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COMMUTATORS WITH SOME SPECIAL ELEMENTS IN CHEVALLEY GROUPS

Abstract. Let $G = \widetilde{G}(K)$ where $\widetilde{G}$ is a simple and simply connected algebraic group that is defined and quasi-split over a field K. We consider commutators in G with some regular elements. In particular, we prove (under some additional condition) that every unipotent regular element of G is conjugate to a commutator $[g, v]$, where g is any fixed semisimple regular element of G, and that every non-central element of G is conjugate to a product $[g, \sigma][u_{\mathrm{reg}}, \tau]$, where g is some special element of the group G and u_{reg} is some regular unipotent element of G.

Introduction

Estimation of the commutator length is a popular topic in group theory. The well-known Ore conjecture claims that the commutator length of a finite simple group is equal to one, that is, every element in a finite simple group is a single commutator. Since for sporadic groups this fact has been established by computer calculations, the question was reduced to the case of finite simple groups of Lie type. Using special properties of the Gauss decomposition and of the Bruhat decomposition of such groups, the authors of the present paper have proved the Ore conjecture for groups of Lie type over fields that contain more than eight elements ([6]). However, such methods do not work for small fields. R. Gow in his elegant paper [8] has shown that every semisimple element of a finite simple group of Lie type G is a commutator $[g, x]$ for some regular semisimple element $g \in G$. However, his proof which is based on a property of the Steinberg representation, does not imply the same fact for unipotent elements. The final proof of the Ore conjecture was obtained by M. W. Liebeck, E. A. O'Brien, A. Shalev, and P. H. Tiep in [9] by means of character theory. The proof for small fields which was done in [9] is very complicated and is based on

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ingenious calculations. In spite of the complete solution of the Ore conjecture there are some open questions connected with the commutator length in finite groups of Lie type and more generally in Chevalley groups. There is the obvious question, if every element in a Chevalley group G can be written as a commutator $[g, x]$ for some special element $g \in G$.

We consider a group $G = \widetilde{G}(K)$ where $\widetilde{G}$ is a simple and simply connected algebraic group that is defined and quasi-split over a field K. If $\widetilde{G}$ is not split over K we assume that K is a finite field. We also assume that the root system corresponding to G is not A_1 or A_2 and that K is a perfect field of a characteristic that is not a bad prime for $\widetilde{G}$. Under these assumptions we prove that every regular unipotent element of G is conjugate to a commutator of the form $[u\dot{w}_c, x]$, where w_c is some special Coxeter element of the Weyl group corresponding to G, and u is any fixed unipotent element from a fixed chosen Borel subgroup of G (Theorem 2.1). In particular, this implies that if, in addition, the homological dimension of $K \leq 1$, then every regular unipotent element is conjugate to a commutator $[g, x]$, where g is a fixed regular semisimple element. This gives some extension of R. Gow's theorem but only for fields of a characteristic which is not bad for the corresponding root system. At the end we prove (under the same conditions) that every non-central element of G is conjugate to a product of the form $[u\dot{w}_c, x][u_{\mathrm{reg}}, y]$, where u_{reg} is some regular unipotent element of G (Theorem 2.4). Possibly, some constructions that are used in this paper will also be helpful in the case when K is a local ring.

§1. Preliminaries

1.1. Simple algebraic groups and Chevalley groups. Below K is a field and $\widetilde{G}$ is a simple and simply connected algebraic group that is defined and quasi-split over K. If $\widetilde{G}$ is quasi-split but not split over K, we will assume that K is a finite field.

We shall identify $\widetilde{G}$ with the point group $\widetilde{G}(\overline{K})$ where $\overline{K}$ is the algebraic closure of K.

The group $\widetilde{G}$ corresponds to an irreducible root system $\widetilde{R}$ of rank $\widetilde{r}$ generated by a fixed simple root system $\widetilde{\Pi} = \{\widetilde{\alpha}_1, \dots, \widetilde{\alpha}_{\widetilde{r}}\}$; the Chevalley group $G = \widetilde{G}(K)$ corresponds to an irreducible root system R ([3, 13.3]) of rank r generated by a fixed simple root system $\Pi = \{\alpha_1, \dots, \alpha_r\}$ (the notation and numerations of roots corresponds to [2, Tables I–X]).

If $\widetilde{G}$ is split, then $\widetilde{R} = R$ and $\widetilde{\Pi} = \Pi$; if $\widetilde{G}$ is not split, then $G = \widetilde{G}(K) = \widetilde{G}(\overline{K})^F$ for an appropriate Frobenius map $F : \widetilde{G}(\overline{K}) \to \widetilde{G}(\overline{K})$ ([4, 1.19]); in the latter case, the set of orbits ${}^m\widetilde{R} = \widetilde{R}/\langle F \rangle$ is associated with the root system R (here F acts on $\widetilde{R}$ as a graph automorphism of the order $m = 2, 3$) ([3, 13.3]); recall the correspondence ${}^m\widetilde{R} \leftrightarrow R$:

$$ {}^2A_{2r} \leftrightarrow B_r, \quad {}^2A_{2r-1} \leftrightarrow C_r, \quad {}^2D_{r+1} \leftrightarrow B_r, \quad {}^3D_4 \leftrightarrow G_2, \quad {}^2E_6 \leftrightarrow F_4. $$

(Below we will not consider the twisted groups of the type ${}^2B_2$, ${}^2G_2$, ${}^2F_4$ which cannot be presented as a point group $\widetilde{G}(K)$ for any simple algebraic group $\widetilde{G}$.)

We fix a Borel subgroup $\widetilde{B}$ of $\widetilde{G}$ and its unipotent radical $\widetilde{U}$. We denote

$$ \widetilde{U}_1 = [\widetilde{U}, \widetilde{U}], \;\; \widetilde{U}_i = [\widetilde{U}, \widetilde{U}_{i-1}]. $$

Further, put $U = \widetilde{U}(K)$.

For $\alpha \in R$ we denote the root subgroup of G corresponding to α by X_α; for $\widetilde{\alpha} \in \widetilde{R}$ we denote the root subgroup of $\widetilde{G}$ corresponding to $\widetilde{\alpha}$ by $\widetilde{X}_{\widetilde{\alpha}}$. If $\widetilde{G}$ is a split group then $\alpha = \widetilde{\alpha}$ and $X_\alpha = G \cap \widetilde{X}_{\widetilde{\alpha}}$.

If $\widetilde{G}$ is not a split group, then every root $\alpha \in R$ corresponds to an F-orbit $\Delta_\alpha = \{\widetilde{\beta}_1, \dots, \widetilde{\beta}_e\}$ of roots $\widetilde{\beta}_i \in \widetilde{R}$, where $e = 1, 2$, or 3. Moreover, if $\alpha \in R^+$ then $\widetilde{\beta}_1, \dots, \widetilde{\beta}_e \in \widetilde{R}^+$. Let

$$ \Phi_\alpha = \{i_1\beta_1 + i_2\beta_2 + \dots + i_e\beta_e \;\;|\;\; 0 \le i_1, \dots, i_e \in \mathbb{Z}\} \cap \widetilde{R}. $$

Then ([3, 13.6.3])

$$ X_\alpha \subset \prod_{\widetilde{\beta} \in \Phi_\alpha} \widetilde{X}_{\widetilde{\beta}}. $$

Put

$$ X^*_\alpha = X_\alpha \cap \prod_{\widetilde{\beta} \in \Phi_\alpha \setminus \Delta_\alpha} \widetilde{X}_{\widetilde{\beta}}. $$

If $u \in X_\alpha$, then $u \equiv u_1 u_2 \cdots u_e \bmod X^*_\alpha$ where $u_i \in \widetilde{X}_{\widetilde{\beta}_i}$. We have

$$ u \notin X^*_\alpha \Leftrightarrow u_1 \ne 1, u_2 \ne 1, \dots, u_e \ne 1 \tag{1.1} $$

(see, [3, 13.6.3]).

Put

$$ \widetilde{U}^* = \langle \widetilde{X}_{\widetilde{\alpha}} \;\;|\;\; \widetilde{\alpha} \in \widetilde{R}^+ \setminus \widetilde{\Pi} \rangle, \;\;\; U^* = \widetilde{U}^*(K). $$

Then $\widetilde{U}^*$ is a connected unipotent algebraic group which is defined and split over K. (Indeed, if $\widetilde{G}$ is split then the variety $\widetilde{U}^*$ is a product of one-parameter groups $\widetilde{X}_{\widetilde{\alpha}}, \widetilde{\alpha} \in \widetilde{R}^+ \setminus \widetilde{\Pi}$ which are defined just over K. If

$\widetilde{G}$ is not split, then we assume that K is a finite field and then the Galois group $\Gamma = \mathrm{Gal}(\overline{K}/K)$ is generated by a Frobenius element. Moreover, if $F : \widetilde{G} \to \widetilde{G}$ is a corresponding Frobenius map the subgroup $\widetilde{U}^*$ is F-stable because the set of roots $\widetilde{R}^+ \setminus \widetilde{\Pi}$ is F-stable. Hence it is K-defined ([10, 11.2.8.b]). Since K is a perfect field the group $\widetilde{U}^*$ is split ([10, 14.2.7]).

Thus, there is an isomorphism of the K-varieties (see, [10, 14.3.10])

$$\widetilde{U}^* \approx A^{|\widetilde{R}^+| - \widetilde{r}} \tag{1.2}$$

(here A^n is the affine space of the dimension n).

1.2. Regular unipotent elements in Chevalley groups. An element $g \in G = \widetilde{G}(K)$ is called *regular* if it is regular in $\widetilde{G}$. A unipotent element $\widetilde{u} \in \widetilde{U}$ is regular in $\widetilde{G}$ if and only if

$$\widetilde{u} \equiv \widetilde{u}_1 \widetilde{u}_2 \cdots \widetilde{u}_{\widetilde{r}} \mod \widetilde{U}^* \tag{1.3}$$

for some $\widetilde{u}_i \in \widetilde{X}_{\widetilde{\alpha}_i}, \widetilde{u}_i \neq 1$ ([4, 5.1.3]); a unipotent element $u \in U$ is regular in G if and only if

$$u \equiv u_1 u_2 \cdots u_r \mod U^* \tag{1.4}$$

for some $u_i \in X_{\alpha_i} \setminus X^*_{\alpha_i}$ (indeed, if $\widetilde{G}$ is split, then (1.4) obviously follows from (1.3); if $\widetilde{G}$ is not split, then each root $\alpha_i \in \Pi$ corresponds to an F-orbit Δ_{α_i} of simple roots in $\widetilde{\Pi}$ and our assertion follows from (1.1) and (1.3)).

Recall ([4, 1.14], [11, I, §1]), that the characteristic $p > 0$ of the field K is called a *bad prime* for the root system $\widetilde{R}$ if

$p = 2$ for $\widetilde{R} = B_{\widetilde{r}}, C_{\widetilde{r}}, D_{\widetilde{r}}$;
$p = 2, 3$ for $\widetilde{R} = G_2, F_4, E_6, E_7$;
$p = 2, 3, 5$ for $\widetilde{R} = E_8$.

We say that the characteristic of the field K is *good* for $\widetilde{R}$ if $\mathrm{char}\, K = 0$ or $\mathrm{char}\, K = p$ is not a bad prime for $\widetilde{R}$.

Now let $u_{\mathrm{reg}} \in \widetilde{U}$ be a regular unipotent element and let $C_{\widetilde{G}}(u_{\mathrm{reg}})$ and $C_{\widetilde{U}}(u_{\mathrm{reg}})$ be its centralizers in $\widetilde{G}$ and $\widetilde{U}$, respectively. Then $C^0_{\widetilde{U}}(u_{\mathrm{reg}})$ is an abelian unipotent group (here $C^0_{\widetilde{U}}(u_{\mathrm{reg}})$ is the identity component of $C_{\widetilde{U}}(u_{\mathrm{reg}})$) and

$$\dim C^0_{\widetilde{U}}(u_{\mathrm{reg}}) = \widetilde{r} \tag{1.5}$$

([11, III,§1, 1.4, 1.14]). Moreover,

$$C_{\widetilde{U}}(u_{\mathrm{reg}}) = C^0_{\widetilde{U}}(u_{\mathrm{reg}}) \Leftrightarrow \mathrm{char}\, K \quad \text{is good for} \quad \widetilde{R} \tag{1.6}$$

([11, III, §1.14]).

Lemma 1.1.

$$[u_{\rm reg}, \widetilde{U}] = \widetilde{U}_1 = \widetilde{U}^*.$$

Proof. Consider the map

$$\Psi_{u_{\rm reg}} : \widetilde{U} \to \widetilde{U}_1 \tag{1.7}$$

which is defined by the formula $\Psi_{u_{\rm reg}}(x) = [u_{\rm reg}, x]$. The map $\Psi_{u_{\rm reg}}$ is a regular map of varieties. Moreover,

$$\begin{aligned}\Psi_{u_{\rm reg}}(x) = \Psi_{u_{\rm reg}}(y) &\Leftrightarrow [u_{\rm reg}, x] = [u_{\rm reg}, y] \\ &\Leftrightarrow u_{\rm reg} x^{-1} y u_{\rm reg}^{-1} = x^{-1} y \Leftrightarrow x^{-1} y \in C_{\widetilde{U}}(u_{\rm reg}).\end{aligned}$$

Hence every fiber of $\Psi_{u_{\rm reg}}$ is isomorphic (as a variety) to $C_{\widetilde{U}}(u_{\rm reg})$. Therefore (see, also (1.5))

$$\dim\ \Psi_{u_{\rm reg}}(\widetilde{U}) = \dim\ \widetilde{U} - \dim\ C_{\widetilde{U}}(u_{\rm reg}) = \mid \widetilde{R}^+ \mid - \widetilde{r}. \tag{1.8}$$

Since $\widetilde{U}_1 \subset \widetilde{U}^*$ ([4, 5.1.3]) and since $\widetilde{U}_1, \widetilde{U}^*$ are connected algebraic groups, the formulas (1.2) and (1.8) yield

$$\overline{\Psi_{u_{\rm reg}}(\widetilde{U})} = \widetilde{U}_1 = [\widetilde{U}, \widetilde{U}] = \widetilde{U}^* \tag{1.9}$$

(here $\overline{\Psi_{u_{\rm reg}}(\widetilde{U})}$ is the Zariski closure of $\Psi_{u_{\rm reg}}(\widetilde{U})$ in $\widetilde{U}_1$). Hence (1.7) is a dominant map and all fibers of (1.7) have the same dimension. Then (1.7) is an open map ([1, AG, 18.4]). Since $1 \in \Psi_{u_{\rm reg}}(\widetilde{U})$ we have $\widetilde{U}_i \cap \Psi_{u_{\rm reg}}(\widetilde{U}) \neq \varnothing$ for every $i \geq 1$. Hence $\widetilde{U}_i \cap \Psi_{u_{\rm reg}}(\widetilde{U})$ is a dense open subset of $\widetilde{U}_i$ for every i. Let $\widetilde{U}_{i_0}$ be in the center of $\widetilde{U}$. The commutator formula

$$[a, bc] = [a, b](b[a, c]b^{-1}) = [a, b][a, c][[c, a], b] \tag{1.10}$$

shows that the set $\widetilde{U}_{i_0} \cap \Psi_{u_{\rm reg}}(\widetilde{U})$ is closed with respect to the multiplication in $\widetilde{U}_{i_0}$. Since $\widetilde{U}_{i_0} \cap \Psi_{u_{\rm reg}}(\widetilde{U})$ is a dense open subset of $\widetilde{U}_{i_0}$, we obtain

$$\widetilde{U}_{i_0} = (\widetilde{U}_{i_0} \cap \Psi_{u_{\rm reg}}(\widetilde{U}))(\widetilde{U}_{i_0} \cap \Psi_{u_{\rm reg}}(\widetilde{U})) = \widetilde{U}_{i_0} \cap \Psi_{u_{\rm reg}}(\widetilde{U})$$

([1, I, 1.3 (a)]). Assume

$$\widetilde{U}_i = \widetilde{U}_i \cap \Psi_{u_{\rm reg}}(\widetilde{U}) \tag{1.11}$$

for every $i_1 \leq i \leq i_0$. Let $[u_{\rm reg}, u_1], [u_{\rm reg}, u_2] \in \widetilde{U}_{i_1 - 1}$. The commutator formula (1.10) shows $[u_{\rm reg}, u_1][u_{\rm reg}, u_2] = [u_{\rm reg}, u]v_{i_1}$ for some $u \in \widetilde{U}$ and $v_{i_1} \in \widetilde{U}_{i_1}$. Using again (1.10) and the equality (1.11), we get an element

$u' \in \widetilde{U}$ such that $[u_{\mathrm{reg}}, u]v_{i_1} = [u_{\mathrm{reg}}, uu']v_{i_1+1}$ for some $v_{i_1+1} \in \widetilde{U}_{i_1+1}$. Acting in the same way we can obtain the equality $[u_{\mathrm{reg}}, u]v_{i_1} = [u_{\mathrm{reg}}, v]$ for some $v \in \widetilde{U}$ and therefore we get $[u_{\mathrm{reg}}, u_1][u_{\mathrm{reg}}, u_2] \in \widetilde{U}_{i_1-1} \cap \Psi_{u_{\mathrm{reg}}}(\widetilde{U})$. Hence the set $\widetilde{U}_{i_1-1} \cap \Psi_{u_{\mathrm{reg}}}(\widetilde{U})$ is closed with respect to the multiplication in $\widetilde{U}_{i_1-1}$ and, by induction, the set $\widetilde{U}_1 \cap \Psi_{u_{\mathrm{reg}}}(\widetilde{U})$ is closed with respect to the multiplication in $\widetilde{U}_1$. Now, using (1.9), we get

$$\Psi_{u_{\mathrm{reg}}}(\widetilde{U}) = [u_{\mathrm{reg}}, \widetilde{U}] = \widetilde{U}_1 = \widetilde{U}^*. \qquad \square$$

Lemma 1.2. *Let K be a perfect field and let* $\operatorname{char} K$ *be good for $\widetilde{R}$. If $u_{\mathrm{reg}} \in U$, then*

$$[u_{\mathrm{reg}}, U] = U^*.$$

Proof. Let $\mathcal{G} = \mathrm{Gal}(\overline{K}/K)$ be the Galois group of the extension $\overline{K}/K$. Then the K-structure on a K-variety X defines the action of $\mathcal{G}$ on X and

$$X(\overline{K})^{\mathcal{G}} = \{x \in X(\overline{K}) \mid g(x) = x \text{ for every } g \in \mathcal{G}\} = X(K)$$

([10, 11]). Since $C_{\widetilde{U}}(u_{\mathrm{reg}})$ is a connected (because of (1.6)) split unipotent K-group (because K is a perfect field; see, [10, 12.1.2, 14.3.10]) we have

$$H^1(\mathcal{G}, C_{\widetilde{U}}(u_{\mathrm{reg}})) = 1 \tag{1.12}$$

([10, 12.3.5, (3)]). Further, for every $v \in U^* = \widetilde{U}^*(K)$ there exists an element $x \in \widetilde{U}$ such that $v = [u_{\mathrm{reg}}, x]$ (Lemma 1.1). For every $g \in \mathcal{G}$ we get

$$u_{\mathrm{reg}}\, x\, u_{\mathrm{reg}}^{-1}\, x^{-1} = u_{\mathrm{reg}}\, g(x)\, u_{\mathrm{reg}}^{-1}\, g(x^{-1}) \Rightarrow x^{-1}g(x) \in C_{\widetilde{U}}(u_{\mathrm{reg}}).$$

Hence $\{x_g := x^{-1}g(x)\}_{g \in \mathcal{G}}$ is a 1-cocycle on $C_{\widetilde{U}}(u_{\mathrm{reg}})$ and therefore we have from (1.12) $x^{-1}g(x) = y^{-1}g(y)$ for some $y \in C_{\widetilde{U}}(u_{\mathrm{reg}})$. Thus, $g(xy^{-1}) = xy^{-1}$ for every $g \in \mathcal{G}$. Now we have an element $xy^{-1} \in U = \widetilde{U}(\overline{K})^{\mathcal{G}}$ such that

$$[u_{\mathrm{reg}}, xy^{-1}] = u_{\mathrm{reg}}\, xy^{-1}\, u_{\mathrm{reg}}^{-1}\, yx^{-1} = u_{\mathrm{reg}}\, x\, u_{\mathrm{reg}}^{-1}\, y^{-1}yx^{-1} = [u_{\mathrm{reg}}, x] = v.$$

Hence $[u_{\mathrm{reg}}, U] = U^*$. $\square$

Remark 1.3. The assumption on the characteristic of the field K is essential. Indeed, if K is a finite field of a bad characteristic and let $\widetilde{G}$ be a split K-group, then

$$\mid C_{\widetilde{U}}(u_{\mathrm{reg}})(K) \mid > \mid C^0_{\widetilde{U}}(u_{\mathrm{reg}})(K) \mid = K^{\tilde{r}}. \tag{1.13}$$

Further,

$$| U |= K^{|\widetilde{R}^+|}, \ | U^* |= K^{|\widetilde{R}^+|-\widetilde{r}}. \tag{1.14}$$

Since any fiber of the map

$$U \stackrel{[u_{\mathrm{reg}},x]}{\to} U^*$$

has exactly $| C_{\widetilde{U}}(u_{\mathrm{reg}})(K) |$ elements, the formulas (1.13) and (1.14) yield

$$| [u_{\mathrm{reg}}, U] |<| U^* | .$$

1.3. Coxeter cells in the Bruhat decomposition. Let W (respectively, $\widetilde{W}$) be the Weyl group for R (respectively, $\widetilde{R}$). For an element $w \in W$ (respectively, $w \in \widetilde{W}$) we denote by $\dot{w}$ any preimage of w in the group G (respectively, $\widetilde{G}$) ([4, 2.5]). Recall, that a Coxeter element of W (respectively, $\widetilde{W}$) is a product in any order of all simple reflections w_{α_i} (respectively, $w_{\widetilde{\alpha_i}}$), where each such reflection occurs exactly one time.

We have the Bruhat decompositions $G = \bigcup\limits_{w\in W} B\dot{w}B$, $\widetilde{G} = \bigcup\limits_{w\in \widetilde{W}} \widetilde{B}\dot{w}\widetilde{B}$. Note that every element of W can be identified with some product of elements of $\widetilde{W}$ ([3, 13.1.2]). Hence for every $w \in W$ there is some $w' \in \widetilde{W}$ such that $B\dot{w}B \subset \widetilde{B}\dot{w}'\widetilde{B}$. A Bruhat cell $B\dot{w}B$ (respectively, $\widetilde{B}\dot{w}\widetilde{B}$) is called *a Coxeter cell* if w is a Coxeter element in W (respectively, in $\widetilde{W}$). If G is not of type ${}^2A_{2r}$, then every Coxeter element of W is also the Coxeter element of $\widetilde{W}$ (see [3, 13.1.2, 13.3]). Hence, if G is not of type ${}^2A_{2r}$, every Coxeter cell in G corresponds to a Coxeter cell in $\widetilde{G}$ and therefore, in this case, every element in a Coxeter cell of G is regular ([12]).

Now let $\dim K \leq 1$, where $\dim K$ is the homological dimension of K. Then every regular semisimple conjugacy class of G intersects every Coxeter cell $B\dot{w}_cB$ ([7]). Since we can choose an appropriate preimage of w_c, any element of $B\dot{w}_cB$ is conjugate to an element of the form $u\dot{w}_c$ for some $u \in U$.

§2. Main results

Theorem 2.1. *Let $R \neq A_1, A_2$. There exists a Coxeter element $w_c \in W$ that satisfies the following condition: for every $u \in U$ and for every $u_{\alpha_1} \in X_{\alpha_1}, \dots, u_{\alpha_r} \in X_{\alpha_r}$ there is an element $v \in U$ such that $[u\dot{w}_c, v] \in U$ and*

$$[u\dot{w}_c, v] \equiv u_{\alpha_1}u_{\alpha_2}\cdots u_{\alpha_r} \bmod U^*.$$

If, in addition, K is a perfect field and $\operatorname{char} K$ *is good for* $\widetilde{R}$, *and if* $u_{\mathrm{reg}} \in U = \widetilde{U}(K)$ *is any regular unipotent element of* G, *then*

$$[u\dot{w}_c, v] = v' u_{\mathrm{reg}} {v'}^{-1}$$

for some $v, v' \in U$.

Proof.

Lemma 2.2. *Let* $R \neq A_1, A_2$. *There is a numeration of the simple roots* $\Pi = \{\alpha_1, \ldots, \alpha_r\}$ *such that the Coxeter element* $w_c = w_{\alpha_r} w_{\alpha_{r-1}} \cdots w_{\alpha_1}$ *satisfies the following condition: either* $w_c(\alpha_i) > 0$, $w_c(\alpha_i) \notin \Pi$, *or* $w_c^{-1}(\alpha_i) > 0$, $w_c^{-1}(\alpha_i) \notin \Pi$ *for every* $i = 1, \ldots, r$.

Proof. Assume $R = A_r$, $r > 2$. Put $k = [\frac{r+1}{2}]$, $s = [\frac{r}{2}]$ and put

$$\alpha_i = \epsilon_{2i-1} - \epsilon_{2i} \quad \text{for } i = 1, \ldots, k; \quad \alpha_{k+i} = \epsilon_{2i} - \epsilon_{2i+1} \quad \text{for } i = 1, \ldots, s.$$

Then w_c corresponds to the substitution $(1\ 3 \ldots\ 2s+1\ 2k \ldots 2)$. Let $i = 2t-1$, $t \leq k$, $\alpha_i = \epsilon_{2t-1} - \epsilon_{2t}$. Then for $1 < i < k$ and $1 < j < s$ we have

$$w_c^{-1}(\alpha_i) = \epsilon_{2t-3} - \epsilon_{2t+2},\, w_c(\alpha_{k+j}) = \epsilon_{2j-2} - \epsilon_{2j+3} \in R^+ \setminus \Pi$$

and

$$w_c^{-1}(\alpha_1) = \epsilon_2 - \epsilon_4, \;\; w_c^{-1}(\alpha_k) = \epsilon_{2k-3} - \epsilon_{2s+1},$$
$$w_c(\alpha_{k+1}) = \epsilon_1 - \epsilon_5 \text{ or } w_c(\alpha_{k+1}) = \epsilon_1 - \epsilon_4 (r = 3), \;\; w_c(\alpha_{k+s}) = \epsilon_{2s-2} - \epsilon_{2k}.$$

Assume $R = B_r$, C_r.

If $r = 2$, take the natural numeration $\alpha_1 = \epsilon_1 - \epsilon_2, \alpha_2 = \epsilon_2$ or $2\epsilon_2$. We have

$$w_c^{-1}(\alpha_1) = \epsilon_1 + \epsilon_2, w_c(\alpha_2) = \epsilon_1 \text{ or } 2\epsilon_1.$$

If $r = 3$, then $\alpha_1 = \epsilon_1 - \epsilon_2, \alpha_2 = \epsilon_3$ or $2\epsilon_3$ and $\alpha_3 = \epsilon_2 - \epsilon_3$, and

$$w_c^{-1}(\alpha_1) = \epsilon_2 + \epsilon_3, w_c^{-1}(\alpha_2) = \epsilon_1 \text{ or } 2\epsilon_1, \;\; w_c(\alpha_3) = \epsilon_1 + \epsilon_2.$$

If $r > 3$ take the numeration of the roots $\{\alpha_1, \ldots, \alpha_{r-1}\}$ as above in the case $R = A_{r-1}$ and $\alpha_r = \epsilon_r$ or $2\epsilon_r$. Since w_{α_r} changes the sign of ϵ_r we get our statement for every $\alpha_i, i \leq r-1$, from the previous consideration. Further, $w_c(\epsilon_r) = \epsilon_{r-1}$ or ϵ_{r-2} and $w_c(2\epsilon_r) = 2\epsilon_{r-1}$ or $2\epsilon_{r-2}$.

Assume $R = D_r$.

If $r = 4$, then $\alpha_1 = \epsilon_1 - \epsilon_2$, $\alpha_2 = \epsilon_3 - \epsilon_4$, $\alpha_3 = \epsilon_3 + \epsilon_4$, $\alpha_4 = \epsilon_2 - \epsilon_3$, and

$$w_c^{-1}(\alpha_1) = \epsilon_2 + \epsilon_3,\, w_c^{-1}(\alpha_2) = \epsilon_1 + \epsilon_4,\, w_c^{-1}(\alpha_3) = \epsilon_1 - \epsilon_4,\, w_c(\alpha_4) = \epsilon_1 + \epsilon_2.$$

If $r > 4$, take the numeration of the roots $\{\alpha_1, \dots, \alpha_{r-2}\}$ as above in the case $R = A_{r-2}$ and $\alpha_{r-1} = \epsilon_{r-1} - \epsilon_r$ and $\alpha_r = \epsilon_{r-1} + \epsilon_r$. Since $w_{\alpha_r} w_{\alpha_{r-1}}$ changes the sign of ϵ_{r-1}, ϵ_r we get the statement for every α_i, $i \le r-2$ from the previous consideration. Further, $w_c(\alpha_{r-1}) = \epsilon_{r-2} + \epsilon_r$, or $\epsilon_{r-3} + \epsilon_r$; $w_c(\alpha_r) = \epsilon_{r-2} - \epsilon_r$, or $\epsilon_{r-3} - \epsilon_r$.

Assume $R = E_r$, $r = 6, 7, 8$. Let $\Pi = \{\alpha_1, \dots, \alpha_r\}$ be the standard simple root system ([2, Tables V–VII]). The root system generated by $\Pi \setminus \{\alpha_1\}$ is D_r, where $r = 5, 6, 7$, respectively. Let $\widetilde{w}_c$ be the corresponding Coxeter element of the root system D_r which has been constructed in the previous case (constructing the element $\widetilde{w}_c$ we make the new numeration for $\alpha_2, \dots, \alpha_r$; however, in the arguments which we use below we preserve the original numeration of α_i). Let $w_c = \widetilde{w}_c w_{\alpha_1}$. Now, let $\alpha_i \in \Pi$, $i \ne 1, 3$. Then

$$w_c(\alpha_i) = \widetilde{w}_c(\alpha_i) = \pm\epsilon_k \pm \epsilon_l \ne \alpha_1 = \frac{1}{2}(\cdots).$$

If $\widetilde{w}_c(\alpha_i) > 0$ and $\widetilde{w}_c(\alpha_i) \notin \Pi \setminus \{\alpha_1\}$, we have the condition of the Lemma. Let $\beta = \widetilde{w}_c^{-1}(\alpha_i) > 0$ and $\beta \notin \Pi \setminus \{\alpha_1\}$. Obviously, $\beta \in \langle \Pi \setminus \{\alpha_1\} \rangle = D_r$. Since α_1 is the only positive root which becomes negative after transformation by w_{α_1}, the root $w_{\alpha_1}(\beta)$ is positive. Also, $w_c^{-1}(\alpha_i) = w_{\alpha_1}(\beta) = \beta - n(\beta, \alpha_1)\alpha_1 \notin \Pi$ and therefore we again have the condition of the lemma. Further,

$$\gamma = \widetilde{w}_c^{-1}(\alpha_1) = \alpha_1 + \alpha_3 + \alpha_4 \ \ (\text{cases } E_6, E_8),$$

or

$$\gamma = \widetilde{w}_c^{-1}(\alpha_1) = \alpha_1 + \alpha_3 + \alpha_4 + \alpha_5 \ \ (\text{case } E_7)$$

(this follows from the construction of $\widetilde{w}_c$). Hence

$$0 < w_c^{-1}(\alpha_1) = w_{\alpha_1}(\gamma) = \gamma - n(\gamma, \alpha_1)\alpha_1 = \gamma - \alpha_1 \notin \Pi$$

and we have the condition of the lemma. Further,

$$w_c(\alpha_3) = \widetilde{w}_c w_{\alpha_1}(\alpha_3) = \widetilde{w}_c(\alpha_1 + \alpha_3) \ne \alpha_1$$

(because $\widetilde{w}_c^{-1}(\alpha_1) \ne \alpha_1 + \alpha_3$). Also, $\widetilde{w}_c(\alpha_1 + \alpha_3) = \alpha_1 + \cdots$ is a positive root $\notin \Pi \setminus \{\alpha_1\}$ and we again have the condition of the lemma.

Assume $R = F_4$. Put

$$\alpha_1 = \epsilon_2 - \epsilon_3, \alpha_2 = \epsilon_4, \alpha_3 = \epsilon_3 - \epsilon_4, \alpha_4 = \frac{1}{2}(\epsilon_1 - \epsilon_2 - \epsilon_3 - \epsilon_4).$$

Then

$$w_c(\alpha_4) = \epsilon_3, w_c(\alpha_3) = \epsilon_1 - \epsilon_4,\ w_c^{-1}(\alpha_2) = \frac{1}{2}(\epsilon_1 + \epsilon_2 - \epsilon_3 + \epsilon_4),\ w_c^{-1}(\alpha_1) = \epsilon_3 + \epsilon_4.$$

Assume $R = G_2$.
Put $\alpha_1 = \epsilon_1 - \epsilon_2, \alpha_2 = -2\epsilon_1 + \epsilon_2 + \epsilon_3$. Then

$$w_c(\alpha_2) = 3\alpha_1 + 2\alpha_2, w_c^{-1}(\alpha_1) = 2\alpha_1 + \alpha_2. \qquad \square$$

For every $\alpha_i \in \Pi$ put

$$v_{\alpha_i} = \begin{cases} u_{\alpha_i}^{-1} & \text{if } \ w_c(\alpha_i) = \beta > 0, \quad \beta \notin \Pi, \\ \dot{w}_c^{-1} u_{\alpha_i} \dot{w}_c & \text{if } \ w_c^{-1}(\alpha_i) = \gamma > 0, \ \gamma \notin \Pi. \end{cases} \tag{2.1}$$

From (2.1) and Lemma 2.2, we get

$$\dot{w}_c v_{\alpha_i} \dot{w}_c^{-1} v_{\alpha_i}^{-1} \equiv u_{\alpha_i} \bmod U^*. \tag{2.2}$$

Further, put

$$v = \prod_{i=1}^{r} v_{\alpha_i}.$$

We have

$$\begin{aligned}
[u\dot{w}_c, v] = u\dot{w}_c v \dot{w}_c^{-1} u^{-1} v^{-1} &= u\dot{w}_c \left(\prod_{i=1}^{r} v_{\alpha_i} \right) \dot{w}_c^{-1} u^{-1} \left(\prod_{i=r}^{1} v_{\alpha_i}^{-1} \right) \\
&= u \left(\prod_{i=1}^{r} \dot{w}_c v_{\alpha_i} \dot{w}_c^{-1} \right) u^{-1} \left(\prod_{i=r}^{1} v_{\alpha_i}^{-1} \right) \\
&\equiv \left(\prod_{i=1}^{r} \dot{w}_c v_{\alpha_i} \dot{w}_c^{-1} \right) \left(\prod_{i=r}^{1} v_{\alpha_i}^{-1} \right) \bmod U^* \\
&\equiv \left(\prod_{i=1}^{r} \dot{w}_c v_{\alpha_i} \dot{w}_c^{-1} v_{\alpha_i}^{-1} \right) \overset{\text{by } 2.2}{\equiv} \prod_{i=1}^{r} u_{\alpha_i} \bmod U^*.
\end{aligned}$$

Now, let K be a perfect field and let char$K$ be good for $\widetilde{R}$. Further, let $u_{\text{reg}} \in U$ be a regular element such that $u_{\text{reg}} = \prod_{i=1}^{r} u_{\alpha_i} u'$, where $u_{\alpha_i} \in$

$X_{\alpha_i} \setminus X^*_{\alpha_i}$ for every i and $u' \in U^*$. Then the element $[u\dot{w}_c, v] = \prod_{i=1}^{r} u_{\alpha_i} u''$, where $u'' \in U^*$ is regular because of (1.4). We can find (Lemma 1.2) an element $v' \in U$ such that $[[u\dot{w}_c, v]^{-1}, v'^{-1}] = u''^{-1}u'$. Hence

$$[v'^{-1}u\dot{w}_c v', v'^{-1}vv'] = v'^{-1}[u\dot{w}_c, v]v' = [u\dot{w}_c, v][[u\dot{w}_c, v]^{-1}, v'^{-1}]$$
$$= \left(\prod_{i=1}^{r} u_{\alpha_i} u''\right) u''^{-1}u' = u_{\mathrm{reg}}. \qquad \square$$

Corollary 2.3. *Let* $\dim K \leq 1$ *and let* $R \neq A_1, A_2$. *Further, let* $C \subset G$ *be a conjugacy class of regular semisimple elements. Then for every* $u_{\alpha_1} \in X_{\alpha_1}, \ldots, u_{\alpha_r} \in X_{\alpha_r}$ *there exist elements* $g \in C, v \in U$ *such that* $[g, v] \in U$ *and*

$$[g, v] \equiv u_{\alpha_1} u_{\alpha_2} \cdots u_{\alpha_r} \bmod U^*.$$

If, in addition, K *is a perfect field and* $\mathrm{char} K$ *is good for* $\widetilde{R}$ *and if* $u_{\mathrm{reg}} \in U$ *is any regular unipotent element, then there exist elements* $g \in C, v, v' \in U$ *such that*

$$[g, v] = v' u_{\mathrm{reg}} v'^{-1}.$$

Proof. If $\dim K \leq 1$ then the conjugacy class C intersects the Bruhat cells $B\dot{w}_c B$ for every Coxeter element $w_c \in W$ ([7]). Hence we may assume $g = u\dot{w}_c \in C$ where w_c is the Coxeter element from Theorem 2.1. Now the result follows directly from Theorem 2.1. $\square$

Theorem 2.4. *Let* $R \neq A_1, A_2$, *let* K *be a perfect field, and let* $\mathrm{char}\, K$ *be good for* $\widetilde{R}$. *Further, let* w_c *be the Coxeter element from Theorem 2.1,* $u \in U$. *Then every non-central element of the group* G *is conjugate to a product* $[u\dot{w}_c, \sigma][u_{\mathrm{reg}}, \tau]$ *for some regular unipotent element* u_{reg} *of* G *and some* $\sigma, \tau \in G$.

Proof. Let $w_0 \in W$ be the longest element in the Weyl group and let $V = \dot{w}_0 U \dot{w}_0^{-1}$. Then $v_{\mathrm{reg}} = \dot{w}_0 u_{\mathrm{reg}} \dot{w}_0^{-1}$ is a regular element from the group V. Let $x \in U, y \in V$. Then $x = x_1 x_2, y = y_2 y_1$, where $x_2 \in U^*, y_2 \in V^* = \dot{w}_0 U^* \dot{w}_0^{-1}$ and

$$x_1 = \prod_{i=1}^{r} u_{\alpha_i},\ u_{\alpha_i} \in X_{\alpha_i}; \quad y_1 = \prod_{i=1}^{r} v_{-\alpha_i},\ v_{-\alpha_i} \in X_{-\alpha_i}.$$

Put $g = u\dot{w}_c$. By Theorem 2.1 we can find $v \in U, x' \in U^*$ such that

$$(gvg^{-1}v^{-1}) = x_1 u_{\mathrm{reg}} x'. \qquad (2.3)$$

By Lemma 1.2, we can find an element $\widetilde{u} \in U$ such that

$$u_{\mathrm{reg}}(x'\widetilde{u})u_{\mathrm{reg}}^{-1}(\widetilde{u}^{-1}x'^{-1})x' = x_2. \qquad (2.4)$$

From (2.3), (2.4) we get

$$\begin{aligned}(gvg^{-1}v^{-1})(\widetilde{u}u_{\mathrm{reg}}^{-1}\widetilde{u}^{-1}) &= x_1 u_{\mathrm{reg}} x'(\widetilde{u}u_{\mathrm{reg}}^{-1}\widetilde{u}^{-1}) \\ &= x_1 u_{\mathrm{reg}}(x'\widetilde{u})u_{\mathrm{reg}}^{-1}(\widetilde{u}^{-1}x'^{-1})x' = x_1x_2 = x.\end{aligned}$$

By the same arguments we have

$$y = y_2y_1 = (\widetilde{v}v_{\mathrm{reg}}\widetilde{v}^{-1})\,(v'gv'^{-1}g^{-1})$$

for some $v', \widetilde{v} \in V$. Hence

$$\begin{aligned}y_2y_1x_1x_2 &= (\widetilde{v}v_{\mathrm{reg}}\widetilde{v}^{-1})\,(v'gv'^{-1}g^{-1})(gvg^{-1}v^{-1})(\widetilde{u}u_{\mathrm{reg}}^{-1}\widetilde{u}^{-1}) \\ &= (\widetilde{v}v_{\mathrm{reg}}\widetilde{v}^{-1})\,(v'gv'^{-1})(vg^{-1}v^{-1})(\widetilde{u}u_{\mathrm{reg}}^{-1}\widetilde{u}^{-1}).\end{aligned}$$

Every non-central element of G is conjugate to an element of the form $y_2y_1x_1x_2 \in VU$ ([5]) which, in turn, is conjugate to an element of the form

$$g(v'^{-1}vg^{-1}v^{-1}v')(v'^{-1}\widetilde{u}u_{\mathrm{reg}}^{-1}\widetilde{u}^{-1}v')(v'^{-1}\widetilde{v}v_{\mathrm{reg}}\widetilde{v}^{-1}v'). \qquad (2.5)$$

Note, $(v'^{-1}\widetilde{v}v_{\mathrm{reg}}\widetilde{v}^{-1}v')$ is a regular element which is conjugate to u_{reg} (recall, $v_{\mathrm{reg}} = \dot{w}_0 u_{\mathrm{reg}} \dot{w}_0^{-1}$). Thus, if we put $u_{\mathrm{reg}} := (v'^{-1}\widetilde{u}u_{\mathrm{reg}}^{-1}\widetilde{u}^{-1}v')$ then we can put $\tau u_{\mathrm{reg}}^{-1}\tau^{-1} := (v'^{-1}\widetilde{v}v_{\mathrm{reg}}\widetilde{v}^{-1}v')$ for some $\tau \in G$. If we put $\sigma := v'^{-1}v$ we get the statement from (2.5). □

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