Записки научных
семинаров ПОМИ
Том 414, 2013 г.

I. D. Suprunenko

UNIPOTENT ELEMENTS OF NONPRIME ORDER IN REPRESENTATIONS OF THE CLASSICAL ALGEBRAIC GROUPS: TWO BIG JORDAN BLOCKS

Abstract. For irreducible rational representations of the classical algebraic groups in characteristic $p > 2$ that are not equivalent to a composition of a group morphism and the standard representation, it is proved that usually the image of a unipotent element of order $p^{s+1} > p$ has at least two Jordan blocks of size $> p^s$; all exceptions are indicated explicitly. As a corollary, irreducible rational representations of these groups whose images contain unipotent elements with just one Jordan block of size > 1 are classified.

**To Nikolai Vavilov
with admiration and best wishes
on the occasion of his 60th birthday**

§1. Introduction

For irreducible rational representations of the classical algebraic groups in characteristic $p > 2$ that are not equivalent to a composition of a group morphism and the standard representation, it is proved that usually the image of a unipotent element of order $p^{s+1} > p$ has at least two Jordan blocks of size $> p^s$; all exceptions are indicated explicitly. As a corollary, irreducible rational representations of these groups whose images contain unipotent elements with just one Jordan block of size > 1 are classified.

In what follows K is an algebraically closed field of characteristic $p > 0$, G is a simply connected simple algebraic group of a classical type over K, r is the rank of G, and ω_i, $1 \leqslant i \leqslant r$, are the fundamental weights of G. We assume that $p > 2$ for $G \neq A_r(K)$, $r > 1$ for $G = A_r(K)$ and $C_r(K)$, $r > 2$ for $G = B_r(K)$, and $r > 3$ for $G = D_r(K)$. For representations φ and ψ we write $\varphi \sim \psi$ if φ can be obtained from ψ with the help of a group morphism. Denote by $\varphi(\omega)$ the irreducible representation of G

Key words and phrases: classical groups, irreducible representations, images of unipotent elements, Jordan blocks.

with highest weight ω and by St the standard representation of G. Set $n = \dim \mathrm{St}$.

Theorem 1.1. *Let φ be a nontrivial irreducible representation of G. Assume that $x \in G$ is a unipotent element of order $p^{s+1} > p$. Then $\varphi(x)$ has at least two Jordan blocks of size greater than p^s unless one of the following holds:*

1) $\varphi \sim \mathrm{St}$;

2) $n = p^s + 1$, $G = A_r(K)$ *or* $C_r(K)$, x *is a regular unipotent element, and* $\varphi \sim \varphi(\omega_2)$ *or* $\varphi(2\omega_1)$ *with* $p > 2$ *in the latter case;*

3) $p = 2$, $G = A_3(K)$ *or* $p = 3$ *and* $G \in \{A_4(K), A_5(K), C_3(K)\}$, x *is a regular unipotent element, and* $\varphi \sim \varphi(\omega_2)$;

4) $p = 3$ *or* 5, $G = B_3(K)$, x *is a regular unipotent element, and* $\varphi \sim \varphi(\omega_3)$;

5) $p = 5$ *or* 7, $G = B_4(K)$ *or* $D_5(K)$, x *is a regular unipotent element, and* $\varphi \sim \varphi(\omega_4)$;

In Items $2) - 5)$ $\varphi(x)$ *has only one Jordan block of size greater than* p^s.

Obviously, if $\varphi \sim \mathrm{St}$, then for many elements x of order p^{s+1} their image $\varphi(x)$ has only one Jordan block of size greater than p^s, the number of conjugacy classes of such elements grows with the growth of r. Observe that for $G = D_4(K)$ we have $\varphi(\omega_3) \sim \mathrm{St}$ and $\varphi(\omega_4) \sim \mathrm{St}$.

This research is a part of a more general program of finding in some sense rare unipotent elements in Zariski closed or finite linear groups that can be applied to solving recognition problems based on properties of a single matrix. There are some reasons to suppose that unipotent elements with one Jordan block much bigger than the others form a family of "rare" elements. This is the case for elements of order p with this property and Zariski closed simple subgroups of $GL_m(K)$.

Let $x \in GL_m(K)$ be a unipotent element and $d_1 \geqslant d_2 \geqslant \ldots \geqslant d_t$ be the sizes of all its Jordan blocks (so $d_1 + d_2 + \ldots + d_t = m$). In what follows we assume that $d_2 = 0$ if x has a single Jordan block and denote the order of x by $|x|$. Set

$$\mathrm{Cl}_m = \{SL_m(K); \quad Sp_m(K) \text{ with even } m; \quad SO_m(K) \text{ with even } m \text{ or } p \neq 2\}.$$

Theorem 1.2 ([22, Corollary 1])**.** *Let $S \subset GL_m(K)$ be an irreducible Zariski closed connected simple subgroup of rank greater than 1, and*

$S \notin \mathrm{Cl}_m$. Assume that S is not of type G_2. Let $x \in S$ and $|x| = p$. Then $d_1 - d_2 \leqslant 12$. If S is a group of a classical type, then $d_1 - d_2 \leqslant 6$.

Initially this article was motivated by the author's desire to explore the possibilities of extending Theorem 1.2 to arbitrary unipotent elements and by the question of A. E. Zalesski stated below.

Question. Which modular irreducible representations of simple algebraic groups and finite Chevalley groups satisfy the following condition: the image of some unipotent element has just one Jordan block of size greater than 1?

Here this question is solved for the classical algebraic groups under the restrictions on the ground field characteristic stated at the beginning of the article. In Theorem 1.3 the group G is the same as in Theorem 1.1.

Theorem 1.3. *Let φ be an irreducible representation of G and $x \in G$ be a unipotent element. Assume that $\varphi(x)$ has just one Jordan block of size greater than 1. Then one of the following holds:*

1) $\varphi \sim \mathrm{St}$;

2) $p > 2$, $G = A_2(K)$, *x is a regular unipotent element, and* $\varphi \sim \varphi(2\omega_1)$;

3) $p > 2$, $G = A_3(K)$ *or* $C_2(K)$, *x is a regular unipotent element or has two Jordan blocks of size 2 in the standard realization of G, and* $\varphi \sim \varphi(\omega_2)$;

4) $G = B_3(K)$, *x is a regular unipotent element, and* $\varphi \sim \varphi(\omega_3)$.

The proof of Theorem 1.3 is based on Theorems 1.1 and 1.2 and some results of [20] on the difference $d_1 - d_2$ for certain representations of the classical groups (see details in Section 5). The representations in Item 3) of this theorem are also connected with the standard realizations of certain classical groups. Indeed, for $G = A_3(K)$ or $C_2(K)$ the representation $\varphi(\omega_2)$ can be regarded as the standard realization of $D_3(K)$ or $B_2(K)$, respectively. Unfortunately, elements with two blocks of size 2 in Item 3) of Theorem 1.3 were omitted in the announcement of these results in [26].

It is well known that the image of a nontrivial unipotent element in a tensor indecomposable irreducible representation of the group $A_1(K)$ has a single Jordan block. Corollary 2.9 in Section 2 implies that the image of such element in a tensor decomposable irreducible representation of this group has at least two nontrivial blocks if this representation is not a tensor product of two 2-dimensional representations.

Analogs of Theorems 1.1 and 1.3 for the groups of types C_r and D_r in characteristic 2 will be considered in a subsequent paper. In that case some additional arguments are needed due to a more complicated structure of unipotent conjugacy classes. The groups of type B_r in characteristic 2 can be omitted since they are isomorphic to the groups of type C_r (as abstract groups).

The representations in Item 2) of Theorem 1.1 show that one cannot expect an analogue of Theorem 1.2 for arbitrary unipotent elements with a bigger constant as a bound for $d_1 - d_2$. Indeed, in this case $\varphi(x)$ has one block of size $2p^s - 1$ or $2p^s + 1$ if $\varphi \sim \varphi(\omega_2)$ or $\varphi(2\omega_1)$, respectively, and other block sizes are at most p^s (see the details in Section 3). Now it is not clear whether other classes of representations with similar properties exist. However, if $p > 2$ for $G = C_r(K)$ and $p > 3$ for $G = B_r(K)$ or $D_r(K)$, it is proved that the image of a unipotent element in an irreducible representation of G has at least two blocks of size equal to its order and at least three such blocks for types B_r and D_r if the highest weight of this representation is large enough with respect to p. Furthermore, lower estimates for the number of Jordan blocks of the maximal dimension in the images of unipotent elements of fixed order in such representations are obtained (in terms of the group rank and the order of an element, see [23]). In [23] a p-restricted dominant weight is regarded as large with respect to p if its value on the maximal root of G is at least p, for arbitrary weights the definition of large weights is more involved.

The results of the paper can be easily transferred to representations of finite groups of Lie type in defining characteristic.

It is well known that regular unipotent elements of G have a single Jordan block in its standard realization if $G = A_r(K)$, $B_r(K)$, or $C_r(K)$, and two blocks for $G = D_r(K)$. In the latter case the block sizes are $n - 1$ and 1 for $p > 2$ and $n - 2$ and 2 for $p = 2$. Recently D. Testerman and A. E. Zalesski [29, Theorem 1.2] have shown that a connected reductive subgroup in a simple algebraic group that contains a regular unipotent element of the bigger group cannot lie in a proper parabolic subgroup of this group; they classified semisimple subgroups of these groups containing such elements [29, Theorem 1.4]. (Maximal positive-dimensional subgroups with this property were determined earlier by J. Saxl and G. Seitz [12].) In particular, such subgroups of $SL_m(K)$ are irreducible and hence the image of a reducible indecomposable representation of a semisimple algebraic group cannot contain a unipotent element with a single Jordan block. It would be interesting to find out whether the latter fact can be extended (may be, with some restrictions on p) to unipotent elements considered in Theorem 1.1.

This research has been supported by the Institute of Mathematics of the National Academy of Sciences of Belarus in the framework of the State Basic Research Programme "Mathematical models" (2006–2010) and the State Research Programme "Convergence" (2011–2015). The author thanks the referee for comments that helped to improve the exposition.

§2. Notation and preliminary facts

We keep all the notation fixed in the Introduction. In what follows $\mathbb{Z}$ and $\mathbb{Z}^+$ are the sets of integers and nonnegative integers.

For a semisimple algebraic group S over K denote by $L(S)$, $W(S)$, $\mathbf{X}(S)$, and $R(S)$ the Lie algebra, the Weyl group, the weight and root systems of S, respectively; $r(S)$ is the rank of S. $\mathbf{X}^+(S)$ and $R^+(S)$ are the sets of dominant weights and positive roots (with respect to some fixed maximal torus of S). If $\alpha \in R(S)$ and $t \in K$, then $\mathcal{X}_\alpha$, X_α, and $x_\alpha(t)$ are the root subgroup and the root elements in $L(S)$ and S associated with α and t, $X_{\alpha,d}$ is the element in the hyperalgebra of $L(S)$ associated with α and $d \in \mathbb{Z}^+$. Recall that $X_{\alpha,0} = 1$ and that $X_{\alpha,d} = X_\alpha^d/d!$ for $d < p$. For subgroups $S_1, \dots, S_t$ of S and vectors $v_1, \dots, v_t$ of some vector space we denote by $\langle S_1, \dots, S_t\rangle$ and $\langle v_1, \dots, v_t\rangle$ the subgroup generated by $S_1, \dots, S_t$ and the linear span of $v_1, \dots, v_t$. If $\beta_1, \dots, \beta_l \in R^+(S)$, the subgroup $\langle \mathcal{X}_{\beta_i}, \mathcal{X}_{-\beta_i} \mid 1 \leqslant i \leqslant l\rangle$ is denoted by $S(\beta_1, \dots, \beta_l)$. Put $U^+(S) = \langle \mathcal{X}_\alpha \mid \alpha \in R^+(S)\rangle$.

Only finite dimensional rational representations and modules are considered. In what follows $\operatorname{Irr} S$ is the set of irreducible representations of S considered up to the equivalence, $\varphi|\Gamma$ is the restriction of a representation φ to a subgroup Γ; φ^* is the representation dual to φ, $\dim\varphi$ ($\dim M$) is the dimension of a representation φ (a module M); $\mathbf{X}(\varphi)$ ($\mathbf{X}(M)$) is the set of weights of a representation φ (a module M); if $x \in S$ is unipotent and $\varphi \in \operatorname{Irr} S$, then $d_\varphi(x)$ is the degree of the minimal polynomial of $\varphi(x)$. If $\omega \in \mathbf{X}^+(S)$, then $\varphi(\omega)$ and $M(\omega)$ are the representation in $\operatorname{Irr} S$ and the irreducible S-module with highest weight ω. For an S-module M and a weight vector $m \in M$ the symbol $\omega(m)$ denotes the weight of m, $M_\mu \subset M$ is the weight subspace of weight μ. We use the symbols ω_i and α_j, $1 \leqslant i, j \leqslant r(S)$, to denote the fundamental weights and the simple roots of S, for simple groups the labeling is such as in [2]. The symbol St is used to denote the standard representation for all classical algebraic groups for which such representation is naturally determined. Denote by $\langle \omega, \alpha\rangle$ the value of a weight $\omega \in \mathbf{X}(S)$ at a root $\alpha \in R(S)$ (the canonical pairing in the sense of [17, §3]). By [1, Proposition 5.13], for $m \in M$, $\alpha \in R(S)$, and an element t of the ground field one has

$$x_\alpha(t)m = \sum_{d=0}^{\infty} t^d X_{\alpha,d} m \qquad (1)$$

with $X_{\alpha,d} M_\mu \subset M_{\mu+d\alpha}$.

If $\beta_1, \ldots, \beta_k \in R^+(S)$, $\Gamma = S(\beta_1, \ldots, \beta_k)$, and the roots in brackets form a basis of $R(\Gamma)$, we assume that $\langle \omega_i, \beta_j \rangle = \delta_{ij}$ (the Kronecker symbol) for the fundamental weight $\omega_i \in \mathbf{X}(\Gamma)$. We denote by $\omega_\Gamma(m)$ the weight of m regarded as an element of the Γ-module M.

If $S = S_1 S_2$ where S_i are the simple components of S, the weight system $\mathbf{X}(S)$ can be canonically identified with the set of all pairs (μ_1, μ_2) with $\mu_i \in \mathbf{X}(S_i)$ (here we fix a maximal torus T in S and for a weight $\mu \in \mathbf{X}(S)$ set μ_i equal to the restriction of μ to $T \cap S_i$). Throughout the text we often write $\mu = (\mu_1, \mu_2)$ in this case. It follows from [17, Corollary a) of Lemma 68] that each representation $\rho \in \operatorname{Irr} S$ can be realized as a tensor product $\rho_1 \otimes \rho_2$ where $\rho_i \in \operatorname{Irr} S_i$.

A representation $\varphi \in \operatorname{Irr} S$ is p-restricted if $\omega(\varphi) = \sum_{i=1}^{r(S)} a_i \omega_i$ and all coefficients $a_i < p$. Throughout the text $\operatorname{Irr}_p S \subset \operatorname{Irr} S$ is the set of p-restricted representations. Denote by Fr the Frobenius morphism of S associated with raising the elements of K to the pth power. In what follows $\operatorname{Fr}^j \varphi$ is the representation obtained from a representation $\varphi \in \operatorname{Irr} S$ with the help of the morphism Fr^j. It will always be clear from the context what group S is considered.

If S is a group of type A_1, we identify $\mathbf{X}(S)$ with $\mathbb{Z}$ mapping a weight $a\omega_1 \in \mathbf{X}(S)$ onto $a \in \mathbb{Z}$.

For $S = G$ we often omit the indication of a group in the notation above and write $\mathbf{X}$, R, W, Irr, etc. instead of $\mathbf{X}(G)$, $R(G)$, $W(G)$, $\operatorname{Irr} G$, etc. The weights and roots of G are considered with respect to a fixed maximal torus T. Set $X_{\pm i} = X_{\pm\alpha_i}$, $X_{\pm i,k} = X_{\pm\alpha_i,k}$, $\mathcal{X}_{\pm i} = \mathcal{X}_{\pm\alpha_i}$, and $G(i_1, \ldots, i_k) = G(\alpha_{i_1}, \ldots, \alpha_{i_k})$. For a vector m in a G-module M put $m(i_1 \cdot d_1, \ldots, i_t \cdot d_t) = X_{-i_1,d_1} \ldots X_{-i_t,d_t} m$. If m is a weight vector, set $\omega^i(m) = \langle \omega(m), \alpha_i \rangle$. If M is irreducible, then $v^+ \in M$ is a nonzero highest weight vector. Throughout the text V is the standard G-module where the representation St is realized; ε_i, $1 \leqslant i \leqslant r+1$ for $G = A_r(K)$ and $1 \leqslant i \leqslant r$ otherwise, are weights of V defined in [3, Chap. VIII, §13], their labeling is standard. Recall that $n = \dim V$. We often use a mixed notation $G(1,\dots,i-1,\varepsilon_i)$ for the subgroup $G(\alpha_1,\dots,\alpha_{i-1},\varepsilon_i)\subset B_r(K)$ and a similar notation for other subsystem subgroups of classical groups where the relevant subsystem has a base consisting both of simple and nonsimple roots.

If $u\in G$ is unipotent and $k_1\geqslant k_2\geqslant\dots\geqslant k_t$ are the sizes of all Jordan blocks of u on V, set $J(u)=(k_1,k_2,\dots,k_t)$. If $\varphi\in \mathrm{Irr}$, then $J(\varphi(u))$ denotes the analogous sequence for $(\varphi(u))$ considered as an element of $SL_d(K)$ with $d=\dim\varphi$. This notation will also be used for unipotent elements of other simple algebraic groups of classical types that appear in the paper. Observe that for unipotent $x\in G$ with $J(x)=(k_1,\dots,k_l)$ one has $|x|=p^{s+1}$ if and only if $p^s<k_1\leqslant p^{s+1}$. We call a Jordan block of an element in $GL_m(K)$ of order p^{s+1} an $\mathcal{NS}$-block if its size $>p^s$ (the abbreviation from "not small"). Unipotent blocks of size 1 will be called trivial. Denote by J_l a unipotent Jordan block of size l.

It is well known that the conjugacy classes in $SL_n(K)$ are determined by the canonical Jordan form. Recall the description of unipotent conjugacy classes in other classical simple algebraic groups.

Theorem 2.1. *a) For $G=B_r(K)$ or $D_r(K)$ a unipotent element $z\in SL_n(K)$ is conjugate to an element of G if and only if the multiplicities of all even integers in $J(z)$ are even; for $G=C_r(K)$ such element is conjugate to an element of G if and only if the multiplicities of all odd integers in $J(z)$ are even.*

b) If $G\neq D_r(K)$ or $J(z)$ contains at least one odd integer, then all unipotent elements $u\in G$ with $J(z)=J(u)$ are conjugate.

c) If $G=D_r(K)$ and $J(z)$ contains only even integers, then there are two unipotent conjugacy classes with such $J(z)$ in G, their elements are conjugate in $O_n(K)$. In the latter case z is conjugate to an element of $G(1,\dots,r-1)$ or $G(1,\dots,r-2,r)$.

Proof. All the assertions of the theorem, except the last one, are contained in [9, Corollary 3.6 and Lemma 3.11] and [15, Chap. IV, 2.15 and 2.27 ii]. Let $z\in D_r(K)$ and $J(z)$ consist of even integers. Analyzing the action of the subgroups $G(1,\dots,r-1)$ and $G(1,\dots,r-2,r)$ on V and taking into account that unipotent elements of G with the same Jordan block structure are conjugate in $O_n(K)$, one can conclude that z is conjugate in $O_n(K)$ to elements $z_1=x_1(1)x_{i_2}(1)\dots x_{i_k}(1)x_{r-1}(1)$ and $z_2=x_1(1)x_{i_2}(1)\dots x_{i_k}(1)x_r(1)$ with $1<i_2<\dots<i_k$, $i_2=2$ or 3, $i_{j+1}-i_j<3$, $i_k=r-3$ or $r-2$ (for $G=D_4(K)$ it may occur that $z_1=x_1(1)x_3(1)$ and $z_2=x_1(1)x_4(1)$). One easily observes that z_1 and z_2

are conjugate in $O_n(K)$ with the help of an element with the determinant -1 that interchanges the weight subspaces of weights ε_r and $-\varepsilon_r$ in V. By [15, Chap. IV, 2.27 ii], the centralizers of z_1 and z_2 in $O_n(K)$ belong to $SO_n(K)$. Hence these elements are not conjugate in G. As there are just two conjugacy classes in G containing elements with such canonical Jordan form, this yields that z is conjugate in G to z_1 or z_2. □

Lemma 2.2 below is actually well known. We formulate it for reader's convenience because it plays a key role in the proof of Theorem 1.1.

Lemma 2.2. *Let $x \in GL_m(K)$ and $|x| = p^{s+1} > p$. Set $y = x^{p^s}$. Assume that $J(x) = (k_1, \dots, k_t, \dots, k_l)$ with $k_j = a_j p^s + b_j > p^s$, $a_j, b_j \in \mathbf{Z}^+$, $a_j < p$, $0 < b_j \leqslant p^s$ for $1 \leqslant j \leqslant t$, and $k_j \leqslant p^s$ for $t < j \leqslant l$ (the case $t = l$ is not excluded). Put*

$$f_1 = \dots = f_{b_1} = a_1 + 1,\ f_{b_1+1} = \dots = f_{p^s} = a_1,$$

$$f_{p^s+1} = \dots = f_{p^s+b_2} = a_2 + 1,\ f_{p^s+b_2+1} = \dots = f_{2p^s} = a_2, \dots,$$

$$f_{(t-1)p^s+1} = \dots = f_{(t-1)p^s+b_t} = a_t + 1,\ f_{(t-1)p^s+b_t+1} = \dots = f_{tp^s} = a_t,$$

and $F = (f_1, f_2, \dots, f_{tp^s})$. Then for every integer $z > 1$ the multiplicities of z in the sequences $J(y)$ and F coincide. In particular, if y has more than p^s nontrivial Jordan blocks, then x has at least two $\mathcal{NS}$-blocks.

Proof. The assertion of the lemma follows immediately from Item b) of [24, Proposition 2.5]. □

Corollary 2.3. *Let $\varphi \in \mathrm{Irr}$ be tensor decomposable, $x \in G$, and $|x| = p^{s+1} > p$. Then $\varphi(x)$ has at least two $\mathcal{NS}$-blocks.*

Proof. Set $y = x^{p^s}$. We have $\varphi = \varphi_1 \otimes \varphi_2$ where $\varphi_i \in \mathrm{Irr}$ are nontrivial. Hence the elements $\varphi_i(x)$ have $\mathcal{NS}$-blocks. Then Lemma 2.2 implies that $\varphi_i(y)$ has at least p^s Jordan blocks and at least one nontrivial block for $i = 1, 2$. Obviously, $J_a \otimes J_b$ has a nontrivial block if $a > 1$. Hence $\varphi(y)$ has more than p^s nontrivial blocks (consider the tensor products of the maximal block of $\varphi_1(y)$ with all blocks of $\varphi_2(y)$ and that of another block of $\varphi_1(y)$ with the maximal block of $\varphi_2(y)$). Another application of Lemma 2.2 yields that $\varphi(x)$ has at least two $\mathcal{NS}$-blocks. □

Lemma 2.4. *Let $g, h \in G$ and h lie in the Zarisky closure of the conjugacy class containing g, l be a positive integer, and M be a G-module. Then*

$\dim(h-1)^l M \leqslant \dim(g-1)^l M$. *In particular, if z is a regular unipotent element of G and $u \in G$ is an arbitrary unipotent element, then*

$$\dim(u-1)^l M \leqslant \dim(z-1)^l M.$$

Proof. The proof of the first claim actually coincides with that of [19, Corollary 2.14] where it was proved for $l = p-1$. It is essential here that for a constant c the subset of all elements $f \in G$ for which

$$\dim(f-1)^l M \leqslant c$$

is Zariski closed. The second claim follows immediately from the first one since every unipotent element of G lies in the Zarisky closure of the conjugacy class consisting of the regular unipotent elements, see, for instance, [4, Proposition 5.1.2]. □

Lemma 2.5. *Let u be a unipotent linear transformation of a space M over K. Assume that u preserves a subspace $S \subset M$. For a $K\langle u\rangle$-module F and a positive integer l denote by $n_l(F)$ the number of Jordan blocks of size $\geqslant l$ for u acting on F. Then $n_l(S) \leqslant n_l(M)$ and $n_l(M/S) \leqslant n_l(M)$.*

Proof. Let F^u be the subspace of eigenvectors for u in F. Set $F_l = (u-1)^{l-1}F \cap F^u$ and $F'_l = (u-1)^{l-1}F/(u-1)^l F$ (we assume that $(u-1)^0 = 1$). Using a base of F where u has a canonical Jordan form and analyzing the action of u on F, one easily observes that $n_l(F) = \dim F_l = \dim F'_l$. Obviously, $S_l \subset M_l$. So the lemma holds for $n_l(S)$. We claim that $\dim(M/S)'_l \leqslant \dim M'_l$. Indeed, the composition of the natural homomorphisms from M onto M/S and from $(u-1)^{l-1}(M/S)$ onto $(M/S)'_l$ determines a surjection from $(u-1)^{l-1}M$ onto $(M/S)'_l$ with the kernel equal to $(u-1)^l M + S \cap (u-1)^{l-1}M$. This yields the claim and completes the proof. □

Corollary 2.6. *Let a, b, c, d, k be positive integers, $a \geqslant c$, and $b \geqslant d$. Then $J_a \otimes J_b$ has no less Jordan blocks of size $\geqslant k$ than $J_c \otimes J_d$.*

Proof. Set $u = J_a \otimes J_b$ and $u_1 = J_c \otimes J_d$. Regard u as a transformation of a K-module N of dimension ab and observe that the $K\langle u\rangle$-module N has a submodule where u acts as u_1. Now it remains to apply Lemma 2.5. □

In what follows we apply [5, Theorem 2.7] to describe the canonical Jordan form of the tensor product $J_a \otimes J_b$ with $a, b \leqslant p$ and [6, Lemma 6.14 and Theorem 6.4] to do this if a or $b > p$. We write $J_a \otimes J_b \cong J_{i_1} \oplus \ldots \oplus J_{i_t}$ if $J_{i_1}, \ldots, J_{i_t}$ constitute the complete collection of the Jordan blocks of the canonical Jordan form of the matrix $J_a \otimes J_b$ (multiplicities taken into account); in this case we also write kJ_a instead of a sum $J_a \oplus \ldots \oplus J_a$ (k times).

Theorem 2.7 ([5, Chap. VIII, Theorem 2.7]). *Let $1 \leqslant f \leqslant g \leqslant p$. Then*

$$J_f \otimes J_g \cong \bigoplus_{i=0}^{h-1} J_{g-f+2i+1} \oplus NJ_p,$$

where $h = \min\{f, p-g\}$, $N = 0$ if $f+g \leqslant p$, and $N = f+g-p$ if $f+g > p$.

Corollary 2.8. *In the assumptions of Theorem 2.7 if $g > 1$, then $J_g \otimes J_{g-1}$ has $g-1$ nontrivial blocks, $J_g \otimes J_g$ has $g-1$ nontrivial blocks for $g < p$ and g such blocks if $g = p$. If $2 \leqslant a \leqslant p$ and $3 \leqslant b \leqslant p$, then $J_a \otimes J_b$ has at least 2 nontrivial blocks. If $3 \leqslant a \leqslant p$ and $4 \leqslant b \leqslant p$, this tensor product has at least 3 nontrivial blocks.*

Proof. The first claim follows directly from Theorem 2.7. Then apply Corollary 2.6. □

Corollary 2.9. *Let S be a simple algebraic group over K, $u \in S$ be a nontrivial unipotent element, φ_1 and $\varphi_2 \in \operatorname{Irr} S$ be nontrivial representations, and $\varphi = \varphi_1 \otimes \varphi_2$. Then $\varphi(u)$ has at least two nontrivial Jordan blocks, except the case where $p > 2$, $S = A_1(K)$, and $\dim \varphi_1 = \dim \varphi_2 = 2$. In the exceptional case $J(\varphi(u)) = (3, 1)$.*

Proof. It is well known that $S = A_1(K)$ if $\dim \varphi_i = 2$ for $i = 1$ or 2. If $\dim \varphi_1 = \dim \varphi_2 = 2$, apply Theorem 2.7.

Obviously, $\varphi(u)$ has more than one nontrivial block if at least one of the transformations $\varphi_i(u)$ has more than one block. Corollaries 2.8 and 2.6 yield that $\varphi(u)$ has at least two nontrivial blocks if $d_{\varphi_i}(u) > 2$ for $i = 1$ or 2. This implies the corollary. □

Theorem 2.10 ([6, Lemma 6.14 and Theorem 6.4]). i) *Set $q = p^s$, $s \geqslant 1$. Assume that $0 < g$, $h \leqslant q$ and*

$$J_g \otimes J_h \cong \bigoplus_{i=1}^{l} J_{n_i} \oplus NJ_q$$

with all $n_i < q$. Then $l = \min\{g, h, q-g, q-h\}$.

Let $a = uq + g$ *and* $b = vq + h$ *with* $0 \leqslant u \leqslant v < p$. *For* $0 \leqslant j \leqslant u$ *set* $f_j = v - u + 2j$. *If* $a + b \leqslant pq$, *one has*

$$J_a \otimes J_b \cong \bigoplus_{i=1}^{l}\bigoplus_{j=0}^{u} J_{f_j q + n_i} \oplus \bigoplus_{i=1}^{l}\bigoplus_{j=0}^{u-1} J_{(f_j+2)q-n_i} \oplus \bigoplus_{j=0}^{u-1} |g-h| J_{(f_j+2)q}$$
$$\oplus \bigoplus_{j=0}^{u-1} |q-g-h| J_{(f_j+1)q} \oplus P,$$

where

$$P = \begin{cases} 0 & \text{if} \quad l = g, \\ (g-h)J_{(v-u)q} & \text{if} \quad l = h, \\ (g+h-q)J_{(u+v+1)q} & \text{if} \quad l = q-h, \\ (g-h)J_{(v-u)q} \oplus (g+h-q)J_{(u+v+1)q} & \text{if} \quad l = q-g. \end{cases}$$

ii) *Next, let* $a+b > pq$. *Set* $a_1 = pq-a$ *and* $b_1 = pq-b$. *Then* $a_1+b_1 < pq$ *and*

$$J_a \otimes J_b \cong J_{a_1} \otimes J_{b_1} \oplus (a+b-pq)J_{pq}.$$

Lemma 2.11. *Let* $p^s + 2 \leqslant a \leqslant p^{s+1}$ *and* $b \geqslant 2$. *Set* $u = J_a \otimes J_b$. *Then* u *has at least two* $\mathcal{NS}$*-blocks.*

Proof. First assume that $a = p^s+2$ and $b = 2$. By Theorem 2.7, $J_2 \otimes J_2 \cong J_3 \oplus J_1$ if $p > 2$ and $2J_2$ for $p = 2$. Now Theorem 2.10 a) implies that u has blocks of sizes $p^s + 3$ and $p^s + 1$ if $p > 2$ and two blocks of size $p^s + 2$ for $p = 2$. To complete the proof, apply Corollary 2.6. □

We need some notation to distinguish certain representations of the classical groups that have relatively small dimensions. Let $t > 1$ and Γ be a simple algebraic group of rank t over K. Assume that $t > 2$ for $\Gamma = B_t(K)$, $t > 3$ for $\Gamma = D_t(K)$, and $p > 2$ for $\Gamma \neq A_t(K)$. Set

$$\mathcal{S}(\Gamma) = \begin{cases} \{0, \omega_1, 2\omega_1, \omega_2, \omega_{t-1}, \omega_t, 2\omega_t, \omega_1 + \omega_t\} & \text{if } \Gamma = A_t(K) \text{ and } p > 2, \\ \{0, \omega_1, \omega_2, \omega_{t-1}, \omega_t, \omega_1 + \omega_t\} & \text{if } \Gamma = A_t(K) \text{ and } p = 2, \\ \{0, \omega_1, 2\omega_1, \omega_2\} & \text{otherwise.} \end{cases}$$

Denote by $n(\Gamma)$ the dimension of the standard Γ-module.

Theorem 2.12 ([10, Theorem 5.1])**.** 1) *Let* $t > 11$, $\psi \in \mathrm{Irr}_p(\Gamma)$, *and* $\omega(\psi) \notin \mathcal{S}(\Gamma)$. *Then* $\dim\psi > t^3/8$ *for* $\Gamma = A_t(K)$ *and* $\dim\psi > t^3$ *otherwise.*

2) *For all t satisfying our assumptions we have*

$$\dim\varphi(\omega_2)=\begin{cases} t(t+1)/2 & \text{for}\quad \Gamma=A_t(K),\\ 2t^2+t & \text{for}\quad \Gamma=B_t(K),\\ 2t^2-t-1 & \text{if}\quad \Gamma=C_t(K) \text{ and } p\nmid t,\\ 2t^2-t-2 & \text{if}\quad \Gamma=C_t(K) \text{ and } p\mid t,\\ 2t^2-t & \text{for}\quad \Gamma=D_t(K);\end{cases}$$

$$\dim\varphi(2\omega_1)=\begin{cases} (t+1)(t+2)/2 & \text{for}\quad \Gamma=A_t(K),\ p>2,\\ 2t^2+3t & \text{if}\quad \Gamma=B_t(K) \text{ and } p\nmid(2t+1),\\ 2t^2+3t-1 & \text{if}\quad \Gamma=B_t(K) \text{ and } p\mid(2t+1),\\ 2t^2+t & \text{for}\quad \Gamma=C_t(K),\\ 2t^2+t-1 & \text{if}\quad \Gamma=D_t(K) \text{ and } p\nmid t,\\ 2t^2+t-2 & \text{if}\quad \Gamma=D_t(K) \text{ and } p\mid t.\end{cases}$$

If $\Gamma=A_t(K)$, *then* $\dim\varphi(\omega_1+\omega_t)=t^2+2t$ *if* $p\nmid(t+1)$ *and* t^2+2t-1 *if* $p\mid(t+1)$.

Lemma 2.13. *Let* $\psi\in\mathrm{Irr}_p(\Gamma)$ *be nontrivial and* $\psi\not\sim\mathrm{St}$.

1) *Assume that* $p>2$, $p^s\neq 3$ *for* $G\neq A_t(K)$ *or* $C_t(K)$, $n(\Gamma)\geqslant\max\{4,2p^s-4\}$ *for* $\Gamma=C_t(K)$, *and* $n(\Gamma)\geqslant 2p^s-2$ *otherwise. Then* $\dim\psi>p^{s+1}$ *or one of the following holds:*

a) $p^s=3$, $\Gamma=A_3(K)$ *or* $C_2(K)$, $\psi=\varphi(\omega_2)$;
b) $p^s=5$, $\Gamma=B_4(K)$, $\psi=\varphi(\omega_4)$;
c) $p^s=5$, $\Gamma=C_3(K)$, $\omega(\psi)\in\{\omega_2,\omega_3,2\omega_1\}$;
d) $p^s=5$, $\Gamma=D_5(K)$, $\psi\sim\varphi(\omega_4)$;
e) $p^s=7$, $\Gamma=C_5(K)$, $\psi=\varphi(\omega_2)$;
f) $p^s=7$, $\Gamma=D_6(K)$, $\psi\sim\varphi(\omega_5)$.

2) *Suppose that* $\Gamma\neq D_t(K)$, $p^s>2$ *for* $\Gamma=A_t(K)$, $p^s>3$ *for* $\Gamma=C_t(K)$, $p^s\geqslant 7$ *for* $\Gamma=B_3(K)$ *or* $B_4(K)$, $p^s>7$ *for* $\Gamma=B_t(K)$ *with* $t>4$, $\max\{3,p^s-1\}\leqslant n(\Gamma)<2p^s-2$ *for* $\Gamma=A_t(K)$, $\max\{7,p^s-2\}\leqslant n(\Gamma)<2p^s-1$ *for* $\Gamma=B_t(K)$, *and* $p^s-1\leqslant n(\Gamma)<2p^s-4$ *for* $\Gamma=C_t(K)$. *Then* $\dim\psi>p^{s+1}$ *or one of the following holds:*

a) $\omega(\psi)\in\mathcal{S}(\Gamma)$;
b) $p^s=5$, $\Gamma=A_3(K)$ *or* $C_2(K)$, $\psi\sim\varphi(3\omega_1)$, $\varphi(\omega_1+\omega_2)$, *or* $\varphi(2\omega_2)$;
c) $p^s=5$ *or* 7, $\Gamma=A_5(K)$ *or* $C_3(K)$, $\psi=\varphi(\omega_3)$;
d) $p^s=7$, $\Gamma=A_6(K)$, $\psi\sim\varphi(\omega_3)$;
e) $p^s=7$, $\Gamma=B_3(K)$, $\psi=\varphi(2\omega_3)$ *or* $\varphi(\omega_1+\omega_3)$;
f) $p^s=7$, $\Gamma=C_4(K)$, $\psi=\varphi(\omega_3)$ *or* $\varphi(\omega_4)$;

g) $p^s = 11$, $\Gamma = A_9(K)$ *or* $C_5(K)$, $\psi \sim \varphi(\omega_3)$*;*
h) $p^s = 11$, $\Gamma = B_4(K)$, $\psi = \varphi(\omega_3)$*;*
i) $p^s = 13$, $\Gamma = B_5(K)$, $\psi = \varphi(\omega_3)$*;*
j) $p^s \in \{9, 11, 13, 17, 19\}$, $\Gamma = B_t(K)$ *with* $t = (p^s - 3)/2$, $\psi = \varphi(\omega_t)$*;*
k) $p^s \in \{7, 9, 11, 13, 17\}$, $\Gamma = B_l(K)$ *with* $l = (p^s - 1)/2$, $\psi = \varphi(\omega_l)$*;*
l) $p^s \in \{11, 13, 17, 19\}$, $\Gamma = B_m(K)$ *with* $m = (p^s + 1)/2$, $\psi = \varphi(\omega_m)$*.*

Proof. The proof is based on Lubeck's results on small-dimensional representations of the classical groups (Theorem 2.12 and the tables in [10, Sec. 6]). Observe that under the assumptions of Item 1 we have $r(\Gamma) \geqslant 2p^s - 3$ for $\Gamma = A_t(K)$, $r(\Gamma) \geqslant p^s - 1$ for $\Gamma = B_t(K)$ or $D_t(K)$, and $r(\Gamma) \geqslant p^s - 2$ for $\Gamma = C_t(K)$; for the groups in Item 2 we have $r(\Gamma) \geqslant p^s - 2$ if $\Gamma = A_t(K)$, $r(\Gamma) \geqslant (p^s - 3)/2$ if $\Gamma = B_t(K)$, and $r(\Gamma) \geqslant (p^s - 1)/2$ if $\Gamma = C_t(K)$. Hence in Item 1 we have $r(\Gamma) > 11$ if $p^s > 13$, or $\Gamma \neq C_t(K)$ and $p^s > 11$, or $\Gamma = A_t(K)$ and $p^s > 7$, and in Item 2 we have $r(\Gamma) > 11$ if $p^s > 25$, or $\Gamma \neq B_t(K)$ and $p^s > 23$, or $\Gamma = A_t(K)$ and $p^s > 13$. If l is an integer, one can immediately check that $(2l - 3)^3 > 8l^2$ for $l \geqslant 5$, $(l - 1)^3 > l^2$ for $l > 3$, $(l - 2)^3 > l^2$ for $l > 4$, $(l - 2)^3 > 8l^2$ for $l > 13$, $(l - 3)^3 > 8l^2$ for $l > 15$, and $(l - 1)^3 > 8l^2$ for $l > 10$. Now it is clear that in all cases, where p^s is large enough to guarantee that $r(\Gamma) > 11$, the lemma follows from Theorem 2.12. To complete the proof, we consider all smaller values of p^s separately and apply the tables from [10, Section 6] together with the theorem cited above. Notice that groups of types A_t and D_t have graph morphisms and groups of types B_t and C_t in an odd characteristic have no nontrivial morphisms whose composition with a nontrivial p-restricted representation yields a p-restricted representation. So in the list of exceptions for the former two types we write $\psi \sim \varphi(\lambda)$ for relevant weights λ and in such list for the latter two types we indicate ψ explicitly. □

Corollary 2.14. *Let* p^s *and* Γ *be such as in Item* 2 *of Lemma* 2.13*. Assume that* $\rho \in \operatorname{Irr}\Gamma$, $\rho \cong \operatorname{Fr}^j \psi \otimes \tau$, $\psi \in \operatorname{Irr}_p \Gamma$, $\psi \not\sim \operatorname{St}$, $\tau \in \operatorname{Irr}\Gamma$, *and* $\omega(\psi), \omega(\tau) \neq 0$*. Then* $\dim \rho > p^{s+1}$ *or* $p^s = 5$, $\Gamma = A_3(K)$ *or* $C_2(K)$, *and* $\rho \sim (\varphi(p^j\omega_1) \otimes \varphi(p^k\omega_2))$ *with* $j \neq k$*.*

Proof. It is clear that $\dim \rho \geqslant n(\Gamma) \dim \psi$. Lemma 2.13 implies that it suffices to consider the cases where $\omega(\psi) \in \mathcal{S}(\Gamma)$ or ψ is one of the representations from Items 2c)–2l) of that lemma. Let $\omega(\psi) \in \mathcal{S}(\Gamma)$. Applying Theorem 2.12, one easily observes that $\dim \rho \geqslant (p^s - 1)^2(p^s - 2)/2$ for $\Gamma = A_t(K)$, $\dim \rho \geqslant (p^s - 2)^2(p^s - 3)/2$ if $\Gamma = B_t(K)$, $\dim \rho \geqslant$

$(p^s-1)((p^s-1)(p^s-2)/2-2)$ for $\Gamma = C_t(K)$, $\dim\rho \geqslant 24$ if $p=2$ and $\Gamma = A_t(K)$, $\dim\rho \geqslant 50$ for $\Gamma = A_t(K)$ with $t \geqslant 4$, and $\dim\rho \geqslant 78$ for $\Gamma = C_t(K)$ with $t \geqslant 3$. This implies that $\dim\rho > p^{s+1}$ for our ψ unless $p^s = 5$ and $\Gamma = A_3(K)$ or $C_2(K)$. To complete the proof, use the tables of small-dimensional representations from [10, Section 6]. □

For reader's convenience we state explicit formulae for computing $d_\rho(x)$ for elements $x \in \Gamma$ of order p and representations $\rho \in \operatorname{Irr}_p\Gamma$. Let a unipotent element $x \in \Gamma$ belong to a conjugacy class C. Let $J(x) = (d_1, d_2, \ldots, d_l)$. Naturally, this sequence is the same for all $x \in C$. Set

$$N(C) = (d_1-1, d_1-3, \ldots, 1-d_1, d_2-1, \ldots, 1-d_2, \ldots, d_l-1, \ldots, 1-d_l).$$

Formulae for the minimal polynomials of elements of order p. Define the integers $b_i(C)$ with $1 \leqslant i \leqslant t$ as follows. If $\Gamma = A_t(K)$ or $C_t(K)$, or $\Gamma = B_t(K)$ and $i < t$, or $\Gamma = D_t(K)$ and $i < t-1$, let $b_i(C)$ be equal to the sum of the i biggest members of the sequence $N(C)$. Denote by Σ the sum of all positive members of $N(C)$ (taking into account their multiplicities). Set $b_t(C) = \Sigma/2$ for $\Gamma = B_t(K)$. If $\Gamma = D_t(K)$ and at least one of the integers d_j is odd, put $b_{t-1}(C) = b_t(C) = \Sigma/2$. If $\Gamma = D_t(K)$ and d_j are all even, there exist two distinct unipotent classes $C_1, C_2 \subset \Gamma$ with the same block sizes $d_1, \ldots, d_l$. Naturally, $N(C_1) = N(C_2)$. One can assume that the following holds for the labels δ_i^j on the labelled Dynkin diagrams of the classes C_j corresponding to the roots α_i ($1 \leqslant i \leqslant t$, $j = 1,2$): $\delta_i^1 = \delta_i^2$ for $1 \leqslant i \leqslant t-2$, $\delta_{t-1}^1 = \delta_t^2 = 0$, $\delta_{t-1}^2 = \delta_t^1 = 2$. Set $b_t(C_1) = \Sigma/2$, $b_t(C_2) = b_t(C_1) - 1$, and $b_{t-1}(C_i) = b_t(C_j)$ for $\{i,j\} = \{1,2\}$.

Theorem 2.15 ([18, Theorem 1.1, Propositions 1.3 and 2.6, and Algorithm 1.4]). *Let $u \in \Gamma$ be an element of order p lying in a conjugacy class C and $\rho \in \operatorname{Irr}_p\Gamma$ be the representation with highest weight $\sum_{i=1}^t a_i\omega_i$. Then $d_\rho(u) = \min\{p, 1 + \sum_{i=1}^t a_i b_i(C)\}$.*

Set

$$D(\rho, u) = 1 + \sum_{i=1}^t a_i b_i(C). \tag{2}$$

To prove the main results of the paper, we need to show the existence of composition factors with certain properties in restrictions of irreducible

G-modules to subsystem subgroups with two simple components and in restrictions of such modules for $D_r(K)$ to naturally embedded subgroups of types B_{r-1} and $B_k \times B_l$.

Theorem 2.16 (Jantzen [7], Smith [14]). *Let $H = G(i_1, \dots, i_j) \subset G$, and M be an irreducible G-module. Then KHv^+ is an irreducible H-module with highest weight $\omega_H(v^+)$ and a direct summand of the H-module M.*

Theorem 2.17 ([28, Theorem 1]). *Let H_1 and $H_2 \subset G$ be commuting subsystem subgroups, $\varphi \in \mathrm{Irr}$ be nontrivial, and $\varphi \not\sim \mathrm{St}$. If $G = B_r(K)$ or $D_r(K)$, assume that either $H_1H_2 \neq G(\varepsilon_i - \varepsilon_j, \varepsilon_i + \varepsilon_j)$ where $1 \leqslant i < j \leqslant r$, or $\varphi \not\sim \varphi(\omega_r)$. Then the restriction $\varphi|H_1H_2$ has a composition factor nontrivial for both H_1 and H_2.*

Vectors constructed in Lemmas 2.18 and 2.20 below will be helpful for finding required factors in restrictions of representations of G to subsystem subgroups.

Lemma 2.18 ([13, 1.5]). *Let M be a G-module and $v \in M$ be a vector of weight λ. Assume that $\langle \lambda, \alpha \rangle = m < p$ for a root α of G and that $\mathcal{X}_\alpha$ fixes v. Then $X_{-\alpha,k}v \neq 0$ for $0 \leqslant k \leqslant m$.*

Definition 2.19. *Let $\omega = a_1\omega_1 + \dots + a_r\omega_r \in \mathbf{X}^+$ and M be an irreducible G-module with highest weight ω. Put $y_k = -\langle \alpha_{k-1}, \alpha_k \rangle$ and $z_k = -\langle \alpha_{k+1}, \alpha_k \rangle$. Suppose that the roots α_t with t in the interval with the ends i and j ($1 \leqslant i, j \leqslant r$) form a chain on the Dynkin diagram of G. For an integer d with $0 < d \leqslant a_j$ define the vector $v(i,j,d)$ as follows. Put $d_j = d$. If $i > j$, put $d_k = a_k + d_{k-1}y_k$ for $i \geqslant k > j$. If $i < j$, set $d_k = a_k + d_{k+1}z_k$ for $i \leqslant k < j$. Now define*

$$v(i,j,d) = X_{-i,d_i} \dots X_{-k,d_k} \dots X_{-j,d}v^+.$$

Lemma 2.20 ([19, Lemma 2.9]). *Under the assumptions of Definition 2.19 let $a_j < p$. Then $v(i,j,d) \neq 0$ and $X_{l,b}v(i,j,d) = 0$ for $l \neq i$ and $b > 0$. Hence $\mathcal{X}_l$ fixes $v(i,j,d)$.*

Proposition 2.21. *Let $\varphi \in \mathrm{Irr}_p$ and $\omega(\varphi) \notin \mathcal{S}(G)$. Assume that one of the following holds:*

1) *$G = A_r(K)$, $H_1 = G(1, \dots, i)$, $H_2 = G(i+2, \dots, r)$, $2 \leqslant i \leqslant r-2$;*
2) *$G = B_r(K)$, $H_1 = G(1, \dots, i-1, \varepsilon_{i-1} + \varepsilon_i)$, $H_2 = G(i+1, \dots, r)$, $4 \leqslant i < r$;*
3) *$G = B_r(K)$, $H_1 = G(1, \dots, i-1, \varepsilon_i)$, $H_2 = G(i+1, \dots, r-1, \varepsilon_{r-1} + \varepsilon_r)$, $3 \leqslant i < r-1$;*

4) $G = C_r(K)$, $H_1 = G(1, \dots, i-1, 2\varepsilon_i)$, $H_2 = G(i+1, \dots, r)$, $2 \leqslant i < r$.

Suppose also that $\omega(\varphi) \neq \omega_j$ *and* $p \neq 2$, *if* $G = A_r(K)$ *and* $i = 2$, *and that* $\omega(\varphi) \neq \omega_r$ *for* $G = B_r(K)$ *and* $H_1 \cong D_4(K)$. *Set* $H = H_1 H_2$. *Then the restriction* $\varphi|H$ *has a composition factor* $\psi \cong \psi_1 \otimes \psi_2$ *where* $\psi_j \in \operatorname{Irr} H_j$, $j = 1, 2$, $\omega(\psi_j) \neq 0$, *and* $\psi_1 \cong \bigotimes_{t=1}^{k} \operatorname{Fr}^t \rho_t$ *with* $\rho_t \in \operatorname{Irr}_p H_1$ *and some nontrivial* $\rho_t \not\sim \operatorname{St}$.

Proof. Let M be a module affording φ. In the majority of cases we shall construct a nonzero vector m such that $U^+(H)$ fixes m and the irreducible H_j-modules with highest weights $\omega_{H_j}(m)$ have the required properties. It is clear that m generates an indecomposable H-module with highest weight $\omega_H(m)$ that has a composition factor isomorphic to $\varphi(\omega_{H_1}(m)) \otimes \varphi(\omega_{H_2}(m))$. If $m = v(f, l, d)$, we use Lemma 2.20 to conclude that $m \neq 0$ and $U^+(H)$ fixes m. Set $\omega = \omega(\varphi)$.

First assume that 1) holds. Put $\mu_1 = \sum\limits_{l=1}^{i} a_l \omega_l$ and $\mu_2 = \sum\limits_{l=i+2}^{r} a_l \omega_l$. Passing to the dual representation if necessary, one can assume that $a_l \neq 0$ for some $l \leqslant n/2$. Let $\omega = \omega_k$. Then $3 \leqslant k \leqslant n/2$ and $i > 2$. Set $a = \max\{2, k+i+2-n\}$ and $b = k - a$. Then $a < i$ and $0 < b < n-i-1$. Recall that M is isomorphic to $\bigwedge^k V$ (the kth wedge power of V), see, for instance, [8, Part II, Item 2.15]. Taking this into account, one easily concludes that $M|H$ has a direct summand isomorphic to $M(\omega_a) \times M(\omega_b)$ where the first multiplier is an H_1-module and the second one is an H_2-module. This yields a required factor.

Now assume that $\omega \neq \omega_k$. If $\mu_1 \notin \{0, \omega_1, \omega_i\}$ and $\mu_2 \neq 0$, set $m = v^+$. Now let $\mu_2 = 0$. Fix maximal l with $a_l \neq 0$. First suppose that $a_{l-1} \neq p-1$ if $l > 1$. Our assumptions yield that $a_l > 1$ if $\omega = a_l \omega_l$ and $a_1 > 2$, $p > 3$ if $\omega = a_1 \omega_1$. Put $m = X_{-(i+1),2} v^+$ if $l = i+1$ and $m = v(i+1, l, 1)$ otherwise. Next, let $l > 2$ and $a_{l-1} = p-1$. Set $m = v(i+1, l-2, a_{l-2})$ if $a_{l-2} \neq 0$ and $m = v(i+1, l-1, p-1)$ for $a_{l-2} = 0$. In all situations with $m \neq v^+$ considered in this paragraph we have $\omega_{H_1}(m) = \sum\limits_{t=1}^{i} b_t \omega_t$ and $1 < b_t < p$ for some t or at least two of the coefficients b_t are nonzero and less than p. Now let $\omega = (p-1)\omega_1 + a_2 \omega_2$ with $a_2 > 0$. Put $m = v(i+1, 1, p-1)$ if $a_2 > 1$ and $m = v(i+1, 1, 1)$ if $a_2 = 1$ and $p > 2$. In both these cases $1 < \omega^1(m) < p$. Finally, let $p = 2$ and $\omega = \omega_1 + \omega_2$. Then by our assumptions $i > 2$. Put $m_1 = v(i+1, 2, 1)$. We have $\omega^1(m_1) = 2$. Set $m = X_{-1} m_1$. By [19, Lemma 2.10], $m \neq 0$. One can directly check that $X_1 m = 0$ and hence $\mathcal{X}_1$ fixes m. It is clear that the groups $\mathcal{X}_t$ with $t > 1$ and $t \neq i+1$ also fix

m as they commute with X_{-1} and fix m_1 by Lemma 2.20. Hence $U^+(H)$ fixes m and m generates an indecomposable H-module with highest weight $\omega_H(m)$. Observe that $\omega^2(m) = 1$. Obviously, $\omega_{H_2}(m) \neq 0$ in all situations considered above.

Next, let $\mu_2 \neq 0$, but $\mu_1 \in \{0, \omega_1, \omega_i\}$. Suppose that $\omega \neq \omega_i + \omega_r$. Fix minimal $f > i$ with $a_f \neq 0$. If $\mu_1 = \omega_1$ or ω_i, put $m_1 = v(i+1, f, 1)$. Let $\mu_1 = 0$. Since $a_l \neq 0$ for some $l \leqslant n/2$ and $\omega \neq \omega_k$ due to our assumptions, we conclude that $f < r$ and $a_f > 1$ or $a_s \neq 0$ for some $s > f$. If $a_f > 1$, set $m_1 = v(i+1, f, 2)$. If $a_f = 1$, fix minimal $s > f$ with $a_s \neq 0$ and put $m_1 = v(i+1, s, 1)$. Now set $m = m_1$ if $\mu_1 = \omega_1$ or $p \neq 2$ and $m = X_{-i}m_1$ if $\mu_1 = \omega_i$ and $p = 2$. Then $\omega_{H_1}(m) = \omega_1 + \omega_i$ or $2\omega_i$ in the first case and ω_{i-1} in the second one. In the second case we apply [19, Lemmas 2.9 and 2.10] and argue as at the end of the previous paragraph replacing X_{-1} by X_{-i} to show that $m \neq 0$ and is fixed by the groups $\mathcal{X}_t$ with $t \neq i+1$. Recall that $i > 2$ if $p = 2$. As $\omega \notin \{\omega_1 + \omega_r, \omega_i + \omega_r\}$, one easily checks that $\omega_{H_2}(m) \neq 0$.

Finally, let $\omega = \omega_i + \omega_r$. Set $u = v(i+2, r, 1)$, $w = X_{-i}u$, and $m = X_{-(i+1),2}w$. We claim that $m \neq 0$ and is fixed by the groups $\mathcal{X}_t$ with $t \neq i+1$. By Lemma 2.20, $u \neq 0$ and $\mathcal{X}_t$ fixes u for $t \neq i+2$. Since $\omega^i(u) = 1 < p$, Lemma 2.18 implies that $w \neq 0$. Hence w generates an indecomposable $G(i+1)$-module with highest weight 2. Now it follows from well-known facts on representations of the group $A_1(K)$ that $m \neq 0$. Put $\lambda_1 = \omega(m) + \alpha_i$ and $\lambda_2 = \omega(m) + \alpha_{i+2}$. Then $\langle \lambda_1, \alpha_{i+1} \rangle = \langle \lambda_2, \alpha_{i+1} \rangle = -3$ and one easily concludes that $\lambda_1, \lambda_2 \notin \mathbf{X}(M)$. Hence $X_i m = X_{i+2} m = 0$ and the groups $\mathcal{X}_i$ and $\mathcal{X}_{i+2}$ fix m. The groups $\mathcal{X}_t$ with $t \notin \{i, i+1, i+2\}$ also fix m since they fix u and commute with X_{-i} and $X_{-(i+1),2}$. Now our claim is proved. One easily observes that $\omega_{H_1}(m) = \omega_{i-1} + \omega_i$ and $\omega_{H_2}(m) = \omega_1$.

For $G = A_r(K)$ all the possibilities have been considered. The information on $\omega_H(m)$ given above implies that in all cases the H-module generated by m has a required factor.

Now assume that $G \neq A_r(K)$. If $\sum_{j=i+1}^{r} a_j \neq 0$, put $m = v^+$. Let $\sum_{j=i+1}^{r} a_j = 0$. Fix maximal l with $a_l \neq 0$. Since $\omega \notin \mathcal{S}(G)$, we have $a_1 > 2$ if $l = 1$ and $a_1 + a_2 > 1$ for $l = 2$. If $l > 2$ or $a_1 < p-1$, set $m = v(i, l, 1)$. For $\omega = (p-1)\omega_1 + a_2\omega_2$ put $m = v(i, 1, p-1)$ if $a_2 > 1$ and $m = v(i, 1, 1)$

if $a_2 = 1$. Recall that $p > 2$. It is not difficult to observe that KHm has a required factor. This completes the proof. □

Proposition 2.22. *Let $\varphi \in \mathrm{Irr}_p$ be nontrivial and M be a G-module affording φ. Assume that one of the following holds:*

1) *$H = G(2, \dots, r)$ and $\varphi \not\sim \mathrm{St}$;*

2) *$G = A_r(K)$, $H = G(1, \dots, i-1, i+1, \dots, r)$ with $1 < i < r$, $\varphi \not\sim \mathrm{St}$, and $\omega(\varphi) \neq \omega_2$ for $r = 3$;*

3) *$G = B_r(K)$ or $C_r(K)$, $H = G(1, \dots, r-1)$, and $\omega(\varphi) \neq \omega_2$ for $G = C_2(K)$;*

4) *$G = D_r(K)$, $H = G(1, \dots, r-1)$ or $G(1, \dots, r-2, r)$, and $\omega(\varphi) \neq \omega_{r-1}$ or ω_r for $r = 4$.*

Then $M|H$ has a direct summand of the form $N_1 \oplus N_2$ where N_i are H-modules and each of them has at least one nontrivial composition factor.

Proof. There exists a unique index l with $\mathcal{X}_l \not\subset H$. Let $\omega = \omega(\varphi) = \sum_{i=1}^{r} a_i\omega_i$. For a nonnegative integer k set

$$\mathbf{X}_k = \{\mu \in \mathbf{X}(M) \mid \mu = \omega - k\alpha_l - \sum_{i \neq l} b_i\alpha_i\}$$

and $M_k = \bigoplus_{\mu \in \mathbf{X}_k} M_\mu$. Naturally, $\mathbf{X}_k = \varnothing$ and $M_k = 0$ for large enough k. It is clear that M_k are H-modules and $M = \bigoplus_k M_k$. If $\omega \neq a_l\omega_l$, put $N_1 = M_0$. By Theorem 2.16, N_1 is a nontrivial irreducible H-module. Assume that $\omega \neq \omega_2$ for $G = C_2(K)$ and $\omega \neq \omega_1$ for $G = B_r(K)$ (due to the assumptions of the proposition the latter possibility can appear only in Case 3). Then we shall construct a nonzero vector $m \in M_1$ such that $U^+(H)$ fixes m and $\omega_H(m) \neq 0$. Hence the H-module generated by m has a nontrivial composition factor. Set $N_2 = M_1$. In the exceptional cases where $G = C_2(K)$ and $\omega = \omega_2$ or 3) holds, $G = B_r(K)$, and $\omega = \omega_1$, we find a vector m with similar properties in M_2 and put $N_2 = M_2$.

If $\omega = a_l\omega_l$, we set $m_1 = X_{-l}v^+$ and construct a nonzero vector $m_2 \in M_2$ such that $U^+(H)$ fixes m_2 and $\omega_H(m_2) \neq 0$. Arguing as above, we deduce that both M_1 and M_2 have nontrivial composition factors in this situation. Set $N_j = M_j$ for $j = 1, 2$. Thus the construction of required vectors is divided into many subcases. In the majority of them Lemmas 2.18 and 2.20 directly imply that m, m_1 or $m_2 \neq 0$ and is fixed by $U^+(H)$. We add necessary comments.

Assume that $\omega \neq a_l\omega_l$. If $a_l \neq 0$, set $m = X_{-l}v^+$. Now let $a_l = 0$. First consider Case 1) and suppose that $\omega \neq \omega_2$ if $G = C_2(K)$. Fix minimal t with $a_t \neq 0$. If $G \notin \{C_r(K), D_r(K)\}$ or $t < r$, put $m = v(1,t,1)$. If $t = r$, set $m = X_{-1}\dots X_{-(r-1)}X_{-r}v^+$ for $G = C_r(K)$ and $m = X_{-1}\dots X_{-(r-2)}X_{-r}v^+$ if $G = D_r(K)$. If $G = C_r(K)$, we apply Lemma 2.18 several times to show that $m \neq 0$. It is clear that in this situation $\omega(m) + \alpha_j \notin \mathbf{X}(M)$ for $j > 1$. Hence $X_j m = 0$ and the groups $\mathcal{X}_j$ fix m for such j. If $t < r$, we have $\omega^{t+1}m \neq 0$. Let $t = r$. Then our assumptions yield that $a_r > 1$ if $G = A_r(K)$ or $C_2(K)$. So $\omega^r(m) \neq 0$ for $G = A_r(K)$, $B_r(K)$ or $C_2(K)$. One easily concludes that $\omega^{r-1}(m) \neq 0$ for $G = C_r(K)$ with $r > 2$ or $D_r(K)$.

If $G = C_2(K)$, $\omega = \omega_2$, and $H = G(2)$, set $m = v(1,2,1)$. Then $\omega^2(m) \neq 0$.

In Case 2), fix s such that $a_s \neq 0$ and $a_j = 0$ if $|j-i| < |s-i|$ (if there are two indices with this property, choose an arbitrary one). Put $m = v(i,s,1)$. Observe that $\omega^{i-1}(m) \neq 0$ if $s > i$ and $\omega^{i+1}(m) \neq 0$ for $s < i$.

Let 3) hold with $\omega \neq \omega_1$ for $G = B_r(K)$. Fix maximal f with $a_f \neq 0$, then for $G = C_r(K)$ set $m = v(r,f,1)$ and for $G = B_r(K)$ put $v = v(r-1,f,1)$ and $m = X_{-r}v$. Since $p > 2$, the group $\mathcal{X}_r$ fixes v and $\omega^r(v) = 2$, Lemma 2.18 forces that $m \neq 0$ for $G = B_r(K)$. In this situation, the groups $\mathcal{X}_j$ with $j < r-1$ fix m as they fix v and commute with X_{-r}. One easily observes that $\omega(m) + \alpha_{r-1} \notin \mathbf{X}(M)$. Hence $X_{r-1}(m) = 0$ and $\mathcal{X}_{r-1}$ also fixes m. It is clear that $\omega^{r-1}(m) \neq 0$ for $G = C_r(K)$, $\omega^{f-1}(m) \neq 0$ for $G = B_r(K)$ and $f > 1$, and $\omega^1(m) \neq 0$ if $G = B_r(K)$ and $\omega = a_1\omega_1$ (in the latter case $a_1 > 1$).

If $G = B_r(K)$, $\omega = \omega_1$, and $H = G(1,\dots,r-1)$, set $m = v(r,1,1)$ and observe that $\omega^{r-1}(m) \neq 0$.

In Case 4), first assume that $a_j \neq 0$ for some $j < r-1$. Fix maximal such j and set $m = v(r-1,j,1)$ for $l = r-1$ and $m = X_{-r}v(r-2,j,1)$ for $l = r$. Then $\omega^r(m) \neq 0$ for $l = r-1$ and $\omega^{r-1}(m) \neq 0$ for $l = r$. Now let $\omega = a_u\omega_u$ with $\{l,u\} = \{r-1,r\}$. Then put $m = X_{-l}X_{-(r-2)}X_{-u}v^+$ and conclude that $\omega^{r-3}(m) \neq 0$. For $\omega \neq a_l\omega_l$ all possibilities have been considered.

Next, suppose that $\omega = a_l\omega_l$. Recall that $a_l > 1$ if $l = 1$. Put $m_1 = X_{-l}v^+$. If $a_l > 1$, set $m_2 = X_{-l,2}v^+$. Let $a_l = 1$. Then Case 1) cannot occur.

In Case 2), set $m_2 = X_{-i}X_{-(i-1)}X_{-(i+1)}X_{-i}v^+$. One can check that $X_iX_{i+1}m_2 = v(i-1,i) \neq 0$ and $\omega(m_2) + \alpha_{i\pm1} \notin \mathbf{X}(M)$. Hence $m_2 \neq 0$,

$X_{i\pm 1}m_2 = 0$, and $\mathcal{X}_{i\pm 1}$ fixes m_2. Obviously, $\mathcal{X}_j$ fixes m_2 for $j < i-1$ or $j > i+1$. Observe that $i > 2$ or $i < r-1$ as $r > 3$. Hence $\omega_H(m_2) \neq 0$.

In Case 3), put $m_2 = X_{-r}X_{-(r-1)}X_{-r}v^+$ for $G = B_r(K)$ and $m_2 = X_{-r}v(r-1, r)$ for $G = C_r(K)$. One can directly check that $X_{r-1}X_r m_2 \neq 0$ in the first case and $X_r m_2 \neq 0$ in the second one. So $m_2 \neq 0$. In both cases $\omega(m_2) + \alpha_{r-1} \notin \mathbf{X}(M)$, so $X_{r-1}m_2 = 0$ and $\mathcal{X}_{r-1}$ fixes m_2. It is clear that the groups $\mathcal{X}_j$ with $j < r-1$ fix m_2. Recall that $r > 2$ and observe that $\omega^{r-2}(m_2) \neq 0$.

In Case 4), define u as for $\omega \neq \omega_l$ and set

$$m_2 = X_{-l}X_{-(r-2)}X_{-(r-3)}X_{-u}X_{-(r-2)}X_{-l}v^+.$$

Using Lemma 2.20, one can check that $X_u X_{r-2} X_l m_2 \neq 0$. Therefore $m_2 \neq 0$. We claim that $X_k m_2 = 0$ for $k \in \{r-3, r-2, u\}$. Indeed, put

$$w = X_{-(r-2)}X_{-(r-3)}X_{-u}X_{-(r-2)}X_{-l}v^+.$$

Since $\omega - \alpha_l - \alpha_u - 2\alpha_{r-2}$ and $\omega - \alpha_l - 2\alpha_{r-2} - \alpha_{r-3} \notin \mathbf{X}(M)$, we have $X_f w = 0$ for $f = r-3$ or u. So $X_f m_2 = X_{-l}X_f w = 0$. One easily deduces that $\omega(m_2) + \alpha_{r-2} \notin \mathbf{X}(M)$, so $X_{r-2}m_2 = 0$ as well. Hence it is clear that $U^+(H)$ fixes m_2. Recall that $r > 4$ due to our assumptions. Then $\omega^{r-4}(m_2) \neq 0$. Now the required vectors have been constructed in all cases. The proposition is proved. □

Corollary 2.23. *Under the assumptions of Proposition* 2.22 *let $u \in H$ be a nontrivial unipotent element. Then $\varphi(u)$ has at least two $\mathcal{NS}$-blocks.*

Proof. It is clear that $\psi(u)$ has an $\mathcal{NS}$-block for each nontrivial $\psi \in \operatorname{Irr} H$ since $|\psi(u)| = |u|$. So the assertion of the corollary follows immediately from Proposition 2.22. □

In the proofs below M is a module affording a representation φ that is under consideration at a relevant moment.

The following well-known fact is stated explicitly for reader's convenience. In Lemma 2.24 below we assume that the weights ω_{i-1} and ω_i of the group H_1 are the fundamental weights associated with the roots α_{i-1} and $\varepsilon_{i-1} + \varepsilon_i$ and the weight ω_{r-i} of H_2 is the fundamental weight associated with the root α_r (strictly speaking, with the restrictions of the relevant roots to H_1 and H_2, respectively).

Lemma 2.24. *Let $G = B_r(K)$, $H_1 = G(1, \dots, i-1, \varepsilon_{i-1} + \varepsilon_i)$, $H_2 = G(i+1, \dots, r)$, and $2 \leqslant i < r$. Set $H = H_1H_2$ and $\varphi = \varphi(\omega_r)$.*

Then $\varphi|H \cong \varphi(\omega_{i-1})\otimes\varphi(\omega_{r-i})\oplus\varphi(\omega_i)\otimes\varphi(\omega_{r-i})$ (here the first multiplier is a representation of H_1 and the second one is a representation of H_2).

Proof. It is well known that $\mathbf{X}(M) = \{(\pm\varepsilon_1 \pm \ldots \pm \varepsilon_r)/2\}$ (all combinations of the signs "+" and "−" occur). Let $\mathbf{X}_1$ and $\mathbf{X}_2 \subset \mathbf{X}(M)$ be the subsets consisting of all weights with odd or even number of negative coefficients for ε_l with $l \leqslant i$, respectively. Set $M_j = \langle M_\mu \mid \mu \in \mathbf{X}_j\rangle$, $j = 1, 2$. One easily observes that M_j are H-modules. The dimension and the weight structure of M_j imply that they afford the direct summands in the assertion of the lemma. □

We need some details on natural subgroups of types B_{r-1} and $B_l \times B_m$ in $D_r(K)$. In Lemma 2.25 below $\varepsilon_{j,B}$ are the weights ε_j of certain groups of type B_j with the standard labeling. It will be clear from the context what group is considered. In fact the lemma is well known, but we fail to find an explicit reference.

Lemma 2.25. *Let $G = D_r(K)$.*

1) *There exists a subgroup $H_B \subset G$ such that $H_B \cong B_{r-1}(K)$, $T\cap H_B$ is a maximal torus in H_B, $U^+\cap H_B$ is a maximal unipotent subgroup in H_B, and the restriction of weights from T to $T\cap H_B$ induces the homomorphism $\rho : \mathbf{X} \to \mathbf{X}(H_B)$ with $\rho(\varepsilon_i) = \varepsilon_{i,B}$ for $i < r$ and $\rho(\varepsilon_r) = 0$.*

2) *Let l and m be positive integers and $l+m = r-1$. Set $f = \min\{l, m\}$. There exists a subgroup $H_{l,m} \subset G$ such that $H_{l,m} = H_1H_2$, H_1 and H_2 commute, $H_1 \cong B_l(K)$, $H_2 \cong B_m(K)$, $T \cap H_{l,m}$ is a maximal torus in $H_{l,m}$, $U^+ \cap H_{l,m}$ is a maximal unipotent subgroup in $H_{l,m}$, and the restriction of weights from T to $T \cap H_{l,m}$ induces the homomorphism $\tau : \mathbf{X} \to \mathbf{X}(H_{l,m})$ with the following properties:*

$\tau(\varepsilon_{2i-1}) = (\varepsilon_{i,B}, 0)$ and $\tau(\varepsilon_{2i}) = (0, \varepsilon_{i,B})$ for $i \leqslant f$,
$\tau(\varepsilon_r) = 0$,
if $2f < k \leqslant r-1$, then $\tau(\varepsilon_k) = (\varepsilon_{k-f,B}, 0)$ if $l > m$ and $(0, \varepsilon_{k-f,B})$ for $l < m$.

Denote by V_0 the subspace in V generated by the vectors of weights ε_r and $-\varepsilon_r$. The group $H_{l,m}$ preserves subspaces V_1 and $V_2 \subset V$ such that V_1 is generated by the weight vectors of weights ε_{2i-1} with $1 \leqslant i \leqslant f$ and ε_k with $2f < k \leqslant r-1$ (if such k exist) and a vector from V_0, V_2 is generated by the weight vectors of weights ε_{2i} with $1 \leqslant i \leqslant f$ and a vector from V_0, $V = V_1 \oplus V_2$.

Proof. This lemma is actually a part of [24, Lemma 2.23]. The facts on maximal unipotent subgroups are not formulated explicitly in the citation above, but follow immediately from the construction described there. □

The notation H_B and $H_{l,m}$ is used throughout the text.

Corollary 2.26. *Under the assumptions of Lemma* 2.25*, we have*

$$U^+ \cap H_{l,m} \subset \langle \mathcal{X}_\alpha \mid \alpha \in R^+, \alpha \neq \alpha_1 \rangle.$$

Proof. Set $U_H = U^+ \cap H_{l,m}$ and $U_1 = \langle \mathcal{X}_\alpha \mid \alpha \in R^+, \alpha \neq \alpha_1 \rangle$. It follows from [17, Lemma 17] that each element $u \in U^+$ can be written in the form $u = x_1(t)u_1$ where $u_1 \in U_1$. If $u \in U_H$, then u preserves the subspace V_2 described in Lemma 2.25. Let $v \in V_2$ be a nonzero vector of weight ε_2. Then u_1 fixes v. One easily observes that $u(v) \notin V_2$ if $t \neq 0$. This yields the corollary. □

In Corollaries 2.27 and 2.28 and Lemmas 2.32 and 2.33 below $\omega = \omega(\varphi)$.

Corollary 2.27. *Let* $G = D_r(K)$ *and* $\varphi \in \mathrm{Irr}$.

1) *If* $\omega = \sum_{i=1}^{r} a_i\omega_i$*, the restriction* $\varphi|H_B$ *has a composition factor with highest weight* $\left(\sum_{i=1}^{r-2} a_i\omega_i\right) + (a_{r-1} + a_r)\omega_{r-1}$.

2) *If* $\varphi \in \mathrm{Irr}_p$ *and* $\omega \neq 0$ *or* ω_1*, the restriction* $\varphi|H_{l,m}$ *has a composition factor* $\varphi_1 \otimes \varphi_2$ *where* $\varphi_j \in \mathrm{Irr}\, H_j$ *and* $\omega(\varphi_j) \neq 0$ *for* $j = 1, 2$ *(here* H_1 *and* H_2 *are the subgroups mentioned in Lemma* 2.25*).*

Proof. We use the notation of Lemma 2.25. Set $H = H_{l,m}$. The formulae for ρ and τ in Lemma 2.25 imply that the H_B-module generated by v^+ has a required composition factor and an analogous H-module has such factor if $\omega \neq a_1\omega_1$. Now let $\omega = a_1\omega_1$ with $1 < a_1 < p$. By Lemma 2.18, $m = X_{-1}v^+ \neq 0$. Obviously, $\mathcal{X}_\beta$ fixes m for every positive root $\beta \neq \alpha_1$. Now Corollary 2.26 implies that the group $U^+ \cap H$ fixes m. So m generates an indecomposable H-module with highest weight $\omega_H(m)$. Since $\omega(m) = (a_1 - 2)\omega_1 + \omega_2$, the weight $\omega_H(m) = ((a_1 - 1)\omega_1, \omega_1)$ if both l and $m > 1$. If $l = 1$, the first component is replaced by $2(a_1 - 1)\omega_1$, and if $m = 1$, the second one is replaced by $2\omega_1$. As $a_1 > 1$, in all cases the both components are nonzero. This completes the proof. □

Corollary 2.28. *Let* $G = D_r(K)$, $\varphi \in \mathrm{Irr}_p$, $\omega(\varphi) = a_{r-1}\omega_{r-1} + a_r\omega_r$*, and* $a_{r-1}a_r \neq 0$*. Then* $\varphi|H_B$ *has a composition factor with highest weight* $\omega_{r-2} + (a_{r-1} + a_r - 2)\omega_{r-1}$.

Proof. By Lemma 2.18, $m_1 = X_{-(r-1)}v^+ \neq 0$ and $m_2 = X_{-r}v^+ \neq 0$. Set $M_1 = \langle m_1, m_2\rangle$. Let ρ be such as in Lemma 2.25. For $1 \leqslant i \leqslant r-1$ put $\beta_i = \rho(\alpha_i)$ and observe that $\beta_1, \dots, \beta_{r-1}$ form a basis in $R(H_B)$. It is clear that $\rho(\omega)$ is the maximal weight of the H_B-module M and M_1 is the weight subspace of weight $\rho(\omega) - \beta_{r-1}$ in this module. Hence $X_{\beta_{r-1}}M_1 \subset \langle v^+\rangle$. So there exists a nonzero vector $m \in M_1$ with $X_{\beta_{r-1}}m = 0$. Hence the group $\mathcal{X}_{\beta_{r-1}}$ fixes m. This yields that $U^+(H_B)$ fixes m since, obviously, $\mathcal{X}_\gamma$ fixes m for $m \in R^+(H_B) \setminus \{\beta_{r-1}\}$. So m generates an indecomposable H_B-module with highest weight $\rho(\omega) - \beta_{r-1} = \omega_{r-2} + (a_{r-1} + a_r - 2)\omega_{r-1}$ and $M|H_B$ has the desired factor. □

Throughout the paper we use the traditional definition of a transvection: a transvection is a unipotent element in $GL_m(K)$ with a single Jordan block of size 2 and other blocks of size 1.

Lemma 2.29. *Let $\Gamma = B_t(K)$ with $t > 2$, $C_t(K)$, or $D_t(K)$ with $t > 3$ and $\psi \in \operatorname{Irr}\Gamma$. Then $\psi(\Gamma)$ contains no transvections for $\Gamma = B_t(K)$ or $D_t(K)$. If $\Gamma = C_t(K)$ and $\psi(\Gamma)$ contains transvections, then $\psi \sim \mathrm{St}$.*

Proof. Let $q = p^l$ be such that all the coefficients in the expansion of $\omega(\psi)$ as a linear combination of the fundamental weights are smaller than q, and $\mathbf{F}$ be the Frobenius morphism of Γ associated with raising the elements of K to the qth power. Denote by $\Gamma^{\mathbf{F}}$ the group of fixed points for $\mathbf{F}$. By a famous theorem of Steinberg [16], $\psi|\Gamma^{\mathbf{F}}$ is irreducible. It is well known that $\Gamma^{\mathbf{F}}$ contains a representative of every unipotent conjugacy class in Γ (this follows, for instance, from [15, I.2.7(a)]). Therefore $\psi(\Gamma)$ and $\psi(\Gamma^{\mathbf{F}})$ simultaneously contain or do not contain transvections. It remains to apply results of [30] and [31] on finite linear groups generated by transvections. □

The following theorem gives estimates for the number of Jordan blocks of size p in the images of elements of order p in irreducible representations of the classical algebraic groups with large highest weights with respect to p.

Theorem 2.30 (a part of [19, Theorem 1.1]). *Let $\varphi \in \operatorname{Irr}_p$ and $\langle\omega(\varphi), \alpha\rangle \geqslant p$ for the maximal root α of G. Set $f(r) = 2r - 2$ for $G = A_r(K)$; $f(r) = 6r - 7$ for $G = B_r(K)$, $p = 3$; $f(r) = 8r - 10$ for $G = B_r(K)$, $p > 3$; $f(r) = 4r - 4$ for $G = C_r(K)$; $f(r) = 6r - 10$ for $G = D_r(K)$, $p = 3$; and $f(r) = 8r - 16$ for $G = D_r(K)$, $p > 3$. Then for each element $z \in G$ of order p the image $\varphi(z)$ has at least $f(r)$ Jordan blocks of size p.*

Lemma 2.31. *Let* $\omega = \sum_{i=1}^{r} a_i \in \mathbf{X}$. *Then*

$$\langle \omega, \alpha \rangle = \begin{cases} \sum_{i=1}^{r} a_i & \text{for } G = A_r(K) \text{ or } C_r(K), \\ a_1 + 2(\sum_{i=2}^{r-1} a_i) + a_r & \text{for } G = B_r(K), \\ a_1 + 2(\sum_{i=2}^{r-2} a_i) + a_{r-1} + a_r & \text{for } G = D_r(K). \end{cases}$$

Proof. This follows directly from the formulae for the maximal root of G in [2, Tables I–IV]. □

We need some information on the block structure of the images of quadratic unipotent elements in certain representations of the classical algebraic groups.

Lemma 2.32. *Let* $G = A_r(K)$, $B_r(K)$, *or* $C_r(K)$, *and* $n = p^s + b$ *with* $1 \leqslant b \leqslant p^s$. *Assume that* $y \in G$ *is a unipotent element that has just* b *blocks of size* 2 *and other blocks of size* 1 *on* V.

1) *Let* $\varphi = \varphi(\omega_2)$. *Assume that* $p > 2$ *for* $G \neq A_r(K)$ *and also for* $b > 1$. *Then* $\varphi(y)$ *has* $b(b-1)/2$ *blocks of size* 3 *and* $b(p^s - b)$ *blocks of size* 2, *other blocks are of size* 1. *Hence* $\varphi(y)$ *has more than* p^s *nontrivial blocks if* $b > 1$ *and* $p^s > 3$, 3 *such blocks if* $b = 2$ *or* 3 *and* $p^s = 3$, *and* $p^s - 1$ *such blocks for* $b = 1$.

2) *Let* $p = 2$, $G = A_r(K)$, $n = 2^s + b$ *with* $2 \leqslant b \leqslant p^s$, *and* $\varphi = \varphi(\omega_2)$. *Then* $\varphi(y)$ *has more than* 2^s *nontrivial blocks for* $n > 4$ *and* 2 *such blocks for* $n = 4$.

3) *Let* $p > 2$ *and* $\varphi = \varphi(2\omega_1)$. *Then* $\varphi(y)$ *has* $b(b+1)/2$ *blocks of size* 3 *and* $b(p^s - b)$ *blocks of size* 2, *other blocks are of size* 1. *Hence* $\varphi(y)$ *has more than* p^s *nontrivial blocks if* $b > 1$ *and* p^s *such blocks for* $b = 1$.

Proof. Set $M^+ = \bigwedge^2 V$ (the wedge square) for $\omega = \omega_2$ and $M^+ = S^2 V$ (the symmetric square) for $\omega = 2\omega_1$. It is well known that M is isomorphic to the unique nontrivial composition factor of M^+, see, for instance, the proof of [10, Theorem 5.1]. Hence for each nonzero weight $\mu \in \mathbf{X}(M)$ we have $\dim M_\mu = \dim M^+_\mu$. First assume that $p > 2$ if $b > 1$. Now we embed y into a closed subgroup $A \subset G$ such that $A \cong A_1(K)$ and $M^+|A$ is a direct sum of irreducible p-restricted A-modules. Hence $M|A$ is such direct sum as well. Then it follows from the representation theory of $A_1(K)$ that the number of nontrivial blocks of $\varphi(y)$ is equal to the number of nontrivial irreducible summands in $M|A$ that coincides with the analogous number for $M^+|A$. For $1 \leqslant j \leqslant b$ set $\beta_j = \varepsilon_1 - \varepsilon_{n-b+j}$ if $G = A_r(K)$ and $\beta_j = 2\varepsilon_j$ if $G = C_r(K)$. If $G = B_r(K)$, Theorem 2.1 a) implies that $b = 2t$. In this case set $\beta_j = \varepsilon_{2j-1} + \varepsilon_{2j}$ for $1 \leqslant j \leqslant t$. Put $d = b$ for $G = A_r(K)$ or $C_r(K)$ and $d = t$ if $G = B_r(K)$. For $f \in K$ set $x(f) = \Pi_{j=1}^{d} x_{\beta_j}(f)$ and $y(f) = \Pi_{j=1}^{d} x_{-\beta_j}(f)$. Analyzing the action of the elements $x(f)$ and $y(f)$ on V, one easily observes that x is conjugate to these elements if $f \neq 0$. Put $A = \langle x(f), y(f) \mid f \in K\rangle$. It is not difficult to see that $A \cong A_1(K)$, the group $T_A = T \cap A$ is a maximal torus in A, and the restriction of weights from T to T_A determines a homomorphism $\rho : \mathbf{X} \to \mathbb{Z}$ such that in all cases $\rho(\varepsilon_j) = 1$ for $1 \leqslant j \leqslant b$, for $G = A_r(K)$ one has $\rho(\varepsilon_k) = 0$ for $b < k < n - b + 1$ and $\rho(\varepsilon_l) = -1$ for $n - b + 1 \leqslant l \leqslant n$, for other groups $\rho(\varepsilon_k) = 0$ for $b < k \leqslant r$ (this completely determines the homomorphism ρ). Obviously, for $\omega = \omega_2$ we have

$$\mathbf{X}(M^+) = \begin{cases} \{\varepsilon_i + \varepsilon_j \mid 1 \leqslant i < j \leqslant n\} & \text{for } G = A_r(K),\\ \{\pm\varepsilon_i \pm \varepsilon_j,\ \pm\varepsilon_k,\ 0 \mid 1 \leqslant i < j \leqslant r, 1 \leqslant k \leqslant r\} & \text{for } G = B_r(K),\\ \{\pm\varepsilon_i \pm \varepsilon_j,\ 0 \mid 1 \leqslant i < j \leqslant r\} & \text{for } G = C_r(K);\end{cases}$$

if $\omega = 2\omega_1$, then

$$\mathbf{X}(M^+) = \begin{cases} \{\varepsilon_i + \varepsilon_j \mid 1 \leqslant i \leqslant j \leqslant n\} & \text{for } G = A_r(K),\\ \{\pm\varepsilon_i \pm \varepsilon_j,\ \pm\varepsilon_i \mid 1 \leqslant i \leqslant j \leqslant r\} & \text{for } G = B_r(K),\\ \{\pm\varepsilon_i \pm \varepsilon_j \mid 1 \leqslant i \leqslant j \leqslant r\} & \text{for } G = C_r(K).\end{cases}$$

This yields that for a weight $\rho \in \mathbf{X}(M^+)$ the integer $|\rho(\mu)| \leqslant 2$ if $b > 1$ and $|\rho(\mu)| \leqslant 1$ for $b = 1$. Hence $|\rho(\mu)| < p$. Then $M^+|A$ is completely reducible by [25, Lemma 7]. Obviously, all its irreducible components are p-restricted. It is clear that $\dim M^+_\mu = 1$ for $\mu \neq 0$. Now one can observe that $\dim(M^+|A)_2 = b(b-1)/2$ for $\omega = \omega_2$ and $b(b+1)/2$ for $\omega = 2\omega_1$, and $\dim(M^+|A)_1 = b(p^s - b)$ in both cases. This completes the proof of assertions 1) and 3).

Finally, let $p = 2$ and $b > 1$. If $n > 4$, observe that $p^s > 2$ and $n > 5$ since $b \leqslant 2^s$. Set $H_1 = G(1,2,3)$, $H_2 = G(5,\dots,r)$ for $n > 4$ and $H_1 = G(1)$, $H_2 = G(3)$ for $n = 4$. In both cases put $H = H_1H_2$. We can assume that $y = y_1y_2$ where $y_i \in H_i$, y_i are regular unipotent elements in H_i for $n = 4$, for $n > 4$ the element y_1 has 2 blocks of size 2 on V, y_2 has $b - 2$ such blocks, other blocks of these elements are trivial. Indeed, it is clear that y is conjugate to such product. Let V_i be the standard H_i-module, $i = 1, 2$. As $M \cong \bigwedge^2 V$, one easily deduces that $M|H \cong M_1 \oplus M_2 \oplus M'$ where $M_i|H_i \cong \bigwedge^2 V_i$, H_j acts trivially on M_i for $i \neq j$, and $M' \cong V_1 \otimes V_2$.

Observe that M_i is trivial if $\dim V_i = 2$, in particular, for $n = 4$. By Theorem 2.7, for $n = 4$ the element y has 2 blocks of size 2 on M'; if $n > 4$, then y has $2(n-4) = 2(p^s + b - 4) \geqslant 2(p^s - 2) \geqslant 2^s$ such blocks. Now it is clear that y has just 2 blocks of size 2 on M for $n = 4$ and on M_1 for $n > 4$. Hence for $n > 4$ the element y has at least $2^s + 2$ blocks of size 2 on M. The lemma is proved. □

Lemma 2.33. *Let $G = C_r(K)$, $n > p^s + 1$, and $z \in G$ be a nonzero long root element. Assume that $\varphi = \varphi(\omega_2)$ or $\varphi(2\omega_1)$. Then $\varphi(z)$ has more than p^s nontrivial Jordan blocks.*

Proof. Set $A = G(2\varepsilon_1)$. Then the restriction of weights from T to $T \cap A$ determines a homomorphism $\tau : \mathbf{X} \to \mathbb{Z}$ with $\tau(\varepsilon_1) = 1$ and $\tau(\varepsilon_j) = 0$ for $j > 1$. Arguing as in the proof of Lemma 2.32, we conclude that $|\tau(\mu)| \leqslant 2$ for $\mu \in \mathbf{X}(M^+)$ and that it suffices to show that $\dim(M^+|A)_1 > p^s$. Using the information on M^+ given in that proof, one easily concludes that $\dim(M^+|A)_1 = n - 2 > p^s$ since n is even and so $n \geqslant p^s + 3$. This completes the proof. □

Lemma 2.34. *Let $G = B_3(K)$, $S = G(1,3)$, u be a regular unipotent element of S, and $\varphi = \varphi(\omega_3)$. Then $J(\varphi(u)) = (3, 2, 2, 1)$.*

Proof. One easily concludes that $J(u) = (3, 2, 2)$. Then by Theorem 2.15, $d_\varphi(u) = 3$ and so $\varphi(u)$ has a block of size 3. Let M be a module affording φ and M_1 be an irreducible S-module where $G(1)$ acts trivially and $G(3)$ acts as on the standard module. Theorem 2.16 yields that M has a direct summand isomorphic to the direct sum of two copies of M_1. Hence $\varphi(u)$ has two blocks of size 2. Since $\dim \varphi = 8$, this completely determines all block sizes of $\varphi(u)$. □

In Sections 3–5 we always assume that $\varphi \in \mathrm{Irr}$, $\omega = \omega(\varphi) = \sum_{i=1}^{r} a_i\omega_i \neq 0$, and $\varphi \not\sim \mathrm{St}$.

§3. Proof of Theorem 1.1 for regular unipotent elements of the groups $A_r(K)$, $B_r(K)$, and $C_r(K)$

In this and the next sections, Theorem 1.1 is proved. Here $x \in G$ is an element of order $p^{s+1} > p$ and $y = x^{p^s}$. Since the morphism Fr does not change the Jordan block structure of unipotent elements, Corollary 2.3 allows us to assume that $\varphi \in \mathrm{Irr}_p$.

Lemma 3.1. *Theorem* 1.1 *holds for* $G = A_2(K)$.

Proof. Let $G = A_2(K)$. Then $p = 2$ and x is a regular unipotent element. Since φ is nontrivial and $\varphi \not\sim \mathrm{St}$, we get that $\omega = \omega_1 + \omega_2$. So M is the p-restricted Steinberg module. It is well known that in this case $\dim \varphi = 8$ and for each unipotent element z the image $\varphi(z)$ has only blocks of size equal to $|z|$. Thus $\varphi(x)$ has two $\mathcal{NS}$-blocks. □

Until the end of Sec. 4, we assume that $G \neq A_2(K)$. Write $n = ap^s + b$ with $1 \leqslant a < p$ and $0 < b \leqslant p^s$. Recall that by Lemma 2.2, y has b blocks of size $a+1$ and $p^s - b$ blocks of size a on V. Fix the notation a and b until the end of this section. Observe that $a + 1 < r$ for $G = A_r(K)$; $t < r - 1$ if $G = B_r(K)$ and $a = 2t$; $2t < r$ if $G = B_r(K)$ and $a = 2t - 1$; $t < r$ if $G = C_r(K)$ and $a = 2t-1$; and $a+1 < r$ if $G = C_r(K)$ and $a = 2t$ (here it is essential that $r > 2$ for $G = A_r(K)$ or $B_r(K)$). Hence the groups H_1 and H_2 in the statement of Proposition 3.2 below are correctly determined.

Proposition 3.2. *Set*

$H_1 = G(1, \dots, r-a-1)$ *and* $H_2 = G(r-a+1, \dots, r)$ *for* $G = A_r(K)$;

$H_1 = G(1, \dots, r-t-1, \varepsilon_{r-t-1} + \varepsilon_{r-t})$ *and* $H_2 = G(r-t+1, \dots, r)$ *for* $G = B_r(K)$ *and* $a = 2t$;

$H_1 = G(1, \dots, r-2t-1, \varepsilon_{r-2t})$ *and* $H_2 = G(r-2t+1, \dots, r-1, \varepsilon_{r-1}+\varepsilon_r)$ *for* $G = B_r(K)$ *and* $a = 2t-1$;

$H_1 = G(1, \dots, r-t-1, 2\varepsilon_{r-t})$ *and* $H_2 = G(r-t+1, \dots, r)$ *for* $G = C_r(K)$ *and* $a = 2t-1$;

$H_1 = G(1, \dots, r-a-2, 2\varepsilon_{r-a-1})$ *and* $H_2 = G(r-a, \dots, r)$ *for* $G = C_r(K)$ *and* $a = 2t$.

Then y is conjugate to an element y_1y_2 such that $y_j \in H_j$ and $y_2 \neq 1$.

If $G \neq B_r(K)$ or H_j is generated by long root subgroups, let $\mathbf{X}_j$ consist of all weights of V whose restrictions to $T \cap H_j$ are nonzero, otherwise assume that $\mathbf{X}_j$ consists of all such weights and the zero weight. Set $V_j = \langle V_\mu \mid \mu \in \mathbf{X}_j \rangle$. Then $V = V_1 \oplus V_2$, the groups H_j preserve V_1 and V_2, and H_k acts trivially on V_j for $\{k, j\} = \{1, 2\}$.

If $G = B_r(K)$ and a is odd or $G = C_r(K)$ and a is even, the element y_2 has two blocks of size $a+1$ on V_2 and y_1 has $b-2$ blocks of size $a+1$ and $p^s - b$ blocks of size a on V_1; otherwise y_2 is a regular unipotent element in H_2 and has a single Jordan block of size $a+1$ on V_2, the element y_1 has $b-1$ blocks of size $a+1$ and $p^s - b$ blocks of size a on V_1. In particular, $y_1 = 1$ just in the following cases: a) $G = A_r(K)$ or $C_r(K)$ and $n = p^s + 1$; b) $G = B_r(K)$ and $n = p^s + 2$.

Proof. By Theorem 2.1 b), the conjugacy class of y is completely determined by its Jordan form on V. Therefore one can find unipotent elements $y_j \in H_j$ such that y is conjugate to y_1y_2 and $y_2 \neq 1$. Analyzing the action of H_j on V, we easily get the required information on the subspaces V_1 and V_2 and the canonical Jordan form of y_1 and y_2. □

We keep the notation of Proposition 3.2 until the end of this section.

Proof of Theorem 1.1 **for regular unipotent elements in the groups** $A_r(K)$**,** $B_r(K)$**, and** $C_r(K)$**.** The proof is based on Lemma 2.2. We shall show that almost always $\varphi(y)$ has more than p^s nontrivial blocks. Then apply Lemma 2.2 to reduce the problem to few exceptional cases where special arguments are used. Set $H = H_1H_2$, $y' = y_1y_2$, $n_1 = \dim V_1$, and $r_1 = r(H_1)$. Lemma 2.5 yields that it suffices to find a composition factor or a submodule of $M|H$ where y' has more than p^s nontrivial blocks. Then $\varphi(y)$ has more than p^s such blocks. Suppose that $\varphi|H$ has a composition factor ψ such that $\psi \cong \psi_1 \otimes \psi_2$, $\psi_i \in \operatorname{Irr} H_i$, $\psi_2(y_2) \neq 1$, and $\psi_1(y_1)$ has more than p^s Jordan blocks. Then $\psi(y')$ has more than p^s nontrivial blocks. Naturally, we are done if $\dim \psi_1 > p^s d_{\psi_1}(y_1)$, in particular, if $\dim \psi_1 > p^{s+1}$ or $y_1 = 1$ and $\dim \psi_1 > p^s$. If r_1 is not too small, Proposition 2.21 allows us to get such factor for the majority of representations. We apply Lemma 2.13 and Corollary 2.14 to estimate $\dim \psi_1$. The proof is subdivided into several subcases. We have to handle separately the representations with highest weights from $\mathcal{S}(G)$ and the groups with small r_1 that do not satisfy the assumptions of Proposition 2.21. Furthermore, in some situations $\dim \psi_1$ is not large enough and additional arguments are required to estimate the number of Jordan blocks of $\psi_1(y_1)$. In this proof we sometimes use the term "a long root element" for groups of type A_i in situations where the groups H_1 of different types are considered simultaneously and an element under consideration is a root element if $H_1 \cong A_{r_1}(K)$ and a long root element otherwise.

A. **Special representations for groups of arbitrary ranks.** First assume that $\omega \in \mathcal{S}(G)$. Passing to the dual representation if necessary, one can suppose that $a_i \neq 0$ for some $i \leqslant n/2$ if $G = A_r(K)$. Let $\omega = \omega_2$ or $2\omega_1$ (naturally, $p > 2$ in the latter case). We claim that $M|H$ has a composition factor $M' \cong V_1 \otimes V_2$. If $\omega = 2\omega_1$, the weight $\omega_2 = \omega - \alpha_1 \in \mathbf{X}(M)$ by Lemma 2.18. Fix minimal j with $V_{\varepsilon_j} \subset V_2$. The weight $\lambda = \varepsilon_1 + \varepsilon_j \in \mathbf{X}(M)$ since λ lies in the W-orbit of ω_2. Let $m \in M_\lambda \setminus \{0\}$. One easily observes that $U^+(H)$ fixes m and that $\omega_H(m)$ is equal to the highest weight of the

H-module $V_1 \otimes V_2$. Hence m generates an indecomposable H-module with the required composition factor M'. Using Corollary 2.8, one can deduce that y' has at least $a(p^s - 1)$ nontrivial blocks on M' if y_1 has a single Jordan block on V_1, and at least $2a(p^s - 2)$ such blocks if y_1 has two Jordan blocks on V_1. Recall that $p^s > 2$ if $a > 1$. Now it is clear that y' has more than p^s nontrivial blocks on M' if $a > 1$.

Let $a = 1$. By Lemma 2.32, $\varphi(y)$ has more than p^s nontrivial blocks, except in the following cases:

1) $n = p^s + 1$;

2) $\omega = \omega_2$, $p^s = 2$ and $n = 4$ or $p^s = 3$ and $n = 5$ or 6.

In the exceptional cases $G = A_4(K)$ for $n = 5$ (recall that $r > 2$ for $G = B_r(K)$) and $G = A_r(K)$ or $C_r(K)$ otherwise. Set $d_\varphi(x) = d$ and apply results of [24] for odd p and of [21] for $p = 2$ to find it. Naturally, the size of the biggest Jordan block of $\varphi(x)$ is equal to d. If $n = p^s + 1$, results of [24, Theorem 1.7, Item 3 of the proof of Lemma 5.3, Proposition 1.5, and Algorithm 1.6] for $p > 2$ and [21, Theorem 1] for $p = 2$ imply that d is such as for the image of a regular unipotent element in the irreducible representation with highest weight ω of the analogous group in characteristic 0; we have $d = 2p^s - 1$ for $\omega = \omega_2$ and $d = 2p^s + 1$ for $\omega = 2\omega_1$. Moreover, if $p^s = 3$, $n = 4$, and $\omega = \omega_2$, observe that $d = 5$, $\dim \varphi = 6$ for $G = A_3(K)$, and $\dim \varphi = 5$ for $G = C_2(K)$. Hence in these two cases $\varphi(x)$ has a single nontrivial block. For $\omega = \omega_2$ and small n in Item 2) the results of [21] and [24] cited above yield the following: $d = 4$ for $n = 4$ and $p^s = 2$; $d = 7$ if $p^s = 3$ and $n = 5$, and $d = 9$ for $p^s = 3$ and $n = 6$ (in the latter case [24, Proposition 12.2] is used). Now it follows from the explicit formulae for the number of nontrivial blocks in Lemma 2.32 and Lemma 2.2 that in all these exceptional cases $\varphi(x)$ actually has only one $\mathcal{NS}$-block. Indeed, the number of nontrivial blocks for $\varphi(y)$ is too small for a power of an element with a block of size d and another $\mathcal{NS}$-block. These situations are described in Items 2 and 3 of Theorem 1.1.

Now let $G = A_r(K)$ and $\omega = \omega_1 + \omega_r$. Denote by V_i^* the H_i-module dual to V_i ($i = 1, 2$). Then Theorem 2.16 yields that $M|H$ has a direct summand $N = M_1 \oplus M_2$ with $M_1 \cong V_1 \otimes V_2^*$ and $M_2 \cong V_1^* \otimes V_2$ (M_1 and M_2 are the H-modules generated by a highest weight and a lowest weight vectors, respectively). Using Corollary 2.8 and arguing as before for $\omega = \omega_2$ or $2\omega_1$, we can conclude that y' has at least $2a(p^s - 1)$ nontrivial blocks on N. Hence y' has more than p^s such blocks on N if $p^s > 2$. Let $p^s = 2$. Then $r = 3$ since $p^{s+1} \geqslant n$. One easily observes that $\dim V_i = 2$ and y_i

is a regular unipotent element of H_i. So Corollary 2.8 implies that y' has 4 blocks of size 2 on N. This completes the analysis of the case where $\omega \in \mathcal{S}(G)$.

B. **Other representations.** Next, assume that $\omega \notin \mathcal{S}(G)$. We want to find out when one can apply Proposition 2.21. If this is possible, we say that G is not small. Obviously, G is not small if and only if $r_1 > 1$ for $G = A_r(K)$ or $C_r(K)$ with odd p, $r_1 > 2$ for $G = A_r(K)$ and $p = 2$ and for $G = B_r(K)$ and odd a, and $r_1 > 3$ for $G = B_r(K)$ and even a. It is clear that n_1 completely determines r_1. To estimate n_1, consider different possibilities for a separately. First let $a \geqslant 4$. It is clear that $p \geqslant 5$ and $n_1 \geqslant 2p^s$. If $a = 3$, then $p \geqslant 5$, $n_1 = 3p^s - 6$ for $G = B_r(K)$, $n = 3p^s + 2$ and $n_1 \geqslant 3p^s - 4 > 2p^s$ otherwise. So one can observe that $n_1 > 2p^s$ for $a = 3$, except the case where $p^s = 5$ and $G = B_8(K)$, in the exceptional case $n_1 = 2p^s - 1$. If $a = 2$, we have $p > 2$, $n_1 = 2p^s - 4$ for $G = C_r(K)$, $n = 2p^s + 2$, and $n_1 \geqslant 2p^s - 2$ otherwise. Finally, if $a = 1$, then $n_1 = p^s - 2$ for $G = B_r(K)$, $n = p^s + 2$, and $n_1 \geqslant p^s - 1$ otherwise. Moreover, $n_1 \geqslant p^s$ if $y_1 \neq 1$. Taking into account the construction of H_1 and the information above, it is not difficult to deduce that either G is not small, or one of the following holds:

i) $p = 2$, $G = A_3(K)$ or $A_4(K)$;
ii) $p = 3$, $G \in \{A_3(K), B_3(K), B_4(K), C_2(K), C_4(K)\}$;
iii) $p = 5$, $G = B_3(K)$ or $B_4(K)$;
iv) $p = 7$, $G = B_4(K)$.

We claim that $p^s = 5$ and $G = B_5(K)$ if $H_1 \cong D_4(K)$. Indeed, one easily concludes that in this case $n = 2p^s + b$ and $G = B_r(K)$. Now it follows from the construction of H_1 that $n = 11$. Hence $p^s = 5$ since otherwise $n > p^{s+1}$ or $n < 2p^s$. Furthermore, it is easy to see that $p^s = 3$ and $n = 5$ if $G = A_r(K)$, $n_1 = 3$ and G is not small. In this situation $\omega \neq \omega_j$ as $\omega \notin \mathcal{S}(G)$.

B1. **Not small groups.** Assume that G is not small and $\omega \neq \omega_5$ for $G = B_5(K)$ with $p^s = 5$. If $G = B_r(K)$ and $a = 1$, we can suppose that $y_2 = x_{\varepsilon_{r-1}+\varepsilon_r}(1)$. Now Proposition 2.21 implies that $\varphi|H$ has a composition factor $\psi \cong \psi_1 \otimes \psi_2$ with $\psi_i \in \operatorname{Irr} H_i$, $\omega(\psi_i) \neq 0$, $\psi_1 = \bigotimes_{j=0}^{k} \operatorname{Fr}^j \circ \rho_j$, and some nontrivial $\rho_j \not\sim \operatorname{St}$. Furthermore, the proof of Proposition 2.21 yields that for $G = B_r(K)$ and $a = 1$ one can find such a factor ψ with these properties for which $\langle \omega(\psi_2), \varepsilon_{r-1} + \varepsilon_r \rangle \neq 0$ and hence $\psi_2(y_2) \neq 1$. Define the sequence $J_{V_1}(y_1)$ in the same manner as $J(u)$ for a unipotent element $u \in G$. Then Lemma 2.13 and Corollary 2.14 yield that one of the following holds:

1) $\dim \psi_1 > p^{s+1}$;

2) $a = 1$, $\omega(\psi_1) = p^j\lambda$, $\lambda \in \mathcal{S}(H_1)$, y_1 acts quadratically on V_1;

3) $p^s = 3$, $n_1 = 4$, $H_1 \cong A_3(K)$, $G = A_6(K)$, $J_{V_1}(y_1) = (2,2)$, $\psi_1 \sim \varphi(\omega_2)$;

4) $p^s = 5$, $n_1 = 4$, $y_1 = 1$, $\psi_1 \sim \varphi(p^j\omega_1) \otimes \varphi(p^k\omega_2)$ with $k \neq j$, and either $H_1 \cong A_3(K)$ and $G = A_5(K)$, or $H_1 \cong C_2(K)$ and $G = C_3(K)$;

5) $p^s = 5$, $n_1 = 4$, $y_1 = 1$, $\psi_1 \sim \varphi(3\omega_1)$, $\varphi(\omega_1 + \omega_2)$, or $\varphi(2\omega_2)$, and either $H_1 \cong A_3(K)$, $G = A_5(K)$, or $H_1 \cong C_2(K)$, $G = C_3(K)$;

6) $p^s = 5$, $n_1 = 9$, $H_1 \cong B_4(K)$, $G = B_8(K)$, $J_{V_1}(y_1) = (3,3,3)$, and $\psi_1 \sim \varphi(\omega_4)$;

7) $p^s = 5$, $n_1 = 6$, $H_1 \cong C_3(K)$, $G = C_6(K)$, $J_{V_1}(y_1) = (2,2,2)$, and $\psi_1 \sim \varphi(\omega_2)$, $\varphi(\omega_3)$, or $\varphi(2\omega_1)$;

8) $p^s = 5$, $n_1 = 10$, $H_1 \cong D_5(K)$, $G = B_6(K)$, $J_{V_1}(y_1) = (3,3,2,2)$, and $\psi_1 \sim \varphi(\omega_4)$;

9) $p^s = 5$ or 7, $n_1 = 6$, $\psi_1 = \varphi(\omega_3)$, the element $y_1 = 1$ for $p = 7$ and $J_{V_1}(y_1) = (2,2,1,1)$ for $p = 5$, and either $H_1 \cong A_5(K)$ and $G = A_7(K)$, or $H_1 \cong C_3(K)$ and $G = C_4(K)$;

10) $p^s = 7$, $n_1 = 10$, $H_1 \cong C_5(K)$, $G = C_8(K)$, y_1 has 5 blocks of size 2 on V_1, and $\psi_1 \sim \varphi(\omega_2)$;

11) $p^s = 7$, $n_1 = 7$, $H_1 \cong A_6(K)$, $G = A_8(K)$, y_1 is a root element, and $\psi_1 \sim \varphi(\omega_3)$;

12) $p^s = 7$, $n_1 = 7$, $H_1 \cong B_3(K)$, $G = B_5(K)$, y_1 is a long root element, and $\psi_1 \sim \varphi(2\omega_3)$ or $\varphi(\omega_1 + \omega_3)$;

13) $p^s = 7$, $n_1 = 8$, $H_1 \cong C_4(K)$, $G = C_5(K)$, y_1 is a short root element, and $\psi_1 \sim \varphi(\omega_3)$ or $\varphi(\omega_4)$;

14) $p^s = 7$, $n_1 = 12$, $H_1 \cong D_6(K)$, $G = B_7(K)$, y_1 has 6 blocks of size 2 on V_1, and $\psi_1 \sim \varphi(\omega_5)$;

15) $p^s = 11$, $n_1 = 10$, $y_1 = 1$, $\psi_1 \sim \varphi(\omega_3)$, and either $H_1 \cong A_9(K)$ and $G = A_{11}(K)$, or $H_1 \cong C_5(K)$ and $G = C_6(K)$;

16) $p^s = 11$, $n_1 = 9$, $H_1 \cong B_4(K)$, $G = B_6(K)$, $y_1 = 1$, and $\psi_1 \sim \varphi(\omega_3)$;

17) $p^s = 13$, $n_1 = 11$, $H_1 \cong B_5(K)$, $G = B_7(K)$, $y_1 = 1$, and $\psi_1 \sim \varphi(\omega_3)$;

18) $p^s \in \{9, 11, 13, 17, 19\}$, $n_1 = p^s - 2$, $H_1 \cong B_t(K)$ with $t = (p^s - 3)/2$, $G = B_{t+2}(K)$, $y_1 = 1$, and $\psi_1 \sim \varphi(\omega_t)$;

19) $p^s \in \{7, 9, 11, 13, 17\}$, $n_1 = p^s$, $H_1 \cong B_l(K)$ with $l = (p^s - 1)/2$, $G = B_{l+2}(K)$, y_1 is a long root element, and $\psi_1 \sim \varphi(\omega_l)$;

20) $p^s \in \{11, 13, 17, 19\}, n_1 = p^s + 2$, $H_1 \cong B_m(K)$ with $m = (p^s + 1)/2$, $G = B_{m+2}(K)$, y_1 has 4 blocks of size 2 and other blocks of size 1 on V_1, and $\psi_1 \sim \varphi(\omega_m)$.

Here we take into account that $\dim \psi_1 > p^{s+1}$ if ψ_1 is tensor decomposable and $n_1 > p^s$.

The arguments in the first paragraph of the proof of the theorem imply that it suffices to show that $\psi_1(y_1)$ has more than p^s Jordan blocks. In particular, no further analysis is needed if $\dim \psi_1 > d_{\psi_1}(y_1)p^s$. So Case 1) is settled. Below in this proof we apply Theorem 2.15 to find or estimate $d_{\psi_1}(y_1)$ or other degrees of the minimal polynomials and Theorem 2.12 and [10, the tables in Section 6] to determine $\dim \psi_1$ and other dimensions we need.

I. Assume that 2) holds. Then $d_{\psi_1}(y_1) \leqslant 3$ and $d_{\psi_1}(y_1) = 2$ if $G = A_r(K)$, y_1 is a root element of H_1 and $\psi_1 \sim \varphi(\omega_2)$. Naturally, $d_{\psi_1}(y_1) = 2$ for $p = 2$. Observe that $H_1 \neq D_{r_1}(K)$ since $a = 1$. As $n \leqslant 2p^s$ and G is not small, we have $p^s > 2$; $n_1 = 3$ or 4 if $p^s = 3$; $n_1 > p^s - 1$ if $p^s = 4$; and $p^s > 5$ for $G = B_r(K)$. Furthermore, $p^s > 7$ if $G = B_r(K)$ and $n_1 = p^s - 2$. If $n_1 = p^s$, then y_1 is a root element of H_1. Without loss of generality we can assume that $\omega(\psi_1) \in \{\omega_2, 2\omega_1, \omega_1 + \omega_r\}$ for $G = A_r(K)$ and $\omega(\psi_1) \in \{\omega_2, 2\omega_1\}$ otherwise.

First let $\omega(\psi_1) = \omega_2$. Then $\dim \psi_1 = n_1(n_1 - 1)/2$ for $G = A_r(K)$ or $B_r(K)$ and $\dim \psi_1 \geqslant (n_1(n_1 - 1)/2) - 2$ if $G = C_r(K)$. Our assumptions yield that $n_1 > 3$ because $\psi_1 \not\sim \mathrm{St}$. One easily deduces that $\dim \psi_1 > 3p^s$ if $n_1 > p^s > 5$, if $p^s = 5$ and $n_1 \geqslant 7$, and if $n_1 = p^s > 7$; $\dim \psi_1 > 2p^s$ for $n_1 = p^s > 5$; and $\dim \psi_1 > p^s$ if $p^s = 4$ or $n_1 = p^s - 1$ or $p^s - 2$.

Let $p^s = 3$ and $n_1 = 4$. Then $H_1 \cong A_3(K)$ or $C_2(K)$ and $\dim \psi_1 = 6$ or 5, respectively; $J_{V_1}(y_1) = (2, 2)$. Therefore y_1 is a short root element of H_1 if $H_1 \cong C_2(K)$. Set $N = \bigwedge^2 V_1$. The representation ψ_1 can be realized in N if $H_1 \cong A_3(K)$, and in the unique nontrivial composition factor of N for $H_1 \cong C_2(K)$. Observe that y_1 is conjugate to the element $z = x_{\varepsilon_1 - \varepsilon_3}(1)x_{\varepsilon_2 - \varepsilon_4}(1)$ for $H_1 \cong A_3(K)$ and $x_{\varepsilon_1 + \varepsilon_2}(1)$ for $H_1 \cong C_2(K)$. One easily concludes that z is contained in a Zariski closed subgroup $S \cong A_1(K)$ such that $T_S = T \cap S$ is a maximal torus in T, the restriction of weights from T to T_S determines the homomorphism $\rho : \mathbf{X} \to \mathbb{Z}$ with $\rho(\varepsilon_1) = \rho(\varepsilon_2) = 1$ in both cases and $\rho(\varepsilon_3) = \rho(\varepsilon_4) = -1$ if $H_1 \cong A_3(K)$, $\rho(\mu) < 3$ for all $\mu \in \mathbf{X}(N)$, the weight 2 has multiplicity 1 in N, and 1 is not a weight of N. Arguing as in the proof of Lemma 2.32, one can deduce that $N|S$ is a direct sum of $M(2)$ and 3 copies of the trivial module. So for $H_1 \cong A_3(K)$ we have $J(\psi_1(y_1)) = (3, 1, 1, 1)$ and $\psi_1(y_1)$ has $4 > p^s$ blocks. As $\dim \psi_1 = \dim N - 1$ for $H_1 \cong C_2(K)$, the arguments above yield that in this case $J(\psi_1(y_1)) = (3, 1, 1)$. Since $\psi_2(y_2)$ has at least one nontrivial block, Theorem 2.7 implies that $\psi(y')$ has at least 4 nontrivial blocks as required.

If $p^s = 4$, Theorem 2.7 yields that $\psi(y')$ has at least $\dim \psi_1$ blocks of size 2 since $\psi_2(y_2) \neq 1$. As $\dim \psi_1 > 4$, we are done.

Next, let $n_1 = p^s = 5$ or 7. Then y_1 is a long root element of H_1. It is clear that $H_1 \cong A_4(K)$ and $\dim \psi_1 = 10 = 2p^s$ if $n_1 = 5$ and $H_1 \cong A_6(K)$ or $B_3(K)$ and $\dim \psi_1 = 21 = 3p^s$ for $n_1 = 7$. In both cases y_1 is conjugate to $x_1(1)$ in H_1 and Theorem 2.16 implies that $\psi_1(y_1)$ has a Jordan block of size 1 (the restriction $\psi_1|G(1)$ has a trivial direct summand). Hence not all blocks of $\psi_1(y_1)$ have the maximal possible size 2 or 3 and therefore $\psi_1(y_1)$ has more than p^s Jordan blocks.

It remains to consider the case where $p^s = 5$ and $n_1 = 6$. Then $H_1 \cong A_5(K)$ or $C_3(K)$, $\dim \psi_1 = 15$ or 14, respectively, and $J_{V_1}(y_1) = (2, 2, 1, 1)$. So y_1 is conjugate in H_1 with an element from $G(3, 4, 5)$ if $H_1 \cong A_5(K)$ and with a short root element if $H_1 \cong C_3(K)$. Applying Theorem 2.16 and arguing as in the previous paragraph, we conclude that in both cases $\psi_1(y_1)$ has a Jordan block of size 1. Since $d_{\psi_1}(y_1) = 3$, this implies that $\psi_1(y_1)$ has more than five Jordan blocks. Now for $\psi_1 = \varphi(\omega_2)$ all the possibilities have been considered.

Next, assume that $\omega(\psi_1) = 2\omega_1$ and $p > 2$. Then $\dim \psi_1 = n_1(n_1+1)/2$ for $G = A_r(K)$ or $C_r(K)$ and $\dim \psi_1 \geqslant (n_1(n_1+1)/2) - 2$ for $G = B_r(K)$. One easily observes that $\dim \psi_1 > 3p^s$ if $n_1 > p^s$ or $n_1 = p^s > 5$, and $\dim \psi_1 > p^s$ if $n_1 = p^s - 1$ or $p^s - 2$.

Let $n_1 = p^s = 3$ or 5. Then $H_1 \cong A_2(K)$ or $A_4(K)$, y_1 is a root element, and $\dim \psi_1 = 3p^s$. Applying Theorem 2.16 and arguing as before for $\omega(\psi_1) = \omega_2$, we observe that $\psi_1(y_1)$ has a Jordan block of size 1 and therefore has more than p^s blocks.

Finally, suppose that $G = A_r(K)$ and $\omega(\psi_1) = \omega_1 + \omega_r$. Then $\dim \psi_1 = n_1^2 - 1$ if $p \nmid n_1$ and $n_1^2 - 2$ if $p \mid n_1$. Hence $\dim \psi_1 > 3p^s$ if $n_1 > p^s$ or $n_1 = p^s \geqslant 4$ and $\dim \psi_1 > p^s$ if $n_1 = p^s - 1$. Hence it remains to consider the situation where $n_1 = p^s = 3$ and $\dim \psi_1 = 7$. Then $\psi_1(y_1)$ has at least three Jordan blocks and one of them has size 3. Arguing as in the case where $p^s = 3$, $H_1 \cong C_2(K)$ and $\omega(\psi_1) = \omega_2$, we can deduce that $\psi(y')$

has at least 4 nontrivial Jordan blocks. (In fact $\psi_1(y_1)$ has exactly three blocks, but we do not need this.) This completes the analysis of Case 2).

In Case 3) argue just as for $G = A_5(K)$ with $p^s = 3$ and $\omega(\psi_1) = \omega_2$.

II. Now we write down explicitly all the situations where p^s, n_1, H_1, y_1, and ψ_1 are such as in one of Cases 4)–20) and $\dim \psi_1 > p^s d_{\psi_1}(y_1)$. In Cases 4) and 5) we have $p^s = 5$, $y_1 = 1$, $\dim \psi_1 \geqslant 20$ for $H_1 \cong A_3(K)$ and $\geqslant 13$ for $H_1 \cong C_2(K)$. In Case 7) we have $p^s = 5$ and $d_{\psi_1}(y_1) = 3$; if $\psi_1 \sim \varphi(2\omega_1)$, then $\dim \psi_1 = 21$. In Case 9) for $p^s = 7$ we have $y_1 = 1$ and $\dim \psi_1 \geqslant 14$; if $p^s = 5$ and $H_1 \cong A_5(K)$, then $d_{\psi_1}(y_1) = 3$ and $\dim \psi_1 = 20$. In Cases 10), 12), and 13) we have $p^s = 7$, $d_{\psi_1}(y_1) = 3$, and $\dim \psi_1 \geqslant 26$. In Case 11) we have $p^s = 7$, $d_{\psi_1}(y_1) = 2$, and $\dim \psi_1 = 35$. In Case 14) we have $p^s = 7$, $d_{\psi_1}(y_1) \leqslant 4$, and $\dim \psi_1 = 32$. In Cases 15)–17) we have $p^s \leqslant 13$, $y_1 = 1$, and $\dim \psi_1 \geqslant 84$. In Case 18) we have $y_1 = 1$ and $\dim \psi_1 > p^s$ if $p^s \geqslant 11$. In Case 19) we have $d_{\psi_1}(y_1) = 2$ and $\dim \psi_1 > 2p^s$ if $p^s \geqslant 11$. In Case 20) we have $d_{\psi_1}(y_1) = 3$ and $\dim \psi_1 > 3p^s$. Clearly, in all these cases no further arguments are needed.

III. Let the assumptions of Case 6) hold. Then $d_{\psi_1}(y_1) = 4$ and $\dim \psi_1 = 16$. We claim that $J(\psi_1(y_1)) = (4, 4, 2, 2, 2, 2)$. Set $S_1 = G(1, 2, \varepsilon_2 + \varepsilon_3)$, $S_2 = G(\varepsilon_4)$, and $S = S_1 S_2$. Obviously, S_1 and S_2 are commuting subsystem subgroups of G, $S_1, S_2 \subset H_1$, $S_1 \cong A_3(K) \cong D_3(K)$, and $S_2 \cong A_1(K) \cong B_1(K)$. The Jordan block structure of the element y_1 acting on V_1 implies that y_1 is conjugate to an element $z = z_1 z_2$ where $z_j \in S_j$ are nontrivial unipotent elements. Without loss of generality we can assume that $\omega(\psi_1) = \omega_4$. Denote by π the standard 2-dimensional representation of S_2. By Lemma 2.24, $\psi_1|S \cong \varphi(\omega_2) \otimes \pi \oplus \varphi(\omega_3) \otimes \pi$. Since $d_{\psi_1}(y_1) = 4$ and $\pi(z_2)$ has a single Jordan block of size 2, Theorem 2.7 forces that $d_\sigma(z_1) = 3$ for $\sigma = \varphi(\omega_2)$ or $\varphi(\omega_3)$. Now it is clear that $J(\sigma(z_1)) = (3, 1)$ for such σ. Another application of Theorem 2.7 yields the claim. Hence $\psi_1(y_1)$ has more than p^s Jordan blocks as desired.

IV. Next, suppose that under the assumptions of Case 7) we have $\psi_1 \not\sim \varphi(2\omega_1)$. Then $\dim \psi_1 = 14$. We have $d_{\psi_1}(y_1) = 3$ for $\psi_1 \sim \varphi(\omega_2)$ and 4 for $\psi_1 \sim \varphi(\omega_3)$. Hence $\psi_1(y_1)$ has at least five Jordan blocks in the first case and at least 4 in the second one. Since ψ_2 is nontrivial and y_2 has blocks of size 3 in the standard realization of H_2, Theorems 2.15 and 2.7 yield that $d_{\psi_2}(y_2) \geqslant 3$. Now Corollary 2.8 implies that $d_\psi(y')$ has more than 5 nontrivial Jordan blocks.

V. Now let the assumptions of Case 8) hold. We claim that $\psi_1(y_1)$ has $7 > p^s$ Jordan blocks. Let $S \subset H_1$ be the subgroup constructed as the subgroup $H_{l,m}$ in Lemma 2.25 with $l = 1$ and $m = 3$, S_1 and S_2 be the simple components of S isomorphic to $B_1(K)$ and $B_3(K)$, respectively. The canonical Jordan form of y_1 implies that y_1 is conjugate in H_1 to an element $z = z_1 z_2 \in S$ where $z_i \in S_i$, z_1 is a regular unipotent element of S_1 and z_2 has a Jordan block of size 3 and two blocks of size 2 on the standard S_2-module. By [13, Theorem 1 and Table 1], the restriction $\psi_1|S$ is irreducible and is equivalent to the tensor product $\rho_1 \otimes \rho_2$ where $\rho_i \in \operatorname{Irr} S_i$, $\omega(\rho_1) = \omega_1$, and $\omega(\rho_2) = \omega_3$. Observe that z_2 is conjugate in S_2 to the product of long and short root elements lying in commuting subsystem subgroups of type A_1. Now Lemma 2.25 forces that $J(\rho_2(z_2)) = (3, 2, 2, 1)$. As $\rho_1(z_1)$ has a single block of size 2, our claim follows directly from Theorem 2.7.

VI. Under the assumptions of Case 9) let $p^s = 5$ and $H_1 \cong C_3(K)$. Then $d_{\psi_1}(y_1) = 3$ and y_1 is a short root element. Applying Theorem 2.16 as in Item I of this proof, we conclude that $\psi_1(y_1)$ has a block of size 1. Now it is clear that this element has more than 5 Jordan blocks since $\dim \psi_1 = 14$.

VII. Next, let $G = B_5(K)$ with $p^s = 7$ or 9 and $H_1 \cong B_3(K)$ or $p^s = 9$, $G = B_6(K)$, and $H_1 \cong B_4(K)$. First assume that $\omega \neq \omega_r$. We claim that a composition factor ψ described above can be chosen such that $\psi_1 \not\sim \omega_{r_1}$. For this purpose we slightly modify the proof of Proposition 2.21. As in that proof, we construct a nonzero vector m such that $U^+(H)$ fixes m and an indecomposable H-module generated by m has a required factor. If $\sum_{i=1}^{r-3} a_i \neq 0$ or $\omega = a_r \omega_r$, choose m as in the proof of Proposition 2.21. Now let $\omega = a_{r-2}\omega_{r-2} + a_{r-1}\omega_{r-1} + a_r\omega_r$ with $a_{r-2} + a_{r-1} \neq 0$. Set $m = X_{-(r-2)}v^+$ if $a_{r-2} \neq 0$ and $m = X_{-(r-2)}X_{-(r-1)}v^+$ for $a_{r-2} = 0$ (in this case $a_{r-1} \neq 0$). By Lemma 2.18, in both cases $m \neq 0$. Since $\mathcal{X}_j$ fixes m for $j \neq r-2$, it is clear that $U^+(H)$ fixes m. One easily observes that for all admissible ω the weight $\omega_H(m)$ has the required properties. This yields the desired factor. The arguments in Items I–VI of this proof imply that $\psi(y')$ has more than p^s nontrivial blocks.

Now let $\omega = \omega_r$. If $p^s = 9$ and $G = B_5(K)$, then y' is a long root element. By the results of [11, Section 2], the image of such element in the representation $\varphi(\omega_r)$ of the group $B_r(K)$ is quadratic and has 2^{r-2} Jordan blocks of size 2. Using [24, Theorem 1.7, Proposition 1.5, and Algorithm 1.6], we can check that $d_\varphi(x) = 16$. Since $\varphi(y)$ has 8 blocks of size 2, Lemma 2.2 forces that $\varphi(x)$ has a block of size $10 > p^s$ and hence there are two $\mathcal{NS}$-blocks.

Next, assume that $p^s = 7$ and $G = B_5(K)$ or $p^s = 9$ and $G = B_6(K)$. Then y_1 is a long root element and y_2 is a root element of H_2. Let ρ_1 and ρ_2 be the irreducible representations of H_2 obtained by the extension to H_2 of the standard representations of $G(r-1)$ and $G(\varepsilon_{r-1}+\varepsilon_r)$, respectively. Set $\chi = \varphi(\omega_{r_1}) \in \operatorname{Irr} H_1$. By Lemma 2.24, $\varphi|H \cong \chi \otimes \rho_1 \oplus \chi \otimes \rho_2$. The element $\chi(y_1)$ has 2 blocks of size 2 and 4 blocks of size 1 for $G = B_5(K)$ and 4 blocks of size 2 and 8 blocks of size 1 for $G = B_6(K)$. Since $H_2 \cong D_2(K) \cong A_1(K) \times A_1(K)$, it is clear that one of the elements $\rho_j(y_2)$ is the unity matrix of degree 2 and another one has a single Jordan block of size 2. Now Theorem 2.7 implies that $\varphi(y)$ has 10 nontrivial blocks for $G = B_5(K)$ and 20 such blocks for $G = B_6(K)$. This completes the analysis of Cases 2)–20).

VIII. Finally, let $p^s = 5$, $G = B_5(K)$, and $\omega = \omega_5$. We have $H_1 \cong D_4(K)$, $H_2 = G(r) \cong B_1(K) \cong A_1(K)$, and $J_{V_1}(y_1) = (2,2,2,2)$. Set $\varphi_1 = \varphi(\omega_3)$ and $\varphi_2 = \varphi(\omega_4) \in \operatorname{Irr} H_1$ and denote by π the standard 2-dimensional representation of H_2. By Lemma 2.24, $\varphi|H \cong \varphi_1 \otimes \pi \oplus \varphi_2 \otimes \pi$. We have $\dim \varphi_j = 8$ $(j = 1,2)$ and $\{d_{\varphi_1}(y_1), d_{\varphi_2}(y_1)\} = \{2,3\}$. Therefore one of the elements $\varphi_j(y_1)$ has at least 4 Jordan blocks and another has at least 3. Hence it is clear that $\varphi(y)$ has at least $7 > p^s$ nontrivial blocks. Now the theorem is proved for regular unipotent elements in not small groups.

B2. **Small cases.** It remains to consider groups of a small rank that satisfy one of the conditions i)–iv) at the beginning of Item B. Theorem 2.30 yields that in all cases under consideration $\varphi(y)$ has more than p^s Jordan blocks of size p if $\langle \omega, \alpha \rangle \geqslant p$ for the maximal root α of G. We apply Lemma 2.31 to determine the highest weights that satisfy the latter inequality. So we can and shall assume that $\sum_{i=1}^{r} a_i < p$ for $G \in \{A_3(K), A_4(K), C_2(K), C_4(K)\}$, $a_1 + 2a_2 + a_3 < p$ for $G = B_3(K)$, and $a_1 + 2a_2 + 2a_3 + a_4 < p$ for $G = B_4(K)$. Recall that $\omega \notin \mathcal{S}(G)$.

I. Let $p = 2$ and $G = A_3(K)$ or $A_4(K)$. Then $\langle \omega, \alpha \rangle \geqslant 2$ for all 2-restricted $\omega \notin \mathcal{S}(G)$. So the theorem is proved for these groups.

II. Now let $p = 3$ and $G = A_3(K)$ or $C_2(K)$. Then y is a long root element. Passing to the dual representation if necessary, one may suppose that $a_1 \geqslant a_3$ for $G = A_3(K)$. Then the assumptions at the beginning of Part B2 imply that it remains to consider the cases where $\omega = \omega_1 + \omega_2$ or

$2\omega_2$. Set $G^1 = A_3(K)$ and $G^2 = C_2(K)$. One can regard G^2 as a naturally embedded subgroup of G^1 that contains a regular unipotent element of G^1, long root elements of G^2 are conjugate to root elements of G^1, we can choose a maximal torus $T_1 \subset G^1$ for which $T_2 = T_1 \cap G^2$ is a maximal torus in G^2 and assume that the restriction to T_2 of the weight ε_i of G^1 with $i = 1$ or 2 yields the weight ε_i of G^2. By [13, Theorem 1 and Table 1], for $\varphi = \varphi(\omega_1 + \omega_2) \in \operatorname{Irr} G^1$ the restriction $\varphi|G^2$ is irreducible. Obviously, the restriction to G^2 of the representation $\varphi(2\omega_2) \in \operatorname{Irr} G^1$ has a composition factor with highest weight $2\omega_2$. Now Lemma 2.5 implies that it suffices to prove that $\varphi(y)$ has $\geqslant 4$ nontrivial Jordan blocks if $G = G^1$ and $\omega = \omega_1 + \omega_2$ or $G = G^2$ and $\omega = 2\omega_2$.

Let $G = A_3(K)$ and $\omega = \omega_1 + \omega_2$. Then M is isomorphic to the truncated symmetric cube of V (see, for instance, [27, Proposition 1.2]). Set $S = G(1,3)$ and denote by τ the trivial representation of $A_1(K)$. It follows from [27, Proposition 1.4] that

$$\varphi|S \cong \varphi(\omega_1) \otimes \tau \oplus \varphi(2\omega_1) \otimes \varphi(\omega_1) \oplus \varphi(\omega_1) \otimes \varphi(2\omega_1) \oplus \tau \otimes \varphi(\omega_1).$$

As y is conjugate to $x_1(1)$, the image $\varphi(y)$ has 6 nontrivial Jordan blocks.

Now assume that $G = C_2(K)$ and $\omega = 2\omega_2$. For $0 \leqslant i \leqslant 3$ let $M_i \subset M$ be the subspace generated by all weight subspaces for the weights of the form $\omega - i\alpha_1 - k\alpha_2$ where k is a nonnegative integer. Put $\Delta = G(2)$. Then y is conjugate to a nontrivial unipotent element $z \in \Delta$. Clearly, M_i are Δ-modules and $\bigoplus_{i=1}^{3} M_i$ is a direct summand in $M|\Delta$. Set $m_0 = v^+$, $m_1 = X_{-1}X_{-2}v^+$, $m_2 = X_{-1}^2 X_{-2}v^+$, and $m_3 = X_{-1,3}X_{-2}^2 v^+$. By Lemma 2.18, m_1 and $m_2 \neq 0$. We claim that $m_3 \neq 0$. Indeed, let N be the Δ-module generated by $X_{-2}^2 v^+$. Then N is an indecomposable module with highest weight 4 and a unique maximal submodule. Denote by $\overline{N}$ and $\overline{m_3}$ the quotient module of N by this submodule and the image of m_3 under the canonical surjection of N onto $\overline{N}$. Then $\overline{N} \cong M(4)$ and it follows from the representation theory of the group $A_1(K)$ that $\overline{m_3} \neq 0$. Hence $m_3 \neq 0$. The weight structure of M forces that the group $\mathcal{X}_2$ fixes m_i for $0 \leqslant i \leqslant 3$. One easily observes that $m_i \in M_i$ and $\omega^2(m_i) \neq 0$. Hence the vectors m_i generate nontrivial indecomposable Δ-modules and each of the modules M_i has a nontrivial composition factor. Therefore z has a nontrivial Jordan block on M_i for $0 \leqslant i \leqslant 3$ and so $\varphi(y)$ has at least 4 such blocks. This completes the analysis of the groups $A_3(K)$ and $C_2(K)$.

III. Now let $G = B_3(K)$ and $p = 3$ or 5. If $\omega = \omega_3$, we have $d_\varphi(x) = 7$ by [24, Proposition 12.2 and Table V]. Since $\dim \varphi = 8$, it is clear that $\varphi(x)$ has only one nontrivial (and $\mathcal{NS}$) block of size 7. This representation appears in Items 4 of Theorems 1.1 and 1.3.

Next, assume that $\omega \neq \omega_3$. Denote by α the maximal root of G. Then $\alpha = \alpha_1 + 2\alpha_2 + 2\alpha_3$ and $\langle \omega, \alpha \rangle = a_1 + 2a_2 + a_3$. Set $S_1 = G(1)$, $S_2 = G(\alpha)$, $S_3 = G(3)$, and $S = S_1 S_2 S_3$. The group S is conjugate to H and is equal to H for $p = 3$. So y is conjugate to an element $z \in H$. For all representations under consideration we construct a composition factor ρ of the restriction $\varphi|S$ such that $\rho(z)$ has more than p^s nontrivial Jordan blocks. Then it remains to apply Lemma 2.5. In this item we denote by (i, j, k) the weight of S whose restrictions to S_1, S_2, and S_3 are equal to i, j, and k, respectively, and by $\varphi(i, j, k)$ the irreducible representation of S with this weight. Naturally, $\varphi(i, j, k) \cong \varphi(i) \otimes \varphi(j) \otimes \varphi(k)$ where the first, the second, and the third multipliers are representations of S_1, S_2, and S_3, respectively.

First, let $p = 3$. Then we can take $z = x_\alpha(1)x_3(1)$. Since $\omega \notin \mathcal{S}(G)$, our previous assumptions yield that it suffices to consider the cases where $\omega = \omega_1 + \omega_3$ or $2\omega_3$. Set $m = v^+$ for $\omega = \omega_1 + \omega_3$ and $m = v(2, 3, 1)$ for $\omega = 2\omega_3$. In the first case it is obvious that $m \neq 0$ and is fixed by $U^+(S)$, in the second one this follows from Lemma 2.20. So m generates an indecomposable S-module with highest weight $\omega_S(m)$ and $\varphi|S$ has a composition factor ρ with such highest weight. One has $\rho = \varphi(1, 2, 1)$ for $\omega = \omega_1 + \omega_3$ and $\varphi(1, 2, 2)$ for $\omega = 2\omega_3$. By Theorem 2.7, $\rho(z)$ has 4 nontrivial blocks in the first case and 6 such blocks in the second one.

Next, let $p = 5$. Now z is a long root element of S and so we can suppose that $z = x_\alpha(1)$. It is clear that $\rho(z)$ has more than 5 nontrivial blocks for a representation $\rho = \varphi(i, j, k)$ if $j \neq 0$ and $\dim(\varphi(i) \otimes \varphi(k)) > 5$. Recall that we assume that $a_1 + 2a_2 + a_3 < 5$. Set $m = v^+$ if $a_1 a_3 \neq 0$ and $a_1 + a_3 > 2$, $m = v(2, 1)$ for $\omega = a_1\omega_1$, $m = v(2, 3)$ for $\omega = \omega_1 + \omega_3$ or $a_3\omega_3$, and $m = X_{-2}v^+$ otherwise. Lemmas 2.18 and 2.20 force that in all cases $m \neq 0$ and is fixed by $U^+(S)$. Put $\mu = \omega_S(m)$. Then $\varphi|S$ has a composition factor $\varphi(\mu)$. Write $\mu = (i, j, k)$. Since $\omega \notin \mathcal{S}(G)$, one can conclude that in all cases ik and $j \neq 0$, $i, k < p$, and $i + k > 2$. Hence $\varphi(\mu)$ is a desired factor, and the theorem holds for $G = B_3(K)$.

IV. Suppose that $p = 3$ and $G = B_4(K)$. Then $H_1 = G(1, 2, \varepsilon_2 + \varepsilon_3) \cong A_3(K)$ and $H_2 = G(4)$. Set $\beta_1 = \alpha_2$, $\beta_2 = \alpha_1$, and $\beta_3 = \varepsilon_2 + \varepsilon_3$. Then

$(\beta_1, \beta_2, \beta_3)$ is a base of $R(H_1)$. Denote by ω_i^1 the fundamental weight of H_1 associated with β_i, $i = 1, 2, 3$.

Let $\omega \neq \omega_4$. Put $m = v^+$ if $a_4 \neq 0$, $m = X_{-3}v^+$ if $a_4 = 0$ and $a_3 \neq 0$, $m = v(3,2,1)$ if $a_3 = a_4 = 0$ and $a_2 \neq 0$, and $m = v(3,1,1)$ for $\omega = a_1\omega_1$. Lemmas 2.18 and 2.20 imply that $m \neq 0$ and is fixed by $U^+(H)$. Hence $\varphi|H$ has a composition factor ψ with highest weight $\omega_H(m)$. Write $\psi \cong \psi_1 \otimes \psi_2$ where $\psi_j \in \operatorname{Irr} H_j$, $j = 1, 2$. As $\omega \notin \mathcal{S}(G) \cup \{\omega_4\}$ and $a_1 + 2a_2 + 2a_3 + a_4 < 3$, one can check that in all cases $\omega(\psi_1)$ is p-restricted, $\omega(\psi_1) \neq 0$ or ω_i^1, and $\omega(\psi_2) \neq 0$. According to [10, Table 6.7], $\dim \psi_1 > 9 = p^{s+1}$. So we can apply the arguments from the first paragraph of the proof to handle this case.

Finally, let $\omega = \omega_4$. Set $\sigma_1 = \varphi(\omega_1^1)$ and $\sigma_2 = \varphi(\omega_3^1) \in \operatorname{Irr} H_1$. By Lemma 2.24, $\varphi|H \cong \sigma_1 \otimes \varphi(\omega_1) \oplus \sigma_2 \otimes \varphi(\omega_1)$ where the second multiplier is a representation of H_2. Since $|y_1| = 3$, the elements $\sigma_1(y_1)$ and $\sigma_2(y_1)$ have at least two Jordan blocks. Hence $\varphi(y)$ has at least 4 nontrivial blocks as desired. Now for $p = 3$ and $G = B_4(K)$ all the possibilities have been considered.

V. Next, let $p = 3$ and $G = C_4(K)$. Here the proof is the same as in the general case for not small groups, but the groups H_1 and H_2 interchange their roles. Recall that $H_1 \cong C_1(K)$, $H_2 \cong C_3(K)$, and both y_1 and y_2 are nontrivial. By Proposition 2.21, $\varphi|H$ has a composition factor $\psi \cong \psi_1 \otimes \psi_2$ with $\psi_i \in \operatorname{Irr} H_i$, $\omega(\psi_i) \neq 0$, $\psi_2 = \bigotimes_{j=0}^{k} \operatorname{Fr}^j \circ \rho_j$ and some nontrivial $\rho_j \not\sim$ St. Lemma 2.13 implies that $\dim \psi_2 > 9$. Hence $\psi(y')$ has more than 3 nontrivial Jordan blocks as desired.

VI. Finally, let $G = B_4(K)$ and $p = 5$ or 7. If $\omega = \omega_4$, one has $d_\varphi(x) = 11$ by [24, Proposition 12.2 and Table VI]. As $\dim \varphi = 16$, it is clear that $\varphi(x)$ has only one $\mathcal{NS}$-block. This representation appears in Item 5 of Theorem 1.1.

Now assume that $\omega \neq \omega_4$. We construct a composition factor ψ of the restriction $\varphi|H$ for which $\psi(y')$ has more than p^s nontrivial blocks. One has $H_1 = G(1, \varepsilon_2)$ and $H_2 = G(3, \varepsilon_3 + \varepsilon_4)$. Put $\beta = \varepsilon_3 + \varepsilon_4$ and recall that $y_2 \in \mathcal{X}_\beta$. Set $m = v^+$ if $a_3 + a_4 \neq 0$, $m = X_{-2}v^+$ if $a_3 = a_4 = 0$ and $a_2 \neq 0$, and $m = v(2,1,1)$ for $\omega = a_1\omega_1$. Using Lemmas 2.18 and 2.20 as in Item IV, we can conclude that in all cases $m \neq 0$ and $\varphi|H$ has a composition factor ψ with highest weight $\lambda = \omega_H(m)$. We claim that ψ has the required properties. Put $b_1 = \langle \lambda, \varepsilon_2 \rangle$ and $b_2 = \langle \lambda, \alpha_1 \rangle$. Write $\psi = \psi_1 \otimes \psi_2$ with $\psi_i \in \operatorname{Irr} H_i$. Observe that y_1 is a long root element and $d_{\psi_1}(y_1) = b_1 + b_2 + 1$ for $p = 5$ and $y_1 = 1$ if $p = 7$. One easily concludes that $\langle \lambda, \beta \rangle \neq 0$. It follows from the arguments at the beginning of the proof that our claim holds if $\dim \psi_1 > 25 = p^{s+1}$ for $p = 5$ and $\dim \psi_1 > 7$ for $p = 7$. Since $\omega \notin \mathcal{S}(G) \cup \{\omega_4\}$ and $a_1 + 2a_2 + 2a_3 + a_4 < p$, we can also check the following: ψ is p-restricted; if $a_3 + a_4 \neq 0$, then $b_1 \neq 0$ and $b_1 > 1$ or $b_2 \neq 0$; otherwise $b_2 \neq 0$ and $b_2 > 1$ if $b_1 = 0$. Recall that $H_1 \cong C_2(K)$. In this proof ω_1^1 and ω_2^1 are the fundamental weights of H_1 associated with the roots ε_2 and α_1, respectively. Now [10, Table 6.22] yields that $\dim \psi_1 > 7$ and for $p = 5$ either $\dim \psi_1 > 25$, or

$$\omega(\psi_1) \in \{2\omega_1, \omega_1 + \omega_2, 2\omega_2, 3\omega_1\}. \tag{3}$$

Hence the claim is proved for $p = 7$ and for $p = 5$ it remains to consider the exceptional cases indicated in formula (3). First suppose that $\omega(\psi_1) = 2\omega_1$. Let γ be the maximal root of H_1, $\Gamma = H_1(\gamma)$, and N be the standard $C_2(K)$-module. Analyzing the restriction of the symmetric square S^2N to Γ as in the proofs of Lemmas 2.32 and 2.33, one can conclude that $J(\psi_1(y_1)) = (3, 2, 2, 1, 1, 1)$.

We claim that in other cases under consideration $\psi_1(y_1)$ has at least 5 Jordan blocks. This is quite clear if $\omega(\psi_1) = 2\omega_2$ or $3\omega_1$. Indeed, in the first case $d_{\psi_1}(y_1) = 3$ and $\dim \psi_1 = 13$, in the second one $d_{\psi_1}(y_1) = 4$ and $\dim \psi_1 = 20$. If $\omega(\psi_1) = \omega_1 + \omega_2$, we have $d_{\psi_1}(y_1) = 3$ and $\dim \psi_1 = 12$. It follows from Theorem 2.16 that $\psi_1|G(1)$ has a direct summand isomorphic to $\varphi(1)$ and hence $\psi_1(y_1)$ has a Jordan block of size 2. Therefore this element has at least 5 blocks. Now Theorem 2.7 forces that $\psi(y')$ has more than 5 nontrivial blocks in the special cases indicated in formula (3) as well. All the possibilities have been considered. Theorem 1.1 is proved for regular unipotent elements of the groups $A_r(K)$, $B_r(K)$, and $C_r(K)$. □

§4. Proof of Theorem 1.1 for other elements

In this section the proof of Theorem 1.1 is completed. If $u \in G$ is a nonregular unipotent element and $J(u) = (d_1, d_2, d_3, \dots, d_t)$, set $d_i(u) = d_i$ for $1 \leqslant i \leqslant 3$ putting $d_3(u) = 0$ if $t = 2$ (here $t > 1$ as u is not regular). Recall that $d_1(u) = d_2(u)$ if $G = B_r(K)$ or $D_r(K)$ and $d_1(u)$ is even or $G = C_r(K)$ and $d_1(u)$ is odd. In Lemma 4.1 below $H_{t,r-1-t}$ is the subgroup defined in Lemma 2.25.

Lemma 4.1. *Set*

$$\mathcal{H} = \begin{cases} \{G(2,\dots,r),\ G(1,\dots,i-1, \\ i+1,\dots,r) \ \ \textit{with} \ \ 1 < i < r\} & \textit{for} \ \ G = A_r(K), \\ \{G(2,\dots,r),\ G(1,\dots,r-1)\} & \textit{for} \ \ G = B_r(K) \ \ \textit{or} \ \ C_r(K), \\ \{G(2,\dots,r),\ G(1,\dots,r-1), \\ G(1,\dots,r-2,r)\} & \textit{for} \ \ G = D_r(K). \end{cases}$$

Let $u \in G$ be a nonregular unipotent element and $d_1 = d_1(u)$. Put

$$S_1(u) = G(1,\dots,t-1,\varepsilon_t),\ S_2(u) = G(t+1,\dots,r-1,\varepsilon_{r-1}+\varepsilon_r)$$

if $G = B_r(K)$ *and* $d_1 = 2t+1 < n-2$;

$$S_1(u) = G(1,\dots,2t-1,\varepsilon_{2t-1}+\varepsilon_{2t}),\ S_2(u) = G(2t+1,\dots,r)$$

if $G = B_r(K)$ *and* $d_1 = 2t < n-1$;

$$S_1(u) = G(1,\dots,t-1,2\varepsilon_t),\ S_2(u) = G(t+1,\dots,r)$$

if $G = C_r(K)$ *and* $d_1 = 2t$;

$$S_1(u) = G(1,\dots,t-1,2\varepsilon_{2t+1}),\ S_2(u) = G(2t+2,\dots,r)$$

if $G = C_r(K)$ *and* $d_1 = 2t+1 < r$;

$$S_1(u) = G(1,\dots,2t-1,\varepsilon_{2t-1}+\varepsilon_{2t}),\ S_2(u) = G(2t+1,\dots,r)$$

if $G = D_r(K)$ *and* $d_1 = 2t < r-1$.

In all these cases set $S(u) = S_1(u)S_2(u)$. If $G = D_r(K)$ and $d_1 = 2t+1 < n-1$, put $S(u) = H_{t,r-1-t}$ and denote by $S_1(u)$ and $S_2(u)$ the simple components of $S(u)$ of ranks t and $r-1-t$, respectively (if $2t = r-1$, the numeration of these components does not matter). Then one of the following holds:

1) *u is conjugate to an element of a subgroup $H \in \mathcal{H}$;*

2) *the subgroup $S(u)$ is defined, u is conjugate to z_1z_2, $z_i \in S_i(u)$, $|z_1| = |u|$, and $z_2 \neq 1$.*

Proof. The proof follows immediately from Theorem 2.1, Lemma 2.25, and well-known facts on the action of subsystem subgroups of G on V. $\square$

Observe that in the assumptions of Lemma 4.1 $|u|$ can be equal to p. For some elements both assertions 1) and 2) of this lemma hold, for our aims it is essential that at least one of them is valid. We shall apply Lemma 4.1 in the proof of Theorem 1.3 as well. The notation of this lemma is used until the end of the text.

The end of the proof of Theorem 1.1. Recall that $|x| = p^{s+1} > p$. Set $d_i = d_i(x)$, $1 \leqslant i \leqslant 3$.

I. First assume that $G = D_r(K)$ and x is a regular unipotent element. Recall that $\omega \neq \omega_3$ or ω_4 for $r = 4$ since $\varphi \not\sim$ St. It is well known (and follows easily from the construction of the subgroup H_B in Lemma 2.25) that x is conjugate to a regular unipotent element x' of H_B. If $\omega \notin \{0, \omega_1, \omega_{r-1}, \omega_r\}$, Corollaries 2.27 and 2.28 yield that $\varphi|H_B$ has a composition factor $\rho \cong \bigotimes_{j=1}^{l} \mathrm{Fr}^j \circ \rho_j$ with $\rho_j \in \mathrm{Irr}_p H_B$ and for some j the weight $\omega(\rho_j) \notin \{0, \omega_1, \omega_{r-1}\}$. By [13, Theorem 1.1 and Table 1], $\varphi|H_B \cong \varphi(\omega_{r-1})$ if $\omega = \omega_{r-1}$ or ω_r. Since Theorem 1.1 holds for regular unipotent elements in groups of type B_i (this has been proved in Section 3), Lemma 2.5 implies that $\varphi(x)$ has a unique $\mathcal{NS}$-block if and only if $p = 5$ or 7, $G = D_5(K)$, and $\omega = \omega_4$ or ω_5. For regular unipotent elements Theorem 1.1 is proved.

II. Now let x be not regular. Suppose that x is conjugate to an element of a subgroup $H \in \mathcal{H}$ in the notation of Lemma 4.1. If $G = A_3(K)$, one easily observes that $p = 2$, $d_1 = 3$ and $d_2 = 1$. Then x is conjugate to an element of $G(2,3)$. Since p is odd for $G \neq A_r(K)$, it is clear that $G \neq C_2(K)$. Therefore x has at least two $\mathcal{NS}$-blocks by Corollary 2.23. In particular, this completes the proof of Theorem 1.1 for $G = A_r(K)$.

III. Next, let the assertion 2) of Lemma 4.1 hold for x. Set $z = z_1 z_2$, $S = S(x)$, and $S_i = S_i(x)$, $i = 1, 2$. Suppose that $d_1 \neq 5$ or $d_2 > 3$ for $G = B_r(K)$ or $D_r(K)$. As $|x| > p > 2$, one easily observes that $l > 2$ if $S_1 \cong B_l(K)$ and $d_1 > 5$, $l > 1$ for $S_1 \cong C_l(K)$, and $l > 3$ for $S_1 \cong D_l(K)$. By Theorem 2.17 and Corollary 2.27.2), the restriction $\varphi|S$ has a composition factor $\psi \cong \psi_1 \otimes \psi_2$ where $\psi_i \in \mathrm{Irr}\, S_i$ and $\omega(\psi_i) \neq 0$. Moreover, using Proposition 2.21, we can assume that for $G = C_r(K)$ the representation $\psi_1 \not\sim$ St if $\omega \neq \omega_2$ or $2\omega_1$. Using Lemma 2.11 and arguing as in the proof of Lemma 2.3, one can deduce that $\psi(z)$ has at least two $\mathcal{NS}$-blocks if $d_{\psi_1}(z_1) > p^s + 1$, or $\psi_1(z_1)$ has more than one $\mathcal{NS}$-block, or $|z_2| = |x|$. By Lemma 2.5, in all these situations $\varphi(x)$ has more than one $\mathcal{NS}$-block. So it remains to consider the case where $\psi_1(z_1)$ has a single $\mathcal{NS}$-block of size $p^s + 1$ and $|z_2| \leqslant p^s$. Then $\psi_1(z_1^{p^s})$ is a transvection. Now Lemma 2.29 and our assumptions on ψ and d_1 imply that $G = C_r(K)$, $\psi_1 \sim$ St, and $\omega = \omega_2$ or $2\omega_1$. It is clear that $d_1 = p^s + 1$ and $d_2 \leqslant p^s$. Hence y is a long root element of G. By Lemma 2.33, $\varphi(y)$ has more than p^s nontrivial Jordan blocks. Then we apply Lemma 2.2 and complete the proof of Theorem 1.1 for $G = C_r(K)$.

IV. Finally, let $G = B_r(K)$ or $D_r(K)$, $d_1 = 5$, and $d_2 < 4$. Then $p = 3$ and y is conjugate to a root element of G. If $d_2 = 1$, we have $d_3 = 1$ since $G \neq D_3(K)$. In this case x is conjugate to an element of $G(2, \dots, r)$ and Corollary 2.23 implies that $\varphi(x)$ has at least two $\mathcal{NS}$-blocks. Now assume that $d_2 > 1$. Then $d_3 = 2$ if $d_2 = 2$. This implies that $n > 7$ and hence $r > 3$. By Lemma 2.2, it suffices to prove that $\varphi(y)$ has more than 3 nontrivial blocks.

First let $G = D_4(K)$. Set $l = a_1 + 2a_2 + a_3 + a_4$ and $H = G(1, 2, 3)$. By Theorem 2.30 and Lemma 2.31, $\varphi(y)$ has at least 14 nontrivial blocks if $l \geqslant 3$. Next, let $l < 3$. As $\varphi \not\sim \mathrm{St}$, one easily concludes that $\varphi \sim \varphi(\lambda)$ with $\lambda \in \{\omega_2, 2\omega_1, \omega_1 + \omega_3\}$. So we can assume that ω is one of these three weights and $y \in H$. It follows from Theorem 2.16 that $M|H$ has a direct summand M_H such that $M_H \cong M(\omega_2) \oplus M(\omega_2)$ for $\omega = \omega_2$, $M_H \cong M(2\omega_1) \oplus M(2\omega_3)$ for $\omega = 2\omega_1$, and $M_H \cong M(\omega_1 + \omega_3)$ for $\omega = \omega_1 + \omega_3$. Here in the first two cases M_H is the direct sum of the H-modules generated by nonzero highest and lowest weight vectors. Applying Lemma 2.32 for $\omega = \omega_2$ or $2\omega_1$ and the arguments in Part A of the proof of Theorem 1.1 for regular unipotent elements in Section 3 for $\omega = \omega_1 + \omega_3$, we deduce that in all the cases considered y has at least 4 nontrivial blocks on M_H.

Now let $G \neq D_4(K)$. Set $\Delta_1 = G(\varepsilon_1 + \varepsilon_2)$, $\Delta_2 = G(3, \dots, r)$, and $\Delta_1 = \Delta_1\Delta_2$. We can assume that $y \in \Delta_1$. By Theorem 2.17, $\varphi|\Delta$ has a composition factor $\rho \cong \rho_1 \otimes \rho_2$ where $\rho_i \in \mathrm{Irr}\,\Delta_i$ and $\omega(\rho_i) \neq 0$. One easily observes that $\Delta_2 \cong B_{r-2}(K)$ or $D_{r-2}(K)$ for $G = B_r(K)$ or $D_r(K)$, respectively. So $\dim \rho_2 \geqslant 4$. Since $\rho_1(y_1) \neq 1$, this forces that $\rho(y)$ has at least 4 nontrivial Jordan blocks. To complete the proof, apply Lemma 2.5 and conclude that $\varphi(y)$ has at least 4 such blocks in all cases considered in Item IV. Now Theorem 1.1 is proved. □

§5. Proof of Theorem 1.3

In this section Theorem 1.3 is proved.

Proof of Theorem 1.3. In what follows $u \in G$ is a nontrivial unipotent element. Since $r > 1$, by Corollary 2.9, one can suppose that φ is p-restricted. Set $d_i = d_i(u)$. Passing to the dual representation if necessary, we can and shall assume that $a_i \neq 0$ for some $i \leqslant n/2$ if $G = A_r(K)$.

1. First let $|u| = p$. We proceed to reduce the question to regular unipotent elements. Suppose that u is not regular. Apply Lemma 4.1. If the assertion 1) of this lemma holds, Corollary 2.23 implies that $\varphi(u)$ has at least two nontrivial Jordan blocks or $n = 4$, $G = A_3(K)$ or $C_2(K)$, $d_1 = d_2 = 2$, and $\omega = \omega_2$. (Recall that $\omega \neq \omega_3$ or ω_4 for $G = D_4(K)$ as $\varphi \not\sim$ St.) Now consider these exceptional cases. Let $G = A_3(K)$. Then $M \cong \bigwedge^2 V$. If $p > 2$, arguing as in the proof of Lemma 2.32.1), we can deduce that $\varphi(u)$ has one block of size 3 and other blocks are trivial. For $p = 2$ Lemma 2.32.2) forces that $\varphi(u)$ has two blocks of size 2. If $G = C_2(K)$, then u is a short root element and φ can be regarded as the standard realization of $B_2(K)$. So again $\varphi(u)$ has one block of size 3 and other blocks are trivial. These elements appear in Item 3) of Theorem 1.3.

Next, let the assertion 2) of Lemma 4.1 hold. Put $S_j = S_j(u)$, $S = S(u)$, and $z = z_1 z_2$ where $z_j \in S_j$ are such as in Lemma 4.1. Using Theorem 2.17 and Corollary 2.27.2) as in Item II of the proof of Theorem 1.1 in Section 3, we get that the restriction $\varphi|S$ has a composition factor $\psi \cong \psi_1 \otimes \psi_2$ where $\psi_j \in \operatorname{Irr} S_j$ and $\omega(\psi_j) \neq 0$. Suppose that $\dim \psi_k > 2$ for some $k \in \{1, 2\}$. Then $d_{\psi_k}(z_k) > 2$ or $\psi_k(z_k)$ has at least two Jordan blocks. Since $z_j \neq 1$ and ψ_j are nontrivial for $j = 1, 2$, Corollary 2.8 yields that $\psi(z)$ has at least two nontrivial blocks. Hence the same holds for $\varphi(u)$ by Lemma 2.5. Now let $\dim \psi_1 = \dim \psi_2 = 2$. Then the construction of the groups S_j in Item 2) of Lemma 4.1 implies that $S_j \cong A_1(K)$ or $A_1(K) \times A_1(K)$ and one of the following holds: a) $G = B_3(K)$, $d_1 = 3$, and $d_2 = d_3 = 2$; b) $G = C_2(K)$ and $d_1 = d_2 = 2$; c) $G = D_4(K)$ and $J(u) = (2, 2, 2, 2)$. Theorem 2.1 b) and c) yields that u is conjugate to an element of $G(1)$ in Case b) and to an element of $G(1, 2, 3)$ or $G(1, 2, 4)$ in Case c). So in these cases the assertion 1) of Lemma 4.1 holds and the theorem is already proved for such elements.

It remains to consider Case a). Set $G_1 = G(1, 3)$. By Lemma 2.34, $\varphi(u)$ has 3 nontrivial Jordan blocks for $\omega = \omega_3$. So now assume that $\omega \neq \omega_3$. It follows from the proof of Lemma 2.34 that we can suppose that u is a regular unipotent element of G_1. Put

$$m = \begin{cases} v^+ & \text{if } \ a_1 a_3 \neq 0 \ \text{ and } \ a_1 + a_3 > 2, \\ X_{-2} v^+ & \text{if } \ a_2 \neq 0 \text{ and } \ a_3 = 0 \ \text{ or } \ a_1 = a_3 = 1, \\ v(2,1,1) & \text{for } \ \omega = a_1 \omega_1, \\ X_{-2}^2 v^+ & \text{if } \ a_1 = 0, \ a_2 > 1, \text{ and } \ a_3 \neq 0, \\ v(2,3,1) & \text{if } \ \omega = \omega_1 + \omega_3, \, \omega_2 + a_3 \omega_3 \ \text{ with } \ a_3 > 0, \text{ or } \ a_3 \omega_3. \end{cases}$$

By Lemmas 2.18 and 2.20, in all cases $m \neq 0$ and is fixed by the groups $\mathcal{X}_1$ and $\mathcal{X}_3$. Set $c_1 = \omega^1(m)$ and $c_3 = \omega^3(m)$. We easily conclude that $c_1c_3 \neq 0$ and $1 < c_j < p$ for at least one $j \in \{1,3\}$ (here it is essential that due to our assumptions $k > 1$ if $\omega = k\omega_1$ or $k\omega_3$). It is clear that the indecomposable G_1-module generated by m has a composition factor $M_1 \cong M(c_1) \otimes M(c_3)$ where the first multiplier is a $G(1)$-module and the second one is a $G(3)$-module. Hence m has at least two nontrivial blocks on M_1 by Corollary 2.8. Lemma 2.5 completes the proof for nonregular elements of order p.

Now let u be a regular unipotent element and $D(\varphi, u)$ be the parameter determined by formula (2). By [20, Proposition 2], $\varphi(u)$ has at least two Jordan blocks of size p if $D(\varphi, u) \geqslant p$. So it suffices to consider the cases where $D(\varphi, u) < p$. We apply the formulae for computing the minimal polynomials given before Theorem 2.15 to calculate $D(\varphi, u)$. Observe that $p > 2$ and $G = A_2(K)$ if $p = 3$.

First assume that $p = 3$. As $\varphi \not\sim \mathrm{St}$, the sum $a_1 + a_2 > 1$. Hence $D(\varphi, u) = 2(a_1 + a_2) > p$ for all representations φ under consideration.

Now suppose that $p > 3$. In [20, Theorem 1] the estimates for the difference between $d_\varphi(u)$ and the size of the second biggest Jordan block of $\varphi(u)$ are given. These estimates imply that $\varphi(u)$ has more than one nontrivial block if $d_\varphi(u) > 7$, if $d_\varphi(u) > 5$ and $G = A_r(K)$ or $\varphi \not\sim \varphi(\omega_r)$, and if $d_\varphi(u) > 3$ and $\omega \neq a_i\omega_i$. Therefore Theorem 2.15 yields that it remains to consider the following cases: i) $G = A_2(K)$, $\omega = 2\omega_1$; ii) $G = A_3(K)$ or $C_2(K), \omega = \omega_2$; iii) $G = B_3(K), \omega = \omega_3$. According to [10, Tables 6.6, 6.7, 6.22, and 6.23], in all these cases $\dim\varphi = d_\varphi(u) + 1$. Hence $\varphi(u)$ has only one nontrivial block. This completes the proof of the theorem for elements of order p.

2. Next, let $|u| = p^{s+1} > p$. Theorem 1.1 implies that it suffices to assume that u is regular and consider the representations that appear in Items 2)–5) of that theorem, in all other situations $\varphi(u)$ has at least two nontrivial blocks.

First let $G = A_r(K)$, $n = p^s + 1$, and $\omega = \omega_2$ or $2\omega_1$ ($p > 2$ in the latter case). Set $H = G(1, \ldots, r-1)$ and $M_H = KHv^+$. By Theorem 2.16, M_H is an irreducible H-module with highest weight ω_2 or $2\omega_1$, respectively. We may assume that $u = \prod_{i=1}^{r} x_i(1)$. Since the element $x_r(1)$ fixes all vectors in M_H, it is clear (apply formula (1) stated in the third paragraph of

Sec. 2) that u preserves M_H and acts on M_H as the element $u' = \prod_{i=1}^{r-1} x_i(1)$. Naturally, u' is a regular unipotent element of H. If $\omega = 2\omega_1$ or $n \geqslant 6$, Theorem 1.1 and the facts established in Item 1 of this proof yield that u' and u have at least two nontrivial blocks on M_H. Then Lemma 2.5 implies that $\varphi(u)$ has at least two such blocks.

Now assume that $G = C_r(K)$, n and ω are such as in the previous paragraph, and $n \geqslant 6$ if $\omega = \omega_2$. Set $G^+ = A_{2r-1}(K)$ and denote by M^+ the irreducible G^+-module with highest weight ω_2 or $2\omega_1$ for $\omega = \omega_2$ or $2\omega_1$, respectively. As we have mentioned before, the group G can be naturally embedded into G^+. If $\omega = \omega_2$, it is well known that $M^+|G$ is isomorphic to the direct sum of M and the trivial module, for instance, this is mentioned in the proof of [10, Theorem 5.1]. For $\omega = 2\omega_1$ the module $M \cong M^+|G$ (see [13, Theorem 1.1 and Table 1]). Hence the facts proved in the previous paragraph imply that Theorem 1.3 holds in this case as well.

We have already shown in the proof of Theorem 1.1 in Section 3 that $\varphi(u)$ has just one nontrivial Jordan block if $p = 3$, $G = A_3(K)$ or $C_2(K)$, and $\omega = \omega_2$ or $p = 3$ or 5, $G = B_3(K)$, and $\omega = \omega_3$.

Next, let $p = 2$, $G = A_3(K)$, and $\omega = \omega_2$. Then $\varphi(G)$ is conjugate to the group $SO_6(K)$, i.e., to the standard realization of $D_3(K)$. It is well known that $J(\varphi(u)) = (4, 2)$.

Several cases considered below are settled with the help of Lemma 2.4. Suppose that one of the following holds: a) $p = 3$, $G = A_4(K)$, and $\omega = \omega_2$; b) p and ω are such as in a), $G = C_3(K)$; c) $p = 5$ or 7, $G = B_4(K)$, and $\omega = \omega_4$. Set $H = G(2, \dots, r)$ and let u_H be a regular unipotent element of H. Recall that $d_\varphi(u) = 7$, 9, and 11 in Cases a), b), and c), respectively (see Part A of the proof of Theorem 1.1 in Section 3 for Cases a) and b) and Item B2.VI of the same proof for Case c)). It is well known and follows from [24, Proposition 2.36] that in Case a) $\varphi|H \cong \varphi(\omega_1) \oplus \varphi(\omega_2)$. Using [32, formula (1) and the proof of Theorem 1], one can deduce that in Case b) $\varphi|H$ is a direct sum of two copies of $\varphi(\omega_1)$ and $\varphi(\omega_2)$. It is also well known and follows from Theorem 2.16 that in Case c) $\varphi|H$ is a direct sum of two copies of $\varphi(\omega_3)$. Observe that in each case for every irreducible component ρ of $\varphi|H$ we know all Jordan block sizes of $\rho(u_H)$, here we use the arguments in Part A and Item B2.III of the proof of Theorem 1.1 in Section 3 for the representations $\varphi(\omega_2)$ and $\varphi(\omega_3)$, respectively. So we have $J(\varphi(u_H)) = (5, 4, 1)$ in Case a), $(5, 4, 4)$ in Case b), and $(7, 7, 1, 1)$ in

Case c). Hence $\dim(u_H - 1)M = 7, 10$, and 12, respectively. By Lemma 2.4, $\dim(u-1)M \geqslant \dim(u_H - 1)M$. Taking into account the value of $d_\varphi(u)$, one can conclude that $\varphi(u)$ has more than one nontrivial Jordan block in Cases a), b), and c).

It remains to consider the cases where $p = 3$, $G = A_5(K)$, and $\omega = \omega_2$, or $p = 5$ or 7, $G = D_5(K)$, and $\omega = \omega_4$ (recall that $\varphi(\omega_4) \sim \varphi(\omega_5)$ for $G = D_5(K)$). Here we use the results for $A_4(K)$ and $B_4(K)$, respectively, just proven in the previous paragraph. For $G = A_5(K)$ the arguments are similar to those in this proof that were applied for $n = p^s + 1$ and such ω. For $G = D_5(K)$ we argue as in the proof of Theorem 1.1 for regular unipotent elements of the group $D_r(K)$ (see Section 4, Part I of the proof) applying [13, Theorem 1.1 and Table 1]. Now all the possibilities have been considered. The theorem is proved. □

References

1. A. Borel, *Properties and linear representations of Chevalley groups.* — In: Seminar on Algebraic Groups and Related Finite Groups (eds. A. Borel et al.). Lect. Notes Math. **131** Springer, 1970, pp. 1–55.
2. N. Bourbaki, *Groupes et algèbres de Lie.* Chaps. IV–VI. Paris, Hermann, 1968.
3. N. Bourbaki, *Groupes et algèbres de Lie.* Chaps. VII–VIII. Paris, Hermann, 1975.
4. R. W. Carter, *Finite groups of Lie type: conjugacy classes and complex characters.* Wiley, Chichester, 1985.
5. W. Feit, *The representation theory of finite groups.* North-Holland, Amsterdam, 1982.
6. P. M. Gudivok, V. P. Rudko, *Tensor products of representations of finite groups* (in Russian). Uzhgorod Univ., Uzhgorod, 1985.
7. J.C. Jantzen, *Darstellungen halbeinfacher algebraicher Gruppen und zugeordnetekontravariante Formen.* — Bonner Math. Schr. **67** (1973).
8. J. C. Jantzen, *Representations of Algebraic Groups.* Second Edition, Providence, AMS, 2003.
9. M. W. Liebeck, G. M. Seitz, *Unipotent and nilpotent classes in simple algebraic groups and Lie algebras.* Math. Surv. Monogr. Vol. 180. Providence, AMS, 2012.
10. F. Lubeck, *Small degree representations of finite Chevalley groups in defining characteristic.* — London Math. Soc. J. Comput. Math. **4** (2001), 135–169.
11. A. A. Premet, I. D. Suprunenko, *Quadratic modules for Chevalley groups over fields of odd characteristic.* — Math. Nachrichten **110** (1983), 65–96.
12. J. Saxl, G. M. Seitz, *Subgroups of algebraic groups containing regular unipotent elements.* — J. London Math. Soc. (2) **55** (1997), 370–386.
13. G. M. Seitz, *The maximal subgroups of classical algebraic groups.* Memoirs of the AMS **67** (1987), no. 365.
14. S. Smith, *Irreducible modules and parabolic subgroups.* — J. Algebra **75** (1982), 286–289.

15. T. A. Springer, R. Steinberg, *Conjugacy classes.* — In: Seminar on algebraic groups and related finite groups (eds. A. Borel et al.). Lect. Notes Math. **131** Springer, 1970, pp. 167–266.
16. R. Steinberg, *Representations of algebraic groups.* — Nagoya Math. J. **22** (1963), 33–56.
17. R. Steinberg, *Lectures on Chevalley groups.* Mimeographed Lect. Notes, Yale Univ. Math. Dept., New Haven, Conn., 1968.
18. I. D. Suprunenko, *Minimal polynomials of elements of order p in irreducible representations of Chevalley groups over fields of characteristic p.* — Siberian Adv. Math. **6** (1996), 97–150.
19. I. D. Suprunenko, *On Jordan blocks of elements of order p in irreducible representations of classical groups with p-large highest weights.* — J. Algebra **191** (1997), 589–627.
20. I. D. Suprunenko, *The second Jordan block and recognition of representations.* — Dokl. NAN Belarusi **47** (2003), no. 1, 48–52 (in Russian).
21. I. D. Suprunenko, *The minimal polynomials of unipotent elements in irreducible representations of the special linear group over a field of characteristic two.* — Dokl. NAN Belarusi **47** (2003), no. 1, 9–13 (in Russian).
22. I. D. Suprunenko, *Unipotent elements of prime order in representations of exceptional algebraic groups: the second Jordan block.* — Dokl. NAN Belarusi **49** (2005), no. 4, 5–9 (in Russian).
23. I. D.Suprunenko, *On the behaviour of unipotent elements in representations of the classical groups with large highest weights.* — Dokl. NAN Belarusi, **49** (2005), no. 5, 11–15 (in Russian).
24. I. D. Suprunenko, *The minimal polynomials of unipotent elements in irreducible representations of the classical groups in odd characteristic.* Memoirs of the AMS **200** (2009), no. 939.
25. I. D. Suprunenko, *On the block structure of regular unipotent elements from subsystem subgroups of type $A_1 \times A_2$ in representations of the special linear group.* — Zap. Nauchn. Semin. POMI **388** (2011), 247–269 (in Russian).
26. I. D. Suprunenko, *Unipotent elements of nonprime order in representations of the classical groups: two big Jordan blocks.* — Dokl. NAN Belarusi **55** (2011), no. 4, 21–26 (in Russian).
27. I. D. Suprunenko, A. E. Zalesskii, *Truncated symmetric powers of the natural realizations of the groups $SL_m(P)$ and $Sp_m(P)$ and their restrictions to subgroups.* — Siber. Math. J. **31** (1990), 555–566.
28. I. D. Suprunenko, A. E. Zalesskii, *On restricting representations of simple algebraic groups to semisimple subgroups with two simple components.* — Trudy Inst. Matem. **13** (2005), no. 2, 109–115.
29. D. Testerman, A. E. Zalesski, *Irreducibility in algebraic groups and regular unipotent elements.* To appear in Proceedings of the Amer. Math. Soc.
30. A. Wagner, *Groups generated by elations.* — Abh. Math. Sem. Univ. Hamburg **41** (1974), 190–205.
31. A. E. Zalesskii, V. N. Serezhkin, *Linear groups generated by transvections.* — Math. USSR-Izv. **10**, No. 1 (1976), 25–46.

32. A. E. Zalesskii, I. D. Suprunenko, *Representations of dimensions* $(p^n \pm 1)/2$ *of the symplectic group of degree* $2n$ *over a field of characteristic* p. — Vesti AN BSSR. Ser. Fiz.-Mat. nauk. No. 6 (1987), 9–15 (in Russian).

Institute of Mathematics
National Academy of Sciences of Belarus
Surganova 11, Minsk, 220072, Belarus
E-mail: suprunenko@im.bas-net.by

Поступило 25 октября 2012 г.