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INFINITE GROUPS WITH RANK RESTRICTIONS ON SUBGROUPS

Abstract. Classical results by Mal'cev and Šunkov show that locally nilpotent groups and locally finite groups of infinite rank must contain some abelian subgroups of infinite rank. In recent years, many authors have studied groups in which all subgroups of infinite rank have a given property (which can be either absolute or of embedding type). Results from these researches and some new contributions to this topic are described in this paper.

Dedicated to Nikolai Vavilov on the occasion of his 60th birthday

§1. Introduction

A group G is said to have *finite (Prüfer) rank* r if every finitely generated subgroup of G can be generated by at most r elements, and r is the least positive integer with such property. If there is no such r, the group G has *infinite rank*. Thus groups of rank 1 are just the locally cyclic groups, while any free non-abelian group has infinite rank. A classical theorem of A. I. Mal'cev [14] states that a locally nilpotent group of infinite rank must contain an abelian subgroup of infinite rank, and V. P. Šunkov [27] proved that a similar result holds for locally finite groups. On the other hand, Y. I. Merzljakov [16] has shown that there exist locally soluble groups of infinite rank in which every abelian subgroup has finite rank. These results suggest that the behavior of subgroups of infinite rank in a (generalized) soluble group has a strong influence on the structure of the whole group. Such influence has been investigated in a series of recent papers, and the aim of this paper is to give a survey of results in this area and to announce some new contributions to the topic.

Key words and phrases: rank; locally soluble group; absolute property; embedding property.

Although we will state our results for locally soluble groups, most of them have been proved in the much larger universe formed by the so-called *strongly locally graded groups*, which contains in particular all locally (soluble-by-finite) groups. Such groups can be defined in the following way.

Recall that a group G is *locally graded* if every finitely generated non-trivial subgroup of G contains a proper subgroup of finite index. Let $\mathfrak{D}$ be the class of all periodic locally graded groups, and let $\bar{\mathfrak{D}}$ be the closure of $\mathfrak{D}$ by the operators $\acute{\boldsymbol{P}}, \grave{\boldsymbol{P}}, \boldsymbol{R}, \boldsymbol{L}$ (for the definitions of these and other relevant operators on group classes we refer to the first chapter of [20]). It is easy to prove that any $\bar{\mathfrak{D}}$-group is locally graded, and that the class $\bar{\mathfrak{D}}$ is closed with respect to forming subgroups. Moreover, N.S. Černikov [1] proved that every $\bar{\mathfrak{D}}$-group with finite rank contains a locally soluble subgroup of finite index. Obviously, all residually finite groups belong to $\bar{\mathfrak{D}}$, and hence the consideration of any free non-abelian group shows that the class $\bar{\mathfrak{D}}$ is not closed with respect to homomorphic images. We shall say that a group G is *strongly locally graded* if every section of G is a $\bar{\mathfrak{D}}$-group.

Most of our notation is standard and can be found in [20].

§2. Absolute properties

We shall say that a property θ pertaining to subgroups of a group is *absolute* if in any group G all subgroups isomorphic to some θ-subgroup of G are likewise θ-subgroups. Thus a property θ is absolute if and only if there exists a group class $\mathfrak{X}$ such that in any group G a subgroup X has the property θ if and only if X belongs to $\mathfrak{X}$. The subgroup property θ is called an *embedding property* if in any group G all images of θ-subgroups under automorphisms of G likewise have the property θ. Any absolute property is trivially an embedding property, but many relevant embedding properties (like for instance normality, subnormality, pronormality) are not absolute. In this section we will consider groups in which all proper subgroups of infinite rank have a given absolute property, while the corresponding investigation for embedding properties will be the subject of the next section.

Let $\mathfrak{X}$ be a class of groups. A group G is *minimal non-*$\mathfrak{X}$ if it is not an $\mathfrak{X}$-group, but all its proper subgroups belong to $\mathfrak{X}$. The description of minimal non-$\mathfrak{X}$ groups for a certain group class $\mathfrak{X}$ can of course be considered as the first step in the investigation of groups (of infinite rank) whose proper subgroups of infinite rank belong to $\mathfrak{X}$. On the other hand, it is easy to prove that any locally soluble group, whose proper subgroups are abelian, is either abelian or finite, i.e., locally soluble minimal non-abelian groups are finite.

Recall here that a group G is a *T-group* if all its subnormal subgroups are normal, i.e., if normality is a transitive relation in G, and that G is an *FC-group* if each element of G has only finitely many conjugates. It is known that any soluble minimal non-T group is finite and that any soluble group whose proper subgroups have the FC-property either is an FC-group or has finite rank. All these results suggest that groups with many subgroups in a given group class $\mathfrak{X}$ either belong to $\mathfrak{X}$ or must be small in some sense, at least for several natural choices of $\mathfrak{X}$.

We begin our discussion with a result dealing with nilpotent groups of given class.

Theorem 2.1. (M. R. Dixon–M. J. Evans–H. Smith [9]) *Let c be a positive integer and let G be a locally soluble group whose proper subgroups of infinite rank are nilpotent with class at most c. Then either G has finite rank or it is nilpotent with class at most c.*

For $c = 1$, the above result shows that if G is any locally soluble group of infinite rank, whose proper subgroups of infinite rank are abelian, then G itself is abelian. This fact is also a special case of the following theorem, which proves that also the class of soluble groups with given derived length can be detected from the behavior of subgroups of infinite rank.

Theorem 2.2. (M. R. Dixon–M. J. Evans–H. Smith [8]) *Let k be a positive integer and let G be a soluble group whose proper subgroups of infinite rank have derived length at most k. Then either G has finite rank or it has derived length at most k.*

If G is a group and x is any element of G, the conjugacy class of x is obviously contained in the coset xG' (where G' is the commutator subgroup of G), and so groups with finite commutator subgroup form a relevant class of FC-groups (the so-called *BFC-groups*, or groups with boundedly finite conjugacy classes). Thus our next result can be considered as a first step in the study of groups whose proper subgroups of infinite rank have the FC-property.

Theorem 2.3. (M. De Falco–F. de Giovanni–C. Musella–N. Trabelsi [6]) *Let G be a locally soluble group of infinite rank. If all proper subgroups of infinite rank of G have (locally) finite commutator subgroup, then also the commutator subgroup G' of G is (locally) finite.*

The main result on groups of infinite rank in which all proper subgroups of infinite rank have the FC-property shows that such groups either have the FC-property or are minimal non-FC, at least within the universe of locally soluble groups.

Theorem 2.4. (M. De Falco–F. de Giovanni–C. Musella–N. Trabelsi [7]) *Let G be a locally soluble group whose proper subgroups of infinite rank have the FC-property. Then either G has finite rank or all its proper subgroups have the FC-property.*

A group G is called *metahamiltonian* if all its non-abelian subgroups are normal. Metahamiltonian groups have been introduced by G. M. Romalis and N. F. Sesekin ([21–23]), who proved in particular that locally soluble groups with such property have finite commutator subgroup (of prime-power order), and so they have the FC-property. As metahamiltonian groups form a local class, it is not difficult to show that locally soluble minimal non-metahamiltonian groups must be finite (see for instance [2], Lemma 4.2). Moreover:

Theorem 2.5. (M. De Falco–F. de Giovanni–C. Musella–N. Trabelsi [6]) *Let G be a locally soluble group of infinite rank. If all proper subgroups of infinite rank of G are metahamiltonian, then G itself is metahamiltonian.*

Recall also that a group G is said to be *quasihamiltonian* if $XY = YX$ for all subgroups X and Y of G. A classical result by K. Iwasawa shows that quasihamiltonian groups are locally nilpotent and play a crucial role in the theory of subgroup lattices of groups (the monograph [25] can be used as a general reference for results in this large chapter of group theory).

Theorem 2.6. (M. De Falco–F. de Giovanni–C. Musella–N. Trabelsi [6]) *Let G be a locally soluble group of infinite rank. If all proper subgroups of infinite rank of G are quasihamiltonian, then G itself is quasihamiltonian.*

Since any simple group has the T-property, it is clear that the class of T-groups is not subgroup closed. On the other hand, it is known that any finite soluble T-group is a $\bar{T}$-group, i.e., all its subgroups likewise have the T-property; moreover, all finite $\bar{T}$-groups are soluble, and torsion-free soluble $\bar{T}$-groups are abelian. The main results on infinite soluble T-groups can be found in an important paper by D. J. S. Robinson [19].

Theorem 2.7. (M. De Falco–F. de Giovanni–C. Musella [3]) *Let G be a locally soluble group whose proper subgroups of infinite rank have the T-property. Then either G has finite rank or it is a soluble $\bar{T}$-group.*

§3. Embedding properties

Normality is probably the most natural and relevant embedding property. For this reason many authors have investigated the structure of groups which are rich of normal subgroups in some sense. Recall that a group is called a *Dedekind group* if all its subgroups are normal. It is well known that any Dedekind group is either abelian or a direct product $Q_8 \times A$, where A is a periodic abelian group with no elements of order 4.

Theorem 3.1. (M. J. Evans–Y. Kim [11]) *Let G be a locally soluble group whose subgroups of infinite rank are normal. Then either G has finite rank or it is a Dedekind group.*

Evans and Kim obtained this result as a special case of the following theorem, dealing with subnormal subgroups of bounded defect. It should be seen in relation with a relevant result by J. E. Roseblade [24], who proved that if G is any group in which all subgroups are subnormal with bounded defect, then G is nilpotent with bounded class.

Theorem 3.2. (M. J. Evans–Y. Kim [11]) *For each positive integer k there exists a positive integer $f(k)$, depending only on k, such that, if G is any locally soluble group in which all subgroups of infinite rank are subnormal with defect at most k, then either G has finite rank or it is nilpotent with class at most $f(k)$.*

It is known that in the statement of Roseblade's theorem the bound condition on the defects cannot be omitted (and the same remark of course applies also to Theorem 3.2). On the other hand, without this assumption, W. Möhres proved the following theorem, which is one of the most important results in the theory of infinite groups over the last 25 years.

Theorem 3.3. (W. Möhres [17]) *Let G be a group in which all subgroups are subnormal. Then G is soluble.*

If the subnormality condition (without bound assumption) is imposed only to subgroups of infinite rank, the following can be proved.

Theorem 3.4. (L. A. Kurdachenko–H. Smith [13]) *Let G be a locally (soluble-by-finite) group in which all subgroups of infinite rank are subnormal. Then either G has finite rank or it is soluble.*

In any group we may consider the property for a subgroup to be either abelian or normal, which is obviously related to the definition of metahamiltonian groups. This is of course an embedding property which is not absolute. The case of groups in which such property is imposed to subgroups of infinite rank is completely described by our next result.

Theorem 3.5. (M. De Falco–F. de Giovanni–C. Musella–Y. P. Sysak [4]) *Let G be a locally soluble group in which every non-abelian subgroup of infinite rank is normal. Then either G has finite rank or it is metahamiltonian.*

Recall now that a subgroup X of a group G is said to be *quasinormal* (or *permutable*) if $XY = YX$ for every subgroup Y of G. Thus a group G is quasihamiltonian if and only if all its subgroups are quasinormal. It is known that quasinormal subgroups of arbitrary groups are ascendant (i.e., they belong to some ascending series of the group). For quasinormal subgroups the following result has recently been obtained.

Theorem 3.6. (M. R. Dixon–Y. Karatas [10]) *Let G be a locally soluble group in which all subgroups of infinite rank are quasinormal. Then either G has finite rank or it is quasihamiltonian.*

In relation to T-groups, we consider finally the imposition of the embedding property defined by normality only to subnormal subgroups of infinite rank.

Theorem 3.7. (M. De Falco–F. de Giovanni–C. Musella–Y. P. Sysak [5]) *Let G be a periodic soluble group whose subnormal subgroups of infinite rank are normal. Then either G has finite rank or it is a T-group.*

The hypothesis that the group G is periodic cannot be omitted in the above statement, even in the metabelian case, as the following example shows. Let A be the additive group of rational numbers, and for each prime number p let x_p be the automorphism of A defined by setting $ax_p = pa$ for each element a of A. Then the direct product

$$X = \underset{p}{\mathrm{Dr}}\langle x_p \rangle$$

is a free abelian subgroup of infinite rank of the full automorphism group of A, and the semidirect product

$$G = X \ltimes A$$

is a torsion-free metabelian group. Clearly, A is a minimal normal subgroup of G, and so G is not a T-group. On the other hand, it is easy to prove that any subnormal subgroup of G either is contained in A or contains A; in particular, all subnormal subgroups of infinite rank of G contain A and hence they are normal in G.

On the other hand, for arbitrary soluble groups the situation is better at least when the Fitting subgroup (i.e., the subgroup generated by all nilpotent normal subgroups) has infinite rank. This is proved by the following result.

Theorem 3.8. (M. De Falco–F. de Giovanni–C. Musella–Y. P. Sysak [5]) *Let G be a soluble group whose subnormal subgroups of infinite rank are normal. If the Fitting subgroup of G has infinite rank, then G is a T-group.*

Some further results on the imposition of embedding properties to subgroups of infinite rank can be found in a paper of N.N. Semko and S.N. Kuchmenko [26], where in particular the authors consider almost normality and nearly normality. Here a subgroup X of a group G is said to be *almost normal* in G if it has only finitely many conjugates in G, or equivalently if its normalizer $N_G(X)$ has finite index in G, and X is called *nearly normal* if it has finite index in its normal closure. Two famous theorems of B. H. Neumann [18] prove that in a group G all subgroups are almost normal if and only if the centre $Z(G)$ has finite index, and that groups with finite commutator subgroups are characterized by the property that each subgroup is nearly normal.

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