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INTERACTION OF HECKE–SHIMURA RINGS ACTING ON THETA-SERIES

Abstract. The paper considers principal features of *interactions* (transfer homomorphisms) for Hecke–Shimura rings of integral symplectic groups and integral orthogonal groups of positive definite integral quadratic forms both operating on theta-series of positive definite quadratic forms by Hecke operators.

§1. Introduction

An important tendency of further progress in the Diophantine arithmetic is closely related with study of *interaction* of different representations of Hecke–Shimura rings for arithmetical discrete subgroups of Lie groups on spaces of automorphic forms.

We understand that an automorphic structure on a Lie group consists of an arithmetical discrete subgroup of the group together with a space of automorphic forms for the subgroup and a linear representation of corresponding Hecke–Shimura ring on the space given by Hecke operators. An interaction from one automorphic structure to another is formed by an interaction mapping of the Hecke–Shimura rings together with a mapping of corresponding spaces of automorphic forms compatible with the action of Hecke operators. Examples of interaction are provided by lifts of automorphic structures to analogous groups of higher orders (see, e.g., [An01]) and interactions arising from consideration of Hecke–Shimura rings of Lie groups of different types, say, symplectic and orthogonal (see, e.g., [An06]).

If the spaces of automorphic forms are finite-dimensional then one can look at related common eigenfunctions of Hecke operators and compare corresponding invariants such as zeta-functions, which will be a object of further considerations.

In this paper, we consider in more details interaction maps of Hecke–Shimura rings of certain subgroups of symplectic groups to Hecke–Shimura

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rings of orthogonal groups of integral positive definite quadratic forms in even number of variables which naturally operate on spaces of theta-series of the quadratic forms.

Notation We fix the letters $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{C}$, as usual, for the set of positive rational integers, the ring of rational integers, the field of rational numbers, and the field of complex numbers, respectively.

If $\mathbb{A}$ is a set, $\mathbb{A}_n^m$ denotes the set of all $m \times n$-matrices with elements in $\mathbb{A}$. If $\mathbb{A}$ is a ring with the identity element, 1_n and 0_n denote the unit element and the zero of the ring $\mathbb{A}_n^n$, respectively.

The transpose of a matrix M is denoted by ${}^tM$. For two matrices Q and N of appropriate size we set

$$Q[N] = {}^tNQN.$$

For a complex square matrix A we write

$$\mathbf{e}\{A\} = \exp(\pi\sqrt{-1}\sigma(A)),$$

where $\sigma(A)$ is the sum of diagonal entries of A.

§2. Theta-series

In this section, we shall remind the basic definitions and properties of theta-series of integral positive definite quadratic forms in even number of variables. Let

$$\mathbf{q}(X) = \frac{1}{2}\,{}^tXQX = \frac{1}{2}Q[X] \quad ({}^tX = (x_1, \dots, x_m)) \tag{2.1}$$

be an integral positive definite quadratic form in m variables with the (symmetric) matrix $Q = {}^tQ$. Speaking on quadratic forms, we shall mainly use the equivalent matrix language. For $n = 1, 2, \dots$ the *Theta-series* $\Theta^n(Z;\, Q)$ *of* Q *of genus* n is defined by the series

$$\Theta^n(Z,\, Q) = \sum_{N \in \mathbb{Z}_n^m} \mathbf{e}\{Q[N]Z\}, \tag{2.2}$$

where the variable Z belongs to the (*Siegel*) *upper half-plane of genus* n,

$$\mathbb{H}^n = \left\{Z = X + \sqrt{-1}Y \in \mathbb{C}_n^n \mid {}^tZ = Z,\, Y > 0\right\}. \tag{2.3}$$

The theta-series is convergent absolutely and uniformly on subsets of $\mathbb{H}^n$ of the form

$$\left\{Z = X + \sqrt{-1}Y \in \mathbb{H}^n \mid Y \geqslant \varepsilon 1_n\right\} \text{ with } \varepsilon > 0,$$

and so it defines a holomorphic function of Z on $\mathbb{H}^n$. Since the form $\mathbf{q}$ is *integral*, the matrix Q of the form belongs to the set

$$\mathbb{E}^m = \left\{ Q = (Q_{ij}) \in \mathbb{Z}_m^m \mid Q_{ij} = Q_{ji},\, Q_{ii} \in 2\mathbb{Z} \;\; (i, j = 1, \ldots, m) \right\}$$

of *even* matrices of order m, and the theta-series has Fourier expansion of the form

$$\Theta^n(Z,\, Q) = \sum_{A \in \mathbb{E}^n,\, A \geqslant 0} r(A,\, Q)\, \mathbf{e}\{AZ\}$$

with constant Fourier coefficients expressing the numbers of *integral representations of the quadratic form with matrix A by the form* $\mathbf{q}$, i.e., the number of solutions of the equation $Q[N] = A$ in integral $m \times n-$matrices N.

According to [3, Theorems 4.1–4.3], we can cite the following theorem.

Theorem 2.1. *Let Q be an even positive definite matrix of even order m and let q be the level of Q, i.e., the least $q \in \mathbb{N}$ such that the matrix qQ^{-1} is even. Then the theta-series* (2.2) *of Q of genus $n \geqslant 1$ satisfies the functional equation*

$$\Theta^n\left(M\langle Z\rangle, Q\right) = j_Q(M,\, Z)\Theta^n(Z, Q), \tag{2.4}$$

for each matrix $M = \left(\begin{smallmatrix} A & ; \\ B & ; \end{smallmatrix}\right) C, D.$ of the group

$$\Gamma_0^n(q) = \left\{ M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp_n(\mathbb{Z}) \middle| C \equiv 0 \pmod q \right\}, \tag{2.5}$$

where $M\langle Z\rangle = \left(\begin{smallmatrix} A & ; \\ B & ; \end{smallmatrix}\right) C, D.\langle Z\rangle = (AZ + B)(CZ + D)^{-1}$,

$$j_Q(M,\, Z) = j_Q^n(M,\, Z) = \chi_Q(\det D)(\det(CZ + D))^{m/2}, \tag{2.6}$$

and χ_Q is the character of quadratic form with matrix Q.

We recall here that the *character of an integral nonsingular quadratic form in even number m of variables with matrix Q of level q* is the Dirichlet character modulo q satisfying the conditions

$$\chi_Q(-1) = (-1)^{m/2}, \tag{2.7}$$

$$\chi_Q(p) = \left(\frac{(-1)^{m/2} \det Q}{p} \right) \quad \text{(the Legendre symbol)} \tag{2.8}$$

if p is an odd prime number not dividing q, and

$$\chi_Q(2) = 2^{-m/2} \sum_{R \in \mathbb{Z}^m/2\mathbb{Z}^m} \mathbf{e}\left\{\frac{1}{2}Q[R]\right\}$$

if q is odd.

§3. Symplectic Hecke–Shimura rings and Hecke operators

Following the general pattern of the theory of Hecke operators on modular forms (see, e.g., [2, Chap. 4] or [4, §2]), we shall now remind the basic definitions and the simplest properties of (regular) Hecke–Shimura rings and Hecke operators for the groups $\Gamma_0^n(q)$. Let us denote by

$$\mathcal{H}_0^n(q) = \mathcal{H}(\Gamma_0^n(q),\, \Sigma_0^n(q))$$

the *Hecke–Shimura ring over* $\mathbb{C}$ *of the semigroup*

$$\Sigma_0^n(q) = \left\{ M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathbb{Z}_{2n}^{2n} \;\middle|\; {}^tMJ_nM = \mu(M)J_n,\, \mu(M) > 0, \right.$$
$$\left. \gcd(\det M, q) = 1,\, C \equiv 0_n \pmod{q} \right\} \quad \left(J_n = \begin{pmatrix} 0_n, 1_n, \\ -1_n, 0_n \end{pmatrix} \right)$$

relative to the group $\Gamma_0^n(q)$. The ring $\mathcal{H}_0^n(q)$ consists of all formal finite linear combinations with complex coefficients of the symbols $\tau(M)$, which are in one-to-one correspondence with double cosets $\Gamma_0^n(q)M\Gamma_0^n(q) \subset \Sigma_0^n(q)$. It is convenient to write each of the symbols $\tau(M)$, called also the *double cosets*, as the formal sum of different left cosets it contains (more precisely, of the corresponding symbols),

$$\tau(M) = \sum_{M' \in \Gamma\backslash\Gamma M \Gamma} (\Gamma M') \qquad (\Gamma = \Gamma_0^n(q),\, M \in \Sigma_0^n(q)), \tag{3.1}$$

Then each element $T \in \mathcal{H}_0^n(q)$ can be also written as the formal linear combination of different left cosets,

$$T = \sum_\alpha c_\alpha(\Gamma_0^n(q)M_\alpha) \quad (c_\alpha \in \mathbb{C}). \tag{3.2}$$

These linear combinations can be characterized by the the condition of invariance with respect to all right multiplication by elements of $\Gamma_0^n(q)$:

$$T\gamma = \sum_\alpha c_\alpha(\Gamma_0^n(q)M_\alpha\gamma) = T \quad \text{for all} \quad \gamma \in \Gamma_0^n(q).$$

In this notation, the product in $\mathcal{H}_0^n(q)$ can be defined by

$$TT' = \sum_\alpha c_\alpha(\Gamma_0^n(q)M_\alpha)\sum_\beta c'_\beta(\Gamma_0^n(q)M'_\beta) = \sum_{\alpha,\beta} c_\alpha c'_\beta(\Gamma_0^n(q)M_\alpha M'_\beta).$$

The ring $\mathcal{H}_0^n(q)$ is a commutative $\mathbb{C}$-algebra generated over $\mathbb{C}$ by a denumerable set of algebraically independent elements. As a set of algebraically independent generators one can take, for example, double cosets of the form

$$\begin{cases} T^n(p) &= \tau\big(\operatorname{diag}(\underbrace{1,\dots,1}_{n},\underbrace{p,\dots,p}_{n})\big),\\ T_i^n(p^2) &= \tau\big(\operatorname{diag}(\underbrace{1,\dots,1}_{n-i},\underbrace{p,\dots,p}_{i},\underbrace{p^2,\dots,p^2}_{n-i},\underbrace{p,\dots,p}_{i})\big) \\ & \qquad\qquad\qquad\qquad\qquad (1 \leqslant i \leqslant n), \end{cases} \tag{3.3}$$

where p runs over all prime numbers not dividing q (see [2, Theorem 3.3.23]).

In order to define Hecke operators on theta-series we denote by $\mathfrak{F}^n$ for fixed $n \in \mathbb{N}$ the space of all analytic functions $F = F(Z) : \mathbb{H}^n \mapsto \mathbb{C}$ and define the action of the semigroup $\Sigma_0^n(q)$ on the spaces by *Petersson operators*

$$\Sigma_0^n(q) \ni M : F = F(Z) \mapsto F|_{\mathbf{j}}M = j_Q(M,\, Z)^{-1}F(M\langle Z\rangle), \tag{3.4}$$

where $\mathbf{j} = j_Q(M,\, Z)$ is defined by (2.6). It is well-known that these operators map the space $\mathfrak{F}^n$ into itself and satisfy relations

$$F|_{\mathbf{j}}M|_{\mathbf{j}}M' = F|_{\mathbf{j}}MM' \quad (F \in \mathfrak{F}^n, \quad M, M' \in \Sigma_0^n(q)). \tag{3.5}$$

Thus we can define the standard representation $T \mapsto |_{\mathbf{j}}T$ of the Hecke-Shimura ring $\mathcal{H}_0^n(q) = \mathcal{H}(\Gamma_0^n(q),\, \Sigma_0^n(q))$ on the subspace

$$\mathfrak{M}_{\mathbf{j}}^n = \mathfrak{M}_{\mathbf{j}}^n(\Gamma_0^n(q)) = \{F \in \mathfrak{F}^n \mid F|_{\mathbf{j}}\gamma = F \quad \text{for all} \quad \gamma \in \Gamma_0^n(q)\} \tag{3.6}$$

of all $\Gamma_0^n(q)-$invariant functions, where for $n = 1$ one should add a simple boundary condition. The *Hecke operator* $|_{\mathbf{j}}T$ *on the space* $\mathfrak{M}_{\mathbf{j}}^n(\Gamma_0^n(q))$

corresponding to an element of the form (3.2) is defined by

$$F|_{\mathbf{j}}T = \sum_{\alpha} c_\alpha F\big|_{\mathbf{j}} M_\alpha \quad (F = F(Z) \in \mathfrak{M}^n_{\mathbf{j}}(\Gamma^n_0(q))), \tag{3.7}$$

where $|_{\mathbf{j}}M_\alpha$ are the Petersson operators (3.4) corresponding to $\mathbf{j}{=}j^n_Q(M,Z)$. The Hecke operators are independent of the choice of representatives $M_\alpha \in \Gamma^n_0(q)M_\alpha$ and map the space $\mathfrak{M}^n_{\mathbf{j}}(\Gamma^n_0(q))$ into itself. It follows from the definition of multiplication in the Hecke–Shimura rings and (3.5) that Hecke operators satisfy

$$|_{\mathbf{j}}T|_{\mathbf{j}}T' = |_{\mathbf{j}}TT' \quad \text{for all} \quad T, T' \in \mathcal{H}^n_0(q). \tag{3.8}$$

Hence, the map $T \mapsto |_{\mathbf{j}}T$ is a linear representation of the ring $\mathcal{H}^n_0(q)$ on the space $\mathfrak{M}^n_{\mathbf{j}}(\Gamma^n_0(q))$.

In the notation and under the assumptions of Theorem 2.1, the theta-series $\Theta^n(Z,Q)$ viewed as a function of Z, belongs to the space $\mathfrak{M}^n_{\mathbf{j}}(\Gamma^n_0(q))$. We shall see later that the images of the theta-series under the Hecke operators can be written as finite linear combinations with constant coefficients of similar theta-series. At first we shall express these images as infinite sums with explicitly written coefficients.

We shall begin with two simple technical remarks. By definition, each matrix $M \in \Sigma^n_0(q)$ satisfies the relation ${}^tMJ_nM = \mu(M)J_n$, where $\mu(M)$ is a positive integer coprime with q called the *multiplier of* M. Clearly,

$$\mu(MM') = \mu(M)\mu(M') \quad (M, M' \in \Sigma^n_0(q)), \text{ and } \mu(M) = 1 \Leftrightarrow M \in \Gamma^n_0(q).$$

It follows that the function $M \mapsto \mu(M)$ takes constant value on each left and double coset of the matrix M modulo the group $\Gamma^n_0(q)$. Hence, one can speak on the multiplier of the corresponding cosets, $\mu(\Gamma^n_0(q)M) = \mu(\Gamma^n_0(q)M\Gamma^n_0(q)) = \mu(M)$. We shall say that a nonzero formal finite linear combination T of left or double cosets modulo $\Gamma^n_0(q)$ of matrices in $\Sigma^n_0(q)$ is *homogeneous of the multiplier* $\mu(T) = \mu$ if all of different cosets entering to the linear combination with nonzero coefficients have the same multiplier μ. It is clear that each finite linear combination of the cosets is a sum of homogeneous combinations with different multipliers, called *homogeneous components*, and these components are uniquely determined. This allows us, in particular, to reduce the study of arbitrary Hecke operators $|T$ to the case of homogeneous T.

Another reduction is related to a specific choice of representatives in the left cosets $\Gamma^n_0(q)M \subset \Sigma^n_0(q)$. According, for example, to [2, Lemma 3.3.4], each of the left cosets contains a representative of the form

$$M = \begin{pmatrix} A & B \\ 0_n & D \end{pmatrix}$$
$$\text{with} \quad A, B, D \in \mathbb{Z}_n^n, \ {}^tAD = \mu(M)1_n, \ {}^tBD = {}^tDB. \quad (3.9)$$

Such representatives are convenient for computation of action of Hecke operators and will be referred to as *triangular representatives.*

Proposition 3.1. *Let Q be an even positive definite matrix of even order m and level q. Then the image of the theta-series $\Theta^n(Z,Q)$ under the action of Hecke operator corresponding to an homogeneous element*

$$T = \sum_{\alpha} c_\alpha(\Gamma_0^n(q)M_\alpha) \in \mathcal{H}_0^n(q) \quad (3.10)$$

of a multiplier μ with triangular representatives

$$M_\alpha = \begin{pmatrix} A_\alpha & B_\alpha \\ 0_n & D_\alpha \end{pmatrix} \quad (A_\alpha, B_\alpha, D_\alpha \in \mathbb{Z}_n^n, \ {}^tA_\alpha D_\alpha = \mu 1_n, \ {}^tB_\alpha D_\alpha = {}^tD_\alpha B_\alpha),$$

can be written in the form

$$\Theta^n(Z,Q)|_{\mathbf{j}}T = \sum_{N \in C^n(Q/\mu)} I(N, Q, T)\mathbf{e}\{\mu^{-1}Q[N]Z\}. \quad (3.11)$$

where

$$C^n(Q/\mu) = \{N \in \mathbb{Z}_n^m \mid \mu^{-1}Q[N] \in \mathbb{E}^n\}, \quad (3.12)$$

$$I(N,Q,T) = \sum_{\alpha,\, N\cdot {}^tD_\alpha \equiv 0 \pmod{\mu}} c_\alpha j_Q(D_\alpha)^{-1}\mathbf{e}\{\mu^{-2}Q[N]\cdot {}^tD_\alpha B_\alpha\}, \quad (3.13)$$

and

$$j_Q(D) = \chi_Q(|\det D|)|\det D|^{m/2}. \quad (3.14)$$

Proof. See [4, Proposition 2.1] or [7, Proposition 3.1] with $V = 0_{2n}^m$. Let us introduce the exponent

$$\mathbf{e}(Z,Q;N) = \mathbf{e}\{Q[N]Z\}. \quad (3.15)$$

It is not hard to see that for each matrix M of the form (3.9) the exponent satisfies the relation

$$\mathbf{e}(M < Z >, Q; N) = \mathbf{e}\{Q[N]BD^{-1}\}\mathbf{e}(Z, \mu^{-1}Q; NA).$$

By (3.4) in these notation we get

$$\begin{aligned}\Theta^n(Z,Q)\big|_{\mathbf{j}}T &= \sum_\alpha c_\alpha \Theta^n(Z,Q)\big|_{\mathbf{j}}\begin{pmatrix} A_\alpha & B_\alpha \\ 0_n & D_\alpha\end{pmatrix} \\ &= \sum_{N\in\mathbb{Z}^m_n}\sum_\alpha c_\alpha j_Q(M_\alpha,Z)^{-1}\mathbf{e}(M_\alpha\langle Z\rangle, Q; N).\end{aligned} \tag{3.16}$$

By definition we have $j_Q(M_\alpha,Z)^{-1} = j_Q(D_\alpha)^{-1}$. Hence in the notation (3.15) we can rewrite the sum to the right of (3.16) as

$$= \sum_{N\in\mathbb{Z}^m_n}\sum_\alpha c_\alpha j_Q(D_\alpha)^{-1}\mathbf{e}\{B_\alpha D_\alpha^{-1}Q[N]\}\mathbf{e}(Z,\mu^{-1}Q;NA_\alpha).$$

Collecting here all the terms with a fixed matrix $NA_\alpha = \mu N\,{}^tD_\alpha^{-1} = \widetilde{N}\in\mathbb{Z}^m_n$ and using obvious relations

$$\mathbf{e}\{B_\alpha D_\alpha^{-1}Q[N]\} = \mathbf{e}\{D_\alpha^{-1}Q[\mu^{-1}\widetilde{N}\,{}^tD_\alpha]B_\alpha\} = \mathbf{e}\{\mu^{-2}Q[\widetilde{N}]\,{}^tD_\alpha B_\alpha\},$$

we get the formula

$$\begin{aligned}&\Theta^n(Z,Q)|_{\mathbf{j}}T \\ &= \sum_{\widetilde{N}\in\mathbb{Z}^m_n}\left(\sum_{\alpha,\widetilde{N}A_\alpha^{-1}\in\mathbb{Z}^m_n} c_\alpha j_Q(D_\alpha)^{-1}\mathbf{e}\{\mu^{-2}Q[\widetilde{N}]\,{}^tD_\alpha B_\alpha\}\right)\mathbf{e}(Z,\mu^{-1}Q;\widetilde{N}).\end{aligned}$$

Hence, if we omit the tilde and note that conditions $NA_\alpha^{-1}\in\mathbb{Z}^m_n$ is equivalent to condition $N\,{}^tD_\alpha\in\mu\mathbb{Z}^m_n$, we get formula (3.11). □

Of elementary properties of iSums we note here only the following simple lemma.

Lemma 3.1. *The iSums $I(N, Q, T)$ for $N\in C^n(Q/\mu)$ and $T\in\mathcal{H}^n_0(q)$ are independent of a particular choice of triangular representatives in expansions* (3.10) *of T and satisfy the relations*

$$I(\lambda N\gamma, Q, T) = I(N, Q[\lambda], T) \text{ for all } \lambda\in\mathrm{GL}_m(\mathbb{Z}) \text{ and } \gamma\in\mathrm{GL}_n(\mathbb{Z}).$$

Proof. The independence of the choice of representatives easily follows from definitions. If T is an homogeneous element of the form (3.10) with triangular representatives M_α satisfying $\mu(M_\alpha) = \mu(T) = \mu$, then, by (3.13) we obtain

$$I(\lambda N\gamma, Q, T) = \sum_{\alpha,\,\lambda N\,{}^t(D_\alpha\,{}^t\gamma)\equiv 0 \pmod{\mu}} c_\alpha j_Q(D_\alpha\,{}^t\gamma)^{-1}\mathbf{e}\{\mu^{-2}Q[\lambda][N]\cdot{}^t(D_\alpha\,{}^t\gamma)B_\alpha\,{}^t\gamma\}$$

$$= I\left(N,\, Q[\lambda],\, T\begin{pmatrix}\gamma^{-1} & 0\\ 0 & {}^t\gamma\end{pmatrix}\right) = I(N,\, Q[\lambda],\, T);$$

since T is invariant under right multiplications by elements of $\Gamma_0^n(q)$. □

§4. 4 Siegel operator and Zharkovskaya mapping on theta-series

Here we shall remind definitions of Siegel operators and Zharkovskaya mappings for holomorphic modular of integral weigh for congruence subgroups of the symplectic modular group $\Gamma^n = \mathrm{Sp}_n(\mathbb{Z})$ and their commutation relations (see, e.g., [2, §4.2.4]). This will allow us to link action of Hecke operators on theta-series of different genera and corresponding interaction sums. Final formulas depend on normalization of Hecke operators.

Through this section Q is the matrix of an integral positive definite quadratic form $\mathbf{q}(X)$ in an even number of variables m, q – is the level of Q, and χ_Q – is the corresponding Dirichlet character modulo q.

For a function $F : \mathbb{H}^n \mapsto \mathbb{C}$ and $0 \leqslant r \leqslant n$, we define the *Siegel operator* $\Phi^{n,r}$ by

$$(F\big|\Phi^{n,r})(Z') = \lim_{\lambda\mapsto+\infty} F\left(\begin{pmatrix} Z & ' \\ , & 0\end{pmatrix}^{r}_{n-r}, 0_r^{n-r}, \sqrt{-1}\lambda\cdot 1_{n-r}.\right), \tag{4.1}$$

if $0 < r < n$, where $Z' \in \mathbb{H}^r$,

$$F\big|\Phi^{n,0} = \lim_{\lambda\mapsto+\infty} F(0_{2n}^m,\, \sqrt{-1}\lambda 1_n), \tag{4.2}$$

and

$$F\big|\Phi^{n,n} = F. \tag{4.3}$$

If F is a theta-series (2.2) of genus n of Q and $1 \leqslant r \leqslant n$, then the limits exist and equal to the theta-series of genus r :

$$(\Theta^n\big|\Phi^{n,r})(Z'; Q) = \Theta^r(Z';\, Q) \quad (Z' \in \mathbb{H}^r). \tag{4.4}$$

Also, $\Theta^n\big|\Phi^{n,0} = 1$.

For $n > r \geqslant 1$, the *Zharkovskaya homomorphism* related to the action of Hecke operators on theta-series of genus n of quadratic form in m variables,

$$\Psi^{n,r} = \Psi_Q^{n,r} : \mathcal{H}_0^n(q) \mapsto \mathcal{H}_0^r(q), \tag{4.5}$$

can be defined as follows. If $T \in \mathcal{H}_0^n(q)$ is an element of the form (3.10) with triangular representatives M_α, then after replacing of each representative M_α by $\begin{pmatrix} {}^tU_\alpha^{-1} & 0\\ 0 & U_\alpha\end{pmatrix} M_\alpha$ with suitable $U_\alpha \in \mathrm{GL}_n(\mathbb{Z})$ one may assume that the block D_α of M_α has the form $D_\alpha = \begin{pmatrix} D'_\alpha & * \\ 0 & D''_\alpha \end{pmatrix}$ with $D'_\alpha \in \mathbb{Z}^r_r$ (see, e.g., [2, Lemma 3.2.7]), so that the matrix M_α takes the shape

$$M_\alpha = \begin{pmatrix} \begin{pmatrix} A'_\alpha & 0 \\ * & A''_\alpha \end{pmatrix} & \begin{pmatrix} B'_\alpha & * \\ * & * \end{pmatrix} \\ 0 & \begin{pmatrix} D'_\alpha & * \\ 0 & D''_\alpha \end{pmatrix} \end{pmatrix}, \tag{4.6}$$

where

$$M'_\alpha = \begin{pmatrix} A'_\alpha & B'_\alpha \\ 0 & D'_\alpha \end{pmatrix} \in \Sigma^r_0(q) \text{ with } \mu(M'_\alpha) = \mu(M_\alpha).$$

Then we put

$$\Psi^{n,r}(T) = \Psi^{n,r}_Q(T) = \sum_\alpha c_\alpha j_Q(D''_\alpha)^{-1}(\Gamma^r_0(q)M'_\alpha), \tag{4.7}$$

where j_Q has the form (3.14). It can be easily verified that the linear combination (4.8) belongs to the ring $\mathcal{H}^r_0(q)$, and that the mapping $T \mapsto \Psi^{n,r}(T)$ is a $\mathbb{C}$-linear homomorphism of the rings. It is clear that the Zharkovskaya homomorphism with $r \geqslant 1$ maps homogeneous elements of $\mathcal{H}^n_0(q)$ to homogeneous elements of the same multipliers.

Proposition 4.1 (Zharkovskaya commutation relation for theta-series)**.** *Let Q be a nonsingular positive definite matrix of an even order m and level q, and let $T \in \mathcal{H}^n_0(q)$. Then the following commutation relation holds for the action of Hecke operator $|_{\mathbf{j}}T$ with $\mathbf{j} = j_Q(M, Z)$ of the form* (2.6) *on theta-series* (2.2) *of genus n:*

$$\begin{aligned} ((\Theta^n|_{\mathbf{j}}T)|\Phi^{n,r})(Z'; Q) &= ((\Theta^n|\Phi^{n,r})|_{\mathbf{j}}\Psi^{n,r}(T))(Z'; Q) \\ &= (\Theta^r|_{\mathbf{j}}\Psi^{n,r}(T))(Z'; Q), \end{aligned} \tag{4.8}$$

if $n > r \geqslant 1$, where $Z' \in \mathbb{H}^r$.

The Zharkovskaya homomorphism (4.4) is not always epimorphic (see, e.g., [4, Proposition 3.3]).

The following useful lemma is an elementary consequence of definitions.

Lemma 4.1. *If $T \in \mathcal{H}^n_0(q)$ and $N \in C^r(Q/\mu)$, where $n > r > 0$, then the sums* (3.13) *satisfy the relations*

$$I((N, 0^m_{n-r}), Q, T) = I(N, Q, \Psi^{n,r}(T)), \tag{4.9}$$

where $\Psi^{n,r} = \Psi^{n,r}_Q$ is the Zharkovskaya homomorphism.

Proof. In order to prove (4.9) we may assume that T is an homogeneous element of the form (3.10) with triangular representatives M_α of the form (4.6), where $\mu(M_\alpha) = \mu(T) = \mu$. By (3.13) we have

$$I((N, 0^m_{n-r}), Q, T) = \sum_{\alpha,\, (N, 0^m_{n-r})\begin{pmatrix} {}^tD'_\alpha & 0 \\ * & {}^tD''_\alpha \end{pmatrix} \equiv 0 \pmod{\mu}} c_\alpha j_Q\left(\begin{pmatrix} D'_\alpha & * \\ 0 & D''_\alpha \end{pmatrix}\right)^{-1}$$

$$\times\, \mathbf{e}\left\{\mu^{-2}\begin{pmatrix} Q[N] & 0 \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} {}^tD'_\alpha B'_\alpha & * \\ * & * \end{pmatrix}\right\}$$

$$= \sum_{\alpha,\, N\,{}^tD'_\alpha \equiv 0 \pmod{\mu}} c_\alpha j_Q(D''_\alpha)^{-1} j_Q(D'_\alpha)^{-1}\mathbf{e}\{\mu^{-2}Q[N]\,{}^tD'_\alpha B'_\alpha\} = I(N, Q, \Psi^{n,r}(T)),$$

since

$$\Psi^{n,r}(T) = \sum_\alpha c_\alpha j_Q(D''_\alpha)^{-1}\left(\Gamma^r_0(q)\begin{pmatrix} A'_\alpha & B'_\alpha \\ 0 & D'_\alpha \end{pmatrix}\right).$$

Formula (4.9) directly follows from (3.13) and (4.7). □

§5. Automorphic expansion of interaction sums and explicit formulas

In this section, we again assume that Q the matrix of a fixed integral positive definite quadratic form $\mathbf{q}(X)$ in an even number m of variables, q – is the level of Q, and χ_Q – is the corresponding Dirichlet character modulo q.

We shall say that an integral $m \times m$-matrix D is an *automorph of the matrix* Q (or an *automorph of the form* $\mathbf{q}$) *with the multiplier* $\mu = \mu(D)$ if

$$\mu^{-1}Q[D] \in \mathbb{E}^m \quad \text{and} \quad \det \mu^{-1}Q[D] = \det Q.$$

The set of all automorphes of Q with multiplier μ will be denoted by

$$A(Q, \mu) = \left\{D \in \mathbb{Z}^m_m \;\middle|\; \mu^{-1}Q[D] \in \mathbb{E}^m,\ \det \mu^{-1}Q[D] = \det Q\right\}. \tag{5.1}$$

The set $A(Q, \mu)$ can be empty. It is clear that

$$A(Q, \mu)\Lambda = A(Q, \mu), \quad \text{where} \quad \Lambda = \Lambda^m = \mathrm{GL}_m(\mathbb{Z}),$$

and so the group Λ operates on each of the sets $A(Q, \mu)$ by right multiplications. Since all automorphes of $A(Q, \mu)$ are integral matrices of determinants $\pm\,\mu^{m/2}$, it follows that each *set of classes of automorphes* $A(Q, \mu)/\Lambda$ is finite.

The following lemma plays principal part in formulas for action of Hecke operators on theta-series.

Lemma 5.1 (The automorphic expansion of iSums of genus m)**.** *Let T be an homogeneous nonzero element of the ring $\mathcal{H}_0^m(q)$ with the multiplier $\mu(T)=\mu$. Then the following formula holds for each matrix $N \in C^m(Q/\mu)$ (the set* (3.12) *with $n = m$)*

$$I(N,\,Q,\,T)=\begin{cases}\sum\limits_{D\in A(Q,\,\mu)/\Lambda,\,D|N} I(D,\,Q,\,T) & \text{if } A(Q,\mu)\neq\varnothing,\\ 0 & \text{if } A(Q,\mu)=\varnothing,\end{cases}$$

where the condition $D|N$ means that the matrix $D^{-1}N$ is integral.

Proof. The lemma follows from [1, Theorem 1] and [7, Proposition 5.1]. □

Proposition 5.1 (The automorphic expansion of iSums of arbitrary genus)**.** *Let T be an homogeneous element of $\mathcal{H}_0^n(q)$ for $n=1,2,\dots$ with the multiplier $\mu(T)=\mu$. Suppose that in the case $n<m$ element T satisfies the condition*

$$T \in \Psi_Q^{m,n}(\mathcal{H}_0^m(q)), \tag{5.2}$$

where $\Psi_Q^{m,n}$ is the Zharkovskaya map (4.7)*. Then the following expansion of the iSums holds for each matrix $N \in C^n(Q/\mu)$*

$$I(N,\,Q,\,T)=\begin{cases}\sum\limits_{D\in A(Q,\,\mu)/\Lambda,\,D|N} I(D,Q,\Psi_Q^{n,m}T) & \text{if } A(Q,\mu)\neq\varnothing,\\ 0 & \text{if } A(Q,\mu)=\varnothing,\end{cases} \tag{5.3}$$

where the condition $D|N$ means that the matrix $D^{-1}N$ is integral, and if $n<m$, $\Psi_Q^{n,m}T$ is an inverse image of T under the map $\Psi_Q^{m,n}$.

Proof. Proof By Lemma 5.1, we may assume that $n \neq m$. If $n > m$ and $N \in C^n(Q/\mu)$, then it is clear that there is a matrix γ of $\mathrm{GL}_n(\mathbb{Z})$ such that $N\gamma = (N', 0^m_{n-m})$ with $N' \in C^m(Q/\mu)$. By Lemmas 3.2 and 4.2, we obtain

$$I(N,\,Q,\,T) = I((N', 0^m_{n-m}),\,Q,\,T) = I(N',\,Q,\,\Psi_Q^{n,m}T), \tag{5.4}$$

and, by (5.1) for the case $A(Q,\mu) \neq \varnothing$, we obtain

$$\begin{aligned} I(N', Q, \Psi^{n,m}T) &= \sum_{D \in A(Q,\mu)/\Lambda^m,\, D|N'} I(D, Q, \Psi_Q^{n,m}T) \\ &= \sum_{D \in A(Q,\mu)/\Lambda^m,\, D|N} I(D, Q, \Psi_Q^{n,m}T), \end{aligned}$$

since the conditions $D|N'$ and $D|N$ are clearly equivalent. This proves the first relation (5.1). But if $A(Q,\mu) = \varnothing$, then by (5.4) and second formula (5.1) we see that $I(N, Q, T) = 0$ for all $N \in C^n(Q,\mu)$.

Let now $n < m$ and $N \in C^n(Q/\mu)$. Then clearly $N' = (N, 0^m_{m-n}) \in C^m(Q/\mu)$. If $A(Q,\mu) \neq \varnothing$ and an inverse image $\Psi_Q^{n,m}T \in \mathcal{H}_0^m(q)$ exists, then the matrix N' satisfies the first of relations (5.1):

$$\begin{aligned} I(N', Q, \Psi_Q^{n,m}T) &= \sum_{D \in A(Q,\mu)/\Lambda^m,\, D|N'} I(D, Q, \Psi_Q^{n,m}T) \\ &= \sum_{D \in A(Q,\mu)/\Lambda^m,\, D|N} I(D, Q, \Psi_Q^{n,m}T), \end{aligned}$$

On the other hand, by (4.11),

$$I(N', Q, \Psi^{n,m}T) = I(N, Q, \Psi^{m,n}(\Psi^{n,m}T)) = I(N, Q, T), \tag{5.5}$$

which proves the first relation (5.3) in this case. But if $A(Q,\mu) = \varnothing$ (and an inverse image $\Psi_Q^{n,m}T \in \mathcal{H}_0^m(q)$ exists), then the second of relations (5.1) shows that $I(N', Q, \Psi_Q^{n,m}T) = 0$ which, by (5.5), means that $I(N, Q, T) = 0$. □

Proposition 5.2. *Let T be an homogeneous element of $\mathcal{H}_0^n(q)$ for $n = 1, 2, \ldots$ with the multiplier $\mu(T) = \mu$. Suppose that in the case $n < m$ element T satisfies condition* (5.2). *Then the formula holds*

$$\Theta^n(Z;Q)|_{\mathbf{j}}T = \begin{cases} \sum\limits_{D \in A(Q,\mu)/\Lambda} I(D, Q, \Psi_Q^{n,m}T)\Theta^n(Z; \mu^{-1}Q[D]), \\ 0, \end{cases} \tag{5.6}$$

depending on whether $A(Q,\mu) \neq \varnothing$ or $A(Q,\mu) = \varnothing$, where $\mathbf{j} = j_Q(M, Z)$ is the automorphic factor (2.6).

Proof. From formulas (3.11) and the first formula (5.3) we obtain that the image $\Theta^n(Z;Q)|_{\mathbf{j}}T$ can be written in the form

$$= \sum_{N\in C^n(Q/\mu)} \left(\sum_{D\in A(Q,\mu)/\Lambda,\, D|N} I(D,\, Q,\, \Psi^{n,m}T) \right) \mathbf{e}\{\mu^{-1}Q[N]Z\}$$
$$= \sum_{D\in A(Q,\mu)/\Lambda} I(D,\, Q,\, \Psi^{n,m}T) \sum_{N=DN',\, N'\in\mathbb{Z}_n^m} \mathbf{e}\{\mu^{-1}Q[N]Z\}$$
$$= \sum_{D\in A(Q,\mu)/\Lambda} I(D,\, Q,\, \Psi^{n,m}T) \sum_{N'\in\mathbb{Z}_n^m} \mathbf{e}\{\mu^{-1}Q[D][N']Z\}$$
$$= \sum_{D\in A(Q,\mu)/\Lambda} I(D,\, Q,\, \Psi^{n,m}T)\Theta^n(Z;\mu^{-1}Q[D]),$$

which proves the first formula (5.6). The second formula (5.6) directly follows from the second formula (5.3) and formula (3.11). □

In this section, we have deduced reformulations of transformation formulas for action of Hecke operators on theta-series by using mainly formal computations. However, for complete proofs one can not escape rather deep arithmetical considerations: the proof of Lemma 5.1 is based on Theorem 1 of [1], which uses a complicated techniques of explicit factorizations of certain standard polynomials (*Rankin polynomials*) over symplectic Hecke–Shimura rings under their parabolic embeddings. Perhaps, it could be useful to write a simplified version of the proof, but it would require a plenty of time without any new method or results at the end.

§6. Action on theta-series of Hecke operators $|T^n(p)$ for primes p

By using an elementary approach, similar to the approach of [7] with necessary modifications one can prove the following formulas for the action of Hecke operators $|_{\mathbf{j}}T^n(p)$ with prime p not dividing q on theta-series (2.2) of genus n. Similar formulas were proved by another method in the book [2, §5.2.2].

Theorem 6.1. *Let Q be an even positive definite matrix of even order $m = 2k$ of level q and χ_Q – the corresponding Dirichlet character. Let p be a prime number not dividing the level q. Then the following explicit formulas hold for the action of Hecke operator $|_{\mathbf{j}}T^n(p)$ for $n \geqslant 1$ with* *automorphic factor* $\mathbf{j} = j_Q$ *on the theta-series* (2.2) *of genus* n

$$\Theta^n(Z,\,Q)|_{\mathbf{j}}T^n(p) = p^{n(n+1)/2-k\,n}\xi(n,\,m)\sum_{D\in A(Q,\,p)/\Lambda^m}\Theta^n(Z,\,p^{-1}Q[D])$$

if $\chi_Q(p) = 1$, *where*

$$\xi(n,\,m) = \begin{cases}\prod_{i=1}^{n-k}(1+p^{-i}) & \text{if } n > k,\\ 1 & \text{if } n = k,\\ \prod_{i=1}^{k-n}(1+p^{i-1}) & \text{if } n < k,\end{cases}$$

$A(Q,\,p)$ *is the set* (5.1) *for* $\mu = p$, *and*

$$\Theta^n(Z,\,Q)|_{\mathbf{j}}T^n(p) = 0 \quad \text{if} \quad \chi_Q(p) = -1 \quad \text{and} \quad n \geqslant k.$$

Proof. The formulas follow from formulas (6.27)–(6.28) of Theorem 6.3 [7] for $V = 0^m_{2n}$ and $H = Q$. □

§7. Orthogonal Hecke-Shimura rings

Here we shall briefly recall definition of orthogonal Hecke–Shimura rings. Let Q be again an even positive definite matrix of even order m. We fix a system of representatives

$$\langle Q\rangle = \{Q_1,\ldots,Q_h\} \tag{7.1}$$

of all different classes of integral equivalence of even positive definite matrices of the same order, divisor, level, and determinant as the matrix Q. For such a system and $i,\,j = 1,\ldots,h$ we define groups

$$\mathbf{E}_i = E(Q_i) = \{M \in \mathrm{GL}_m(\mathbb{Z}) \mid Q_i[M] = Q_i\}$$

of *integral units of* Q_i and sets

$$\mathbf{A}_{ij} = \bigcup_{\mu\geqslant 1,\gcd(\mu,q)=1}\mathbf{A}_{ij}(\mu),$$

$$\text{where} \quad \mathbf{A}_{ij}(\mu) = \{D \in \mathbb{Z}^m_m \mid Q_i[D] = \mu Q_j\} \tag{7.2}$$

of (*regular*) *automorphs of* Q_i *to* Q_j. It is easy to verify that groups $\mathbf{E}_i$ and sets $\mathbf{A}_{ij}$ satisfy the following three conditions: $\mathbf{A}_{ij}\mathbf{A}_{jk} \subset \mathbf{A}_{ik}$, $\mathbf{E}_i \subset \mathbf{A}_{ii}$, and each coset $\mathbf{E}_iD\mathbf{E}_j$ with $D \in \mathbf{A}_{ij}$ is a finite union of left cosets modulo $\mathbf{E}_i$. Let us denote by $\mathcal{D}_{ij}$ the set of all finite formal linear combinations with, say, complex coefficients of symbols $(\mathbf{E}_i M)$, corresponding in one-to-one way to different left cosets $\mathbf{E}_i M$ contained in $\mathbf{A}_{ij}$, which are invariant with respect to all right multiplication by element of $\mathbf{E}_j$:

$$\mathcal{D}_{ij} \ni t = \sum_{\alpha} a_\alpha (\mathbf{E}_i M_\alpha), \quad t\lambda = \sum_{\alpha} a_\alpha (\mathbf{E}_i M_\alpha \lambda) = t \quad (\forall \lambda \in \mathbf{E}_j). \tag{7.3}$$

Finally, denote by

$$\mathbf{D} = D(\mathbf{E}_1, \dots, \mathbf{E}_h; \mathbf{A}_{11}, \mathbf{A}_{12}, \dots, , \mathbf{A}_{hh}) \tag{7.4}$$

the set of all $h \times h$-matrices $\mathbf{t} = (t_{ij})$ with $t_{ij} \in \mathcal{D}_{ij}$. With respect to the standard matrix operation, where product of two linear combinations

$$t_{ij} = \sum_{\alpha} a_\alpha (\mathbf{E}_i M_\alpha) \in \mathcal{D}_{ij}, \quad t_{jk} = \sum_{\beta} b_\beta (\mathbf{E}_j N_\beta) \in \mathcal{D}_{jk}$$

is defined by

$$t_{ij} \cdot t_{jk} = \sum_{\alpha, \beta} a_\alpha b_\beta (\mathbf{E}_i M_\alpha N_\beta) \in \mathcal{D}_{ik},$$

the set $\mathbf{D}$ is an associative ring, called the (*regular*) *Hecke-Shimura ring of the system* (7.1).

For example, if Q is a *one-class matrix*, i.e., $h = 1$, then the ring

$$\mathbf{D} = D(\mathbf{E}, \mathbf{A}) = \mathcal{D}, \tag{7.5}$$

where $\mathbf{E} = E(Q)$ and $\mathbf{A} = \mathbf{A}(Q, Q)$, consists of all finite formal linear combinations with complex coefficients of symbols $(\mathbf{E}M)$, corresponding in one-to-one way to those linear combinations different left cosets $\mathbf{E}M \subset \mathbf{A}$, which are invariant with respect to all right multiplication by element of $\mathbf{E}$:

$$\mathbf{D} \ni t = \sum_{\alpha} a_\alpha (\mathbf{E} M_\alpha) \quad \Leftrightarrow \quad t\lambda = \sum_{\alpha} a_\alpha (\mathbf{E} M_\alpha \lambda) = t \quad (\forall \lambda \in \mathbf{E}). \tag{7.6}$$

§8. Interaction mappings

Let $T \in \mathcal{H}_0^n(q)$ be an homogeneous element with multiplier $\mu(T) = \mu$. Let us assume that the a set $A(Q_j, \mu)$ of the form (5.1) is not empty for a matrix Q_j of a system of the form (7.1), then for each matrix $D \in A(Q_j, \mu)$ the matrix $\mu^{-1} Q_j[D]$ is integrally equivalent to one of the matrices Q_i. By choosing an appropriate representative in the coset $D \cdot \Lambda$, where $\Lambda = \Lambda^m = \mathrm{GL}_m(\mathbb{Z})$, one can assume that $\mu^{-1} Q_j[D] = Q_i$, i.e., $Q_j[D] = \mu Q_i$, and the cosets $D \cdot \Lambda$ for such D are reduced to the cosets $D\mathbf{E}_i$ of the group $\mathbf{E}_i = E(Q_i)$ of integral units of Q_i. Thus one can take

$$A(Q_j, \mu)/\Lambda = \sum_{1 \leqslant i \leqslant h} \mathbf{A}_{ji}(\mu)/\mathbf{E}_i \qquad \text{with} \quad \mathbf{A}_{ji}(\mu) = \{D \in \mathbb{Z}_m^m \mid Q_j[D] = \mu Q_i\}. \tag{8.1}$$

Then relation (5.6) under the assumptions of Proposition 5.3 can be written in the form

$$\Theta^n(Z, Q_j)|T = \begin{cases} \sum\limits_{1 \leqslant i \leqslant h} \sum\limits_{D \in \mathbf{A}_{ji}(\mu)/\mathbf{E}_i} I(D,\, Q_j,\, \Psi_Q^{n,m} T)\Theta^n(Z; \mu^{-1}Q_j[D]), \\ 0, \end{cases}$$

where $|T = |_{\mathbf{j}}T$, depending on whether $A(Q_j, \mu)$ is not empty or empty. Since μ is coprime with the level q of Q, the condition $D \in \mathbf{A}_{ji}(\mu)/\mathbf{E}_i$ is equivalent to the condition $\mu D^{-1} \in \mathbf{E}_i \backslash \mathbf{A}_{ij}(\mu)$. Therefore, by replacing $D \mapsto \mu D^{-1}$, the last relations can be rewritten in the form

$$\begin{aligned} &\Theta^n(Z, Q_j)|T \\ &= \begin{cases} \sum\limits_{1 \leqslant i \leqslant h} \sum\limits_{D \in \mathbf{E}_i \backslash \mathbf{A}_{ij}(\mu)} I(\mu D^{-1},\, Q_j,\, \Psi_Q^{n,m} T)\Theta^n(Z; \mu Q_j[D^{-1}]), \\ 0, \end{cases} \\ &= \begin{cases} \sum\limits_{1 \leqslant i \leqslant h} \left(\sum\limits_{D \in \mathbf{E}_i \backslash \mathbf{A}_{ij}(\mu)} I(\mu D^{-1},\, Q_j,\, \Psi_Q^{n,m} T) \right) \Theta^n(Z; Q_i), \\ 0, \end{cases} \end{aligned} \tag{8.2}$$

depending on whether $A(Q_j, \mu)$ is not empty or empty.

For $n \geqslant 1$ and $i, j = 1, \cdots, h$, we set

$$\begin{aligned} &\tau_{ij}^n(T) \\ &= \begin{cases} \sum\limits_{D \in \mathbf{E}_i \backslash \mathbf{A}_{ij}(\mu)} I(\mu D^{-1}, Q_j, \Psi_Q^{n,m} T)(\mathbf{E}_i D) & \text{if } A(Q_j, \mu) \neq 0, \\ 0 & \text{if } A(Q_j, \mu) = 0 \end{cases} \end{aligned} \tag{8.3}$$

and define the action of operator $\circ\tau_{ij}^n(T)$ on theta-series $\Theta^n(Z,\,Q_i)$ by

$$\begin{aligned}\Theta^n(Z,\,Q_i)\circ\tau_{ij}^n(T)\\ &=\begin{cases}\sum\limits_{D\in\mathbf{E}_i\backslash\mathbf{A}_{ij}(\mu)} I(\mu D^{-1},Q_j,\Psi_Q^{n,m}T)\Theta^n(Z,\,\mu^{-1}Q_i[D])\\ \qquad\qquad \text{if}\ \ A(Q_j,\mu)\neq 0,\\ 0 \qquad\qquad \text{if}\ \ A(Q_j,\mu)=0\end{cases}\\ &=\begin{cases}\Big(\sum\limits_{D\in\mathbf{E}_i\backslash\mathbf{A}_{ij}(\mu)} I(\mu D^{-1},Q_j,\Psi_Q^{n,m}T)\Big)\Theta^n(Z,\,Q_j)\\ \qquad\qquad \text{if}\ \ A(Q_j,\mu)\neq 0,\\ 0 \qquad\qquad \text{if}\ \ A(Q_j,\mu)=0\end{cases}\end{aligned}\tag{8.4}$$

It follows from Lemma 3.2 that for each $\gamma\in\mathbf{E}_j$ the product $\tau_{ij}(T)\gamma$ can be written in the form

$$\begin{aligned}&\sum_{D\in\mathbf{E}_i\backslash\mathbf{A}_{ij}(\mu)} I(\mu D^{-1},\,Q_j,\,\Psi_Q^{n,m}T)(\mathbf{E}_iD\gamma)\\ &\quad=\sum_{D\in\mathbf{E}_i\backslash\mathbf{A}_{ij}(\mu)} I(\mu\gamma^{-1}D^{-1},\,Q_j,\,\Psi_Q^{n,m}T)(\mathbf{E}_iD)\\ &\qquad=\sum_{D\in\mathbf{E}_i\backslash\mathbf{A}_{ij}(\mu)} I(\mu D^{-1},\,Q_j[\gamma^{-1}],\,\Psi_Q^{n,m}T)(\mathbf{E}_iD)\end{aligned}$$

and so is equal to $\tau_{ij}^n(T)$. Thus, the linear combination $\tau_{ij}^n(T)$ belongs to the set $\mathcal{D}_{ij}$ defined above and so the matrix

$$\tau^n(T)=\left(\tau_{ij}^n(T)\right).$$

belongs to the Hecke–Shimura ring $\mathbf{D}$ of form (7.4). Extending the mapping $T\mapsto\tau^n(T)$ by linearity to arbitrary $T\in\mathcal{H}_0^n(q)$, we obtain a linear mapping of Hecke–Shimura rings

$$\mathcal{H}_0^n(q)\ni T\mapsto\tau^n(T)\in\mathbf{D}=D(\mathbf{E}_1,\ldots,\mathbf{E}_h;\,\mathbf{A}_{11},\mathbf{A}_{12},\ldots,\mathbf{A}_{hh}).\tag{8.5}$$

Theorem 8.1. *Let Q be an even positive definite matrix of even order m and let $<Q>$ be a system of representatives* (7.1)*. Then for each $n\geqslant m/2$ the component-wise action of Hecke operator $|T=|_{\mathbf{j}}T$ with $T\in\mathcal{H}_0^n(q)$ on* *the theta-vector with components* $\Theta^n(Z, Q_i)$ *can be written in the form*

$$\Big((\Theta^n|T)(Z, Q_1), \dots, (\Theta^n|T)(Z, Q_h)\Big)$$
$$= \left(\sum_{1\leqslant i\leqslant h} \Theta^n(Z, Q_i) \circ \tau_{i1}(T), \dots, \Theta^n(Z, Q_i) \circ \tau_{ih}(T)\right)$$
$$= (\Theta^n(Z, Q_1), \dots, \Theta^n(Z, Q_h)) \circ (\tau^n_{ij}(T)).$$

Proof. If $T \in \mathcal{H}^n_0(q)$ is an homogeneous elements with multiplier μ, then by (8.2) and (8.4) for $j = 1, \dots, h$ we obtain

$$\Theta^n(Z, Q_j)|T = \sum_{1\leqslant i\leqslant h} \Theta^n(Z, Q_i) \circ \tau^n_{ij}(T),$$

which proves the theorem for homogeneous T. The general case follows by linearity. ☐

When $n < m$, the inverse image $\Psi^{n,m}_Q \in \mathcal{H}^m_0(q)$ is not unique, that causes indeterminancy in definition of the mapping (8.5), however, in view of the theorem, it does not affect the action of operator $\circ\tau^n(T)$ on theta-series. We call the mapping $T \mapsto \tau^n(T)$ the *interaction mapping of the Hecke–Shimura rings.*

The following theorem is a direct consequence of [8, Theorem 2].

Theorem 8.2. *Let* Q *be an even positive definite matrix of even order* m. *Then for each* $n \geqslant m$ *the mapping* (8.5) *is a linear ring homomorphism of the Hecke–Shimura rings.*

Note that if $n < m$ the matrices $\tau^n(TT')$ and $\tau^n(T)\tau^n(T')$ for $T, T' \in \mathcal{H}^n_0(q)$ are not necessary equal, but equally operate on theta-vector with components $(\Theta^n|T)(Z, Q_i)$.

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