda} \subseteq M^N} S_{\boldsymbol{\lambda}}(\mathbf{q}) S_{\boldsymbol{\lambda}}(\mathbf{q}/q) = q^{\frac{NM}{2}(1-M)} \det \left(\begin{bmatrix} 2N + i - 1 \\ N + j - 1 \end{bmatrix} \right)_{1 \leqslant i,j \leqslant M} . \tag{7}$$

The entries in (7) are the *q-binomial coefficients*, [20], defined as

$$\begin{bmatrix} R \\ r \end{bmatrix} \equiv \frac{[R]!}{[r]!\,[R-r]!}\,, \tag{8}$$

where $[n]$ is the *q-number*, which is a q-analog of a positive integer $n \in \mathbb{Z}^+$,

$$[n] \equiv \frac{1-q^n}{1-q}\,,$$

and the *q-factorial* $[n]!$ is given by $[n]! \equiv [1]\,[2]\,\ldots\,[n]$, $[0]! \equiv 1$. The determinant in the right-hand side of (7) was directly calculated in [16], and the obtained answer agrees with (5), (6).

§2. Schur functions and lattice paths

In this note, we will give a combinatorial interpretation of Eq. (5). This equation appears in integrable models of strongly correlated bosons, [10], and of free fermions, [15]. It is well known that a combinatorial description of Schur functions can be given in terms of *semistandard Young tableaux*. A filling of the cells of the Young diagram of $\boldsymbol{\lambda}$ with positive integers $n \in \mathbb{N}^+$ is called a *semistandard tableau of shape* $\boldsymbol{\lambda}$ provided that it is weakly increasing along the rows and strictly increasing along the columns. The weight $\mathbf{x}^T$ of a tableau T is defined as

$$\mathbf{x}^T \equiv \prod_{i,j} x_{T_{ij}}\,,$$

where the product is over all entries T_{ij} of the tableau T. An equivalent definition of the Schur function is given by

$$S_{\boldsymbol{\lambda}}(x_1, x_2, \ldots, x_m) = \sum_T \mathbf{x}^T\,, \tag{9}$$

where $m \geqslant N$ and the sum is over all tableaux T of shape $\boldsymbol{\lambda}$ with entries from the set $\{1, 2, \ldots, m\}$.

There is a natural way of representing every semistandard tableau of shape $\boldsymbol{\lambda}$ with entries not exceeding N as a nest of self-avoiding lattice paths with specified start and end points. Let T_{ij} be the entry in the $i^{\rm th}$ row and $j^{\rm th}$ column of a semistandard tableau T. The $i^{\rm th}$ lattice path of the nest C (counted from the top of the nest) encodes the $i^{\rm th}$ row of the tableau ($i = 1, \ldots, N$). It goes from $C_i = (N-i+1, N-i)$ to $(1, \mu_i = \lambda_i + N - i)$ (see Fig. 2). It makes λ_i steps to the north so that the step along the line x_j corresponds to the occurrences of the letter $N-j+1$ in the $i^{\rm th}$ row of

T. The power l_j of x_j in the weight of any particular nest of paths is the number of north steps taken along the vertical line x_j. Thus an equivalent representation of the Schur function takes the form

$$S_{\boldsymbol{\lambda}}(x_1, x_2, \ldots, x_N) = \sum_C \prod_{j=1}^{N} x_j^{l_j}, \qquad (10)$$

where the summation is over all admissible nests C. This representation of Schur functions is natural in the quantum inverse scattering method approach to the solution of the models. The k^{th} path is contained in the

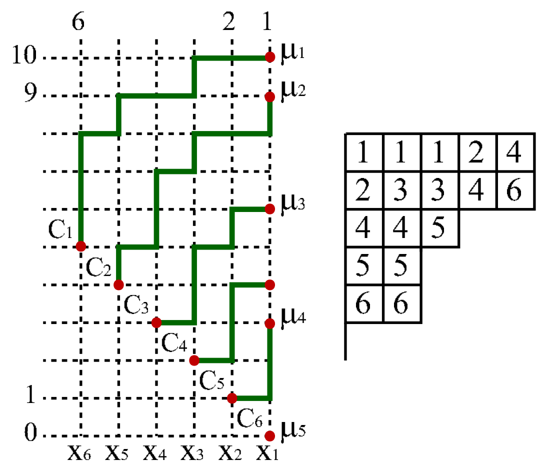

Fig. 2. A semistandard tableau of shape $\boldsymbol{\lambda} = (5, 5, 3, 2, 2, 0)$ is represented as a nest of lattice paths. The vertical steps along the line x_j represent the occurrences of the letter $N - j + 1$, $N = 6$, in the tableau.

$\lambda_k \times (N - k)$ rectangle, $k = 1, \ldots, N$. The starting point of each path is the lower left vertex. We define the volume of a path as the number of squares below it in the corresponding rectangle. The volume of a nest of lattice paths is equal to the volume of the lattice paths in this nest:

$$\mid \zeta \mid_C = \sum_{j=1}^{N} (j-1) l_j .$$

Therefore, the q-parameterized Schur function is the partition function of the described nest:

$$S_{\boldsymbol{\lambda}}(\mathbf{q}/q) = \sum_{C} q^{|\zeta|_C} \,,$$

where the summation is over all admissible nests C. Adding the weight of the partition $|\boldsymbol{\lambda}| = \sum\limits_{k=1}^{N} \lambda_k$ to the volume of the nest, we obtain

$$\mid \xi \mid_C = \mid \lambda \mid + \mid \zeta \mid_C = \sum_{j=1}^{N} j\, l_j \,,$$

and

$$S_{\boldsymbol{\lambda}}(\mathbf{q}) = \sum_{C} q^{|\xi|_C} = q^{|\lambda|} \sum_{C} q^{|\zeta|_C} = q^{|\lambda|} S_{\boldsymbol{\lambda}}(\mathbf{q}/q) \,.$$

Consider the *conjugate* nest of self-avoiding lattice paths (see Fig. 3) from $(1, \mu_i = \lambda_i + N - i)$ to $B_i = (i, N + M - i)$. The i^{th} path consists of $M - \lambda_i$ steps to the north. The representation of the Schur function corresponding to the described nest is

$$S_{\boldsymbol{\lambda}}(x_1, x_2, \ldots, x_N) = \sum_{B} \prod_{j=1}^{N} x_j^{M - l_j} \,, \qquad (11)$$

where the summation is over all admissible nests B of N self-avoiding lattice paths. The k^{th} path is contained in the $(k-1) \times M$ rectangle, $k = 1, \ldots, N$. The end point of each path is the top right vertex. The volume of a path is the number of squares below it in the corresponding rectangle. The volume of a nest of lattice paths is equal to the volume of the paths in this nest:

$$\mid \zeta \mid_B = \sum_{j=1}^{N} (j-1)(M - l_j).$$

In the limit $q \to 1$, the Schur function is equal to the number of nests of self-avoiding lattice paths of type either B or C:

$$S_{\boldsymbol{\lambda}}(1, \ldots, 1) = \sum_{B} 1 = \sum_{C} 1 \,.$$

A summand in the scalar product (3), being the product of two Schur functions, can be graphically expressed as a nest of N self-avoiding lattice paths starting at the equidistant points C_i and terminating at the

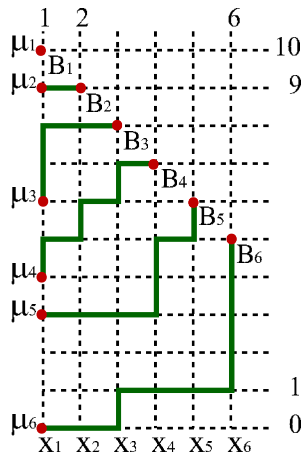

Fig. 3. The conjugated nest of lattice paths.

equidistant points B_i $(i = 1, \ldots, N)$. This configuration, known as a *watermelon*, is presented in Fig. 4. The scalar product (3) is the sum of all such watermelons. Repeating the arguments used above to derive the vol-

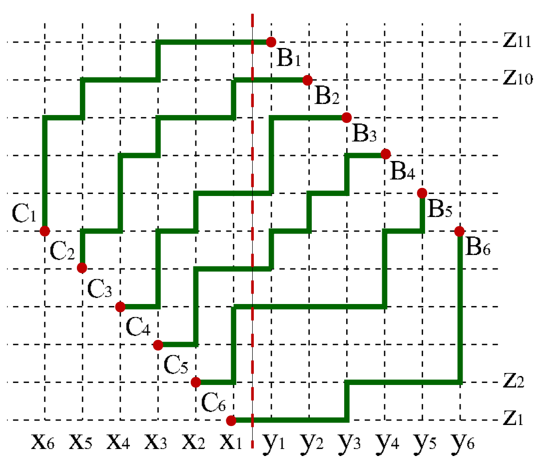

Fig. 4. A watermelon configuration.

umes of lattice paths, it is straightforward to find that the volume of the watermelon is equal to

$$| w |=| \xi |_C + | \zeta |_B \ .$$

The partition (generating) function of watermelons is equal to the left-hand side of (5):

$$\mathsf{W}(N, M) = \sum_W q^{|w|} = \sum_{\boldsymbol{\lambda} \subseteq M^N} S_{\boldsymbol{\lambda}}(\mathbf{q}) S_{\boldsymbol{\lambda}}(\mathbf{q}/q)\,, \tag{12}$$

where the sum $\sum_W$ is taken over all watermelons with the fixed endpoints C_i, B_i, $1 \leqslant i \leqslant N$.

To connect a watermelon with a semistandard tableau, let us now read the watermelon configuration with endpoints $C_i = (N - i + 1, N - i)$, $B_i = (i, N + M - i)$ in the following way. The i^{th} path (counted from the bottom) makes $\lambda_i = N$ steps to the east. The power m_j of z_j in the weight is the number of steps to the east taken along the horizontal line z_j. The Young tableau of such a configuration is the $N \times N$ rectangle. The Schur function of the watermelon is

$$S_{\mathbf{N}}(z_1, z_2, \dots, z_{N+M}) = \sum_W \prod_{j=1}^{N+M} z_j^{m_j}\,, \tag{13}$$

where the summation is over all admissible watermelons, and $\mathbf{N}$ is the partition $(N, N, \dots, N)$ of length N, i.e., $\mathbf{N} \equiv N^N$ in our notation. The volume of the watermelon is equal to

$$| w |= \sum_{j=1}^{M+N} (j-1) m_j - \frac{N^2(N-1)}{2}.$$

The partition function of watermelons can be expressed in terms of the Schur function (13):

$$\mathsf{W}(N, M) = q^{-\frac{N^2}{2}(N-1)}\, S_{\mathbf{N}}(1, q^2, \dots, q^{N+M-1})\,. \tag{14}$$

This function is easy to calculate with the help of a well-known formula (see [1, Chap. 1, Example 1]):

$$S_{\boldsymbol{\lambda}}(1, q^2, \dots, q^{m-1}) = q^{n(\lambda)} \prod_{1 \leqslant i < j \leqslant m} \frac{1 - q^{\lambda_i - \lambda_j - i + j}}{1 - q^{j-i}}\,, \tag{15}$$

where $n(\lambda) = \sum_i (i-1)\lambda_i$. Moreover, if $m > N$, then $\lambda_i = 0$ for $i > N$. We obtain from (15) that

$$\mathsf{W}(N, M) = \prod_{i=1}^{N} \prod_{j=N+1}^{N+M} \frac{1 - q^{N-i+j}}{1 - q^{j-i}}\,. \tag{16}$$

Replacing $j \to N + j$ and $i \to N + 1 - i$, we put (16) into the form

$$\mathsf{W}(N, M) = \prod_{i=1}^{N} \prod_{j=1}^{M} \frac{1 - q^{N+i+j-1}}{1 - q^{j+i-1}} = \prod_{i=1}^{N} \prod_{j=1}^{N} \frac{1 - q^{M+i+j-1}}{1 - q^{j+i-1}}\,. \tag{17}$$

Finally, it is seen that Eqs. (12) and (17) are in agreement with Eqs. (5) and (6).

A watermelon with *deviation* k can be obtained by imposing the boundary condition $l_N = \ldots = l_{N-k+1} = 0$ in (10). The starting points D_i of a watermelon with deviation will be shifted k steps to the east with respect to C_i. A watermelon with deviation is presented in Fig. 5. The boundary condition introduced is equivalent to the following property of the Schur function. Consider a partition $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_{N-k}, \lambda_{N-k+1}, \ldots, \lambda_N)$ with

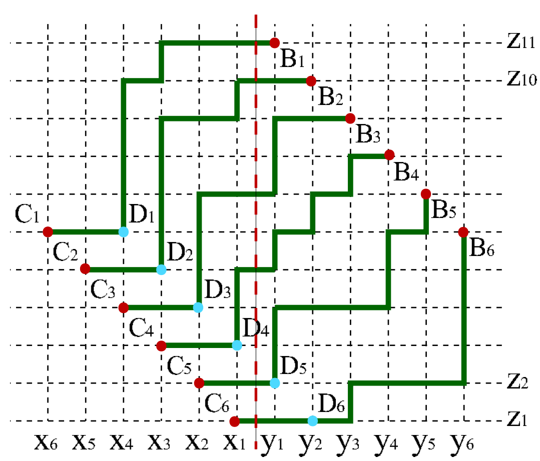

Fig. 5. A watermelon with deviation $k = 2$. The starting points are D_i, the endpoints are B_i.

the last k parts equal to zero, $\lambda_{N-k+1} = \ldots = \lambda_N = 0$. Then the following limiting relation holds:

$$\lim_{x_N\to 0}\cdots\lim_{x_{N-k+1}\to 0} S_{\boldsymbol{\lambda}}(x_1,\dots,x_{N-k},x_{N-k+1},\dots,x_N) = S_{\widetilde{\boldsymbol{\lambda}}}(x_1,\dots,x_{N-k})\,, \quad (18)$$

where the parts of $\widetilde{\boldsymbol{\lambda}}=(\lambda_1,\lambda_2,\dots,\lambda_{N-k})$ satisfy $M\geqslant\lambda_1\geqslant\lambda_2\dots\lambda_{N-k}\geqslant 0$. Taking the limit (18) in (3), we obtain

$$\sum_{\widetilde{\boldsymbol{\lambda}}\subseteq M^{N-k}} S_{\widetilde{\boldsymbol{\lambda}}}(x_1,\dots,x_{N-k})S_{\widehat{\boldsymbol{\lambda}}}(y_1,\dots,y_N) = \left(\prod_{l=1}^{N-k} x_l^{-k}\right)\frac{\det(\widetilde{M}_{kj})_{1\leqslant k,j\leqslant N}}{\mathcal{V}_{N-k}(\mathbf{x})\mathcal{V}_N(\mathbf{y})}\,,$$

where the summation is over all partitions $\widetilde{\boldsymbol{\lambda}}$ with at most $N-k$ parts each of which is at most M. The partition $\widehat{\boldsymbol{\lambda}}$ of length N contains extra zeros $\widehat{\lambda}_{N-k+1}=\widehat{\lambda}_{N-k+2}=\dots\widehat{\lambda}_N=0$, and the entries $\widetilde{M}_{kj}$ are

$$\widetilde{M}_{kj}=M_{kj}\,,\quad 1\leqslant k\leqslant N,\qquad 1\leqslant j\leqslant N-k\,,$$
$$\widetilde{M}_{kj}=y_j^{N-k}\,,\quad 1\leqslant k\leqslant N,\qquad N-k+1\leqslant j\leqslant N\,,$$

where the entries M_{kj} are given by (4).

The semistandard tableau corresponding to a watermelon with deviation consists of N rows of length $L=N-k$. The volume of the watermelon with deviation is

$$|\,w\,|=\sum_{j=1}^{M+N}(j-1)m_j-\frac{NM(M-1)}{2}. \quad (19)$$

In the case of a watermelon with deviation, we obtain a representation analogous to (14):

$$\begin{aligned}\mathsf{W}(N,L,M) &= q^{-\frac{NM(M-1)}{2}}\,S_{\mathbf{L}}(1,q,\dots,q^{N+M-1})\\ &= \sum_{\widetilde{\boldsymbol{\lambda}}\subseteq M^{N-k}} S_{\widetilde{\boldsymbol{\lambda}}}(q,\dots,q^{N-k})S_{\widehat{\boldsymbol{\lambda}}}(1,\dots,q^{N-1})\,, \end{aligned}\quad (20)$$

where $\mathbf{L} = L^N$ for the partition $\mathbf{L}$. Calculating the Schur function $S_{\mathbf{L}}$ with the help of (15), we obtain

$$\mathsf{W}(N,L,M) = \prod_{i=1}^{N} \prod_{j=N+1}^{N+M} \frac{1-q^{L-i+j}}{1-q^{j-i}} = \prod_{i=1}^{N} \prod_{j=1}^{M} \frac{1-q^{L+i+j-1}}{1-q^{j+i-1}}\,. \qquad (21)$$

In the limit $q \to 0$, this formula gives the number of watermelons with deviation:

$$A(N,L,M) = \prod_{i=1}^{N} \prod_{j=1}^{M} \frac{L+i+j-1}{j+i-1}\,. \qquad (22)$$

The Schur function can be expressed as a polynomial in the complete symmetric functions, [1]: $S_{\boldsymbol{\lambda}}(\mathbf{x}) = \det(h_{\lambda_i - i+j}(\mathbf{x}))_{1 \leqslant i,j \leqslant N}$. Under the q-parametrization, the complete symmetric functions are the q-binomial coefficients (8):

$$h_r(\mathbf{q}/q) \;=\; \begin{bmatrix} N+r-1 \\ r \end{bmatrix}, \qquad 1 \leqslant r \leqslant N. \qquad (23)$$

The following determinant with q-binomial entries was calculated in [21]:

$$\det \left(q^{(j-1)(\lambda_i+j-i)} \begin{bmatrix} \lambda_i + m - i \\ m-j \end{bmatrix} \right)_{1 \leqslant i,j \leqslant N} = S_{\boldsymbol{\lambda}}(1, q, \dots, q^{m-1}), \quad m \geqslant N. \qquad (24)$$

Using (23) and the Pascal formula for the q-binomial coefficients,

$$\begin{bmatrix} R \\ r \end{bmatrix} \;=\; \begin{bmatrix} R-1 \\ r-1 \end{bmatrix} \;+\; q^r \begin{bmatrix} R-1 \\ r \end{bmatrix}, \qquad (25)$$

one can re-express the left-hand side of (24) so that the following equation holds:

$$\det \left(h_{\lambda_i - i+j}(1, q, \dots, q^{m-1}) \right)_{1 \leqslant i,j \leqslant N} = S_{\boldsymbol{\lambda}}(1, q, \dots, q^{m-1})\,. \qquad (26)$$

The partition function of watermelons with deviation given by (20) and (21) can be rewritten with regard to the determinantal formulas (24) and (26):

$$\mathsf{W}(N,L,M) \;=\; q^{-\frac{NM(M-1)}{2}} \det \left(q^{(j-1)(L+j-i)} \begin{bmatrix} L+M+N-i \\ M+N-j \end{bmatrix} \right)_{1 \leqslant i,j \leqslant N} \qquad (27)$$

$$\;=\; q^{-\frac{NM(M-1)}{2}} \det \left(h_{L+j-i}(\mathbf{q}/q) \right)_{1 \leqslant i,j \leqslant N}\,. \qquad (28)$$

The number of watermelons with deviation (22) is given by

$$A(N,L,M) \;=\; \det\left(\binom{L+M+N-i}{M+N-j}\right)_{1\leqslant i,j\leqslant N} \tag{29}$$

$$\;=\; \det\left(\binom{L+M+N+j-i-1}{L+j-i}\right)_{1\leqslant i,j\leqslant N}, \tag{30}$$

where the determinant (29) is the *binomial determinant*, [22], while the coincidence of (29) and (30) can be checked independently by means of (25) at $q = 1$.

In the limit $q \to 1$, the Schur function (11) can be expressed with the help of (24):

$$\det\left(\binom{\lambda_i+N-i}{N-j}\right)_{1\leqslant i,j\leqslant N} = S_{\boldsymbol{\lambda}}(1,\ldots,1) \;=\; \sum_B 1 \;=\; \sum_C 1\,. \tag{31}$$

Equation (31) expresses the Gessel–Viennot theorem, [22], connecting the binomial determinant in the left-hand side of (31) with the number of nests of self-avoiding lattice paths of type either B or C.

§3. Plane partitions and watermelons

There exists a bijection between watermelons and plane partitions confined in a box of finite size, [23]. A plane partition is an array $(\pi_{ij})_{1\leqslant i,j}$ of nonnegative integers that are nonincreasing as functions both of i and j, [1,3]. The integers π_{ij} are called the parts of the plane partition, and $|\boldsymbol{\pi}| = \sum_{i,j}\pi_{ij}$ is its volume. Each plane partition has a three-dimensional diagram which can be interpreted as a stack of unit cubes (a three-dimensional Young diagram). The height of the stack with coordinates (i,j) is equal to π_{ij}. One says that a plane partition is contained in the box $\mathcal{B}(N,L,M)$ if $j \leqslant N$, $i \leqslant L$, and $\pi_{ij} \leqslant M$ for all cubes of the Young diagram. The generating function of plane partitions is

$$Z_q(N,L,M) = \sum_{\mathcal{B}(N,L,M)} q^{|\boldsymbol{\pi}|}\,, \tag{32}$$

where the sum is taken over all plane partitions contained in the box $\mathcal{B}(N,L,M)$.

The projection of the gradient lines of a plane partition (see Fig. 6) form a nest of lattice paths that correspond to watermelons (see Fig. 4 and Fig. 5, respectively). By construction, the volume of the watermelon

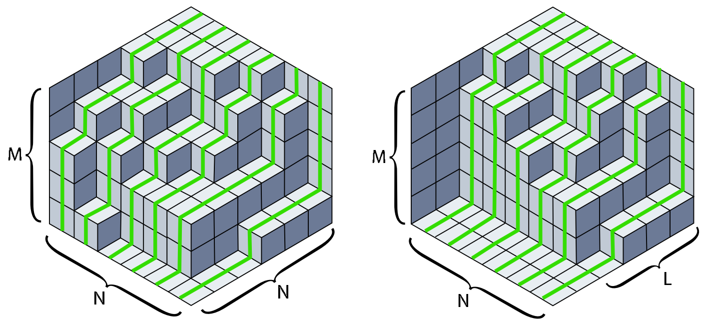

Fig. 6. Plane partitions with gradient lines embedded into a symmetric box $\mathcal{B}(N, N, M)$ and into an arbitrary box $\mathcal{B}(N, L, M)$ obtained as a special limit of symmetric boxes.

(19) coincides with the volume of the plane partition $|\boldsymbol{\pi}|$, and thus

$$Z_q(N, L, M) = \mathsf{W}(N, L, M)\,.$$

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