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ON THE REGULARITY OF SOLUTIONS TO THE EQUATION $-\Delta u + b \cdot \nabla u = 0$

Abstract. The equation $-\Delta u + b \cdot \nabla u = 0$ is considered. The dependence of the local regularity of a solution u on the properties of the coefficient b is investigated.

To the memory of O. A. Ladyzhenskaya

§1. Formulation of the results

Denote by B_R a ball in $\mathbb{R}^n$, $n \geqslant 2$, of radius R centered at the origin. We consider the equation

$$-\Delta u + b \cdot \nabla u = 0 \tag{1.1}$$

in B_R. We always assume that a scalar function $u \in W_2^1(B_R)$, and a vector-valued coefficient $b \in L_p(B_R)$, $p \geqslant 2$. We understand the equation (1.1) in the sense of the integral identity

$$\int\limits_{B_R} \nabla u \cdot (\nabla h + bh)\, dx = 0 \quad \forall h \in C_0^\infty(B_R).$$

We are interested in the dependence of the local regularity of the solution u of (1.1) on the order p of the summability of the coefficient b. The aim of the present paper is to list the results, and the counterexamples which guarantee the sharpness of the results. The brief summary is given in the Table 1 below.

The critical case is $p = n$. If $p > n$, the solution u is continuously differentiable.

Theorem 1.1 ([5], Chapter III, Theorem 15.1). *Let $b \in L_p(B_R)$, $p > n$, and let $u \in W_2^1(B_R)$ be a solution to the equation* (1.1). *Then*

$$u \in W_p^2(B_r) \subset C^{1,1-\frac{n}{p}}(B_r) \quad \forall r < R.$$

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Here and in what follows by $u \in W_p^2(B_r)$ we mean that the restriction of u onto the ball B_r belongs to this space, $u|_{B_r} \in W_p^2(B_r)$.

If $p = n$ the properties of solution depend on the dimension, whether $n = 2$ or $n > 2$.

1.1. Case $n = 2$. Let us consider two simple examples. The first example shows that when $p = n = 2$ a solution u can be unbounded. The second one shows that even if we assume a priori a solution to be bounded, then it can fail to be Hölder continuous.

Example 1. Let $n = 2$, $R = 1/e$,

$$u(x) = \ln|\ln|x||, \quad b(x) = \frac{-x}{|x|^2 \ln|x|}.$$

Then $b \in L_2(B_{1/e})$, $u \in \mathring{W}_2^1(B_{1/e})$, and (1.1) is satisfied, but $u \notin L_\infty(B_{1/e})$.

Example 2. Let $n = 2$, $R = 1/2$,

$$u(x) = \frac{1}{\ln|x|}, \quad b(x) = -\frac{2x}{|x|^2 \ln|x|}.$$

Then $b \in L_2(B_{1/2})$, $u \in W_2^1(B_{1/2}) \cap C(\overline{B_{1/2}})$, and (1.1) is satisfied. But $u \notin C^\alpha(B_{1/2})$ for any $\alpha > 0$.

The situation changes if the coefficient b satisfies an extra condition $\operatorname{div} b = 0$.

Theorem 1.2. *Let $n = 2$, $b \in L_2(B_R)$ and $\operatorname{div} b = 0$. Let $u \in W_2^1(B_R)$ be a solution to equation (1.1). Then*

$$u \in \bigcap_{q<2} W_q^2(B_r) \subset \bigcap_{\alpha<1} C^\alpha(B_r) \quad \forall r < R.$$

We prove Theorem 1.2 in the next section.

Remark 1.3. In [7] a more general equation

$$-\operatorname{div}(a\nabla u) + b \cdot \nabla u = 0 \tag{1.2}$$

is considered. The matrix-coefficient $a(x)$ is assumed to be positive and bounded,

$$0 < \alpha_0 \mathbb{1} \leqslant a(x) \leqslant \alpha_1 \mathbb{1}\,, \tag{1.3}$$

here $\mathbb{1}$ is the identity matrix. If $b \in L_2(B_R)$, $\operatorname{div} b = 0$, then a solution u to (1.2) is Hölder continuous, $u \in C^\alpha$ with some $\alpha > 0$ (see Corollary 2.3 and the comments at the end of §2 in [7]).

Remark 1.4. If the coefficient b satisfies a slightly stronger condition than $b \in L_2$,

$$\int\limits_{B_R} |b(x)|^2 \ln(1 + |b(x)|^2)\, dx < \infty$$

(without the divergence-free condition), then the statement of Theorem 1.2 remains valid, see §4.4 below.

1.2. Case $n \geqslant 3$. In this case, the condition $b \in L_n$ is sufficient for u to be Hölder continuous.

Theorem 1.5. *Let $n \geqslant 3$, $b \in L_n(B_R)$, and $u \in W_2^1(B_R)$ be a solution to equation* (1.1). *Then*

$$u \in \bigcap_{q<n} W_q^2(B_r) \subset \bigcap_{\alpha<1} C^\alpha(B_r) \quad \forall r < R.$$

This theorem is probably known, although we have not found a relevant reference. Theorem 1.5 can be proved in the same way that Theorem 1.2, see Remark 2.8 below.

The following example shows that a solution u can be unbounded when $p < n$.

Example 3. Let $n \geqslant 3$, $R = 1$,

$$u(x) = \ln|x|, \quad b(x) = \frac{(n-2)x}{|x|^2}.$$

Then $b \in L_p(B_1)$ for all $p < n$, $u \in \mathring{W}_2^1(B_1)$, and (1.1) is satisfied, but $u \notin L_\infty(B_1)$.

Furthermore, for $p < n$, if we assume a priori a solution to be bounded, it can be discontinuous, even for divergence-free coefficient $b \in L_p$.

Theorem 1.6. *Let $n \geqslant 3$, $p < n$. There exist a vector-function $b_0 \in L_p(B_{1/2})$, $\operatorname{div} b_0 = 0$, and a scalar function $u_0 \in W_2^1(B_{1/2}) \cap L_\infty(B_{1/2})$ such that the equation* (1.1) *is satisfied, but $u_0 \notin C(B_{1/2})$.*

We prove this Theorem in Section 3.

Remark 1.7. It is easy to construct an example of a bounded solution which is not Hölder continuous for the case $\operatorname{div} b \neq 0$.

Example 4. Let $n \geqslant 3$, $R = 1/2$,

$$u(x) = \frac{1}{\ln|x|}, \quad b(x) = \left((n-2)|x| - \frac{2}{|x|\ln|x|}\right)\frac{x}{|x|}.$$

Then $b \in \cap_{p<n} L_p(B_{1/2})$, $u \in W_2^1(B_{1/2}) \cap C(\overline{B_{1/2}})$, and (1.1) is satisfied. But $u \notin C^\alpha(B_{1/2})$ for any $\alpha > 0$.

For the proof of Theorem 1.6 we follow the approach of the paper [8]. We consider together with (1.1) the equation

$$-\Delta u + \operatorname{div}(bu) = 0. \tag{1.4}$$

We understand this equation in the sense

$$\int\limits_{B_R} u(\Delta h + b \cdot \nabla h)\, dx = 0 \quad \forall h \in C_0^\infty(B_R);$$

the integral is well defined if $u, b \in L_2(B_R)$. It is clear that every solution $u \in W_2^1(B_R)$ to equation (1.1) solves also equation (1.4) if $\operatorname{div} b = 0$. The converse statement is valid for bounded solutions.

Theorem 1.8 ([8], Proposition 4.1). *Let $u \in L_\infty(B_R)$, $b \in L_2(B_R)$, $\operatorname{div} b = 0$, and (1.4) be satisfied. Then $u \in W_2^1(B_r)$ for all $r < R$, u solves the equation (1.1), and the estimate*

$$\|\nabla u\|_{L_2(B_r)} \leqslant C(n, r, R) \left(1 + \|b\|_{L_1(B_R)}\right)^{1/2} \|u\|_{L_\infty(B_R)}$$

holds.

In order to prove Theorem 1.6 we establish

Theorem 1.9. *Let $n \geqslant 3$, $p < n$. There are two positive constants c_0, c_1 such that for any $\varepsilon > 0$ there exist a vector-function $b_\varepsilon \in C^\infty(\overline{B_{1/2}})$, $\operatorname{div} b_\varepsilon = 0$, $\|b_\varepsilon\|_{L_p(B_{1/2})} \leqslant c_0$, and a scalar function $u_\varepsilon \in C^\infty(\overline{B_{1/2}})$, $\|u_\varepsilon\|_{L_\infty(B_{1/2})} \leqslant 1$, $\|u_\varepsilon\|_{W_2^1(B_{1/2})} \leqslant c_0$, which satisfy the equations (1.1) and (1.4), and moreover*

$$u_\varepsilon(0) = 0, \quad u_\varepsilon(0, \ldots, 0, 2\varepsilon) \geqslant c_1.$$

This result was proven in [8] for $n = 3$ and $p = 1$. It is also clear from the construction of b_ε in [8], that one can take any power $p < 2$. However, in order to deduce Theorem 1.6 from Theorem 1.9 one has to get Theorem 1.9 with a power $p \geqslant 2$.

In Section 2 we prove Theorem 1.2. In Section 3 we prove Theorem 1.6 and Theorem 1.9. Some comments are collected in Section 4.

We do not consider the parabolic equation $\partial_t u - \Delta u + b \cdot \nabla u = 0$, and the regularity of a solution in dependence of the properties of a coefficient b. Some results in this direction (under the condition $\operatorname{div} b = 0$) can be found in [7–9] (see also references therein).

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Table 1. The local properties of a solution u to equation (1.1) with $b \in L_p$.

	$n = 2$	$n \geqslant 3$
$p > n$	$u \in C^{1,1-n/p}$	$u \in C^{1,1-n/p}$
$p = n$	In general $u \notin L_\infty$, or $u \in L_\infty$, $u \notin C^\alpha$. If $\operatorname{div} b = 0$,then $u \in C^\alpha$ $\forall \alpha < 1$.	$u \in C^\alpha \quad \forall \alpha < 1$
$p < n$	—	In general $u \notin L_\infty$. It is also possible (even in the case $\operatorname{div} b = 0$) that $u \in L_\infty$, $u \notin C$.

§2. Proof of Theorem 1.2

2.1. Existence of strong solution. First, let us consider the Dirichlet problem for the Laplace equation in a ball

$$-\Delta u = f \text{ in } B_R, \quad u|_{\partial B_R} = 0. \tag{2.1}$$

Explicit formulas for the solution together with the Calderon-Zygmund estimates of singular integrals imply the well known

Theorem 2.1. *Let $f \in L_q(B_R)$, $1 < q < \infty$. There exists a unique function $u \in W_q^2(B_R)$ satisfying (2.1), and $\|u\|_{W_q^2(B_R)} \leqslant C_1 \|f\|_{L_q(B_R)}$.*

Now, let us consider the problem

$$\begin{cases} -\Delta v + b \cdot \nabla v = f \text{ in } B_R, \\ v|_{\partial B_R} = 0. \end{cases} \tag{2.2}$$

The following Lemma is also well known, we give a proof for the reader convenience.

Lemma 2.2. *Let $n \geqslant 2$, $1 < q < n$. There exists a positive number $\varepsilon_0(n,q)$ such that if $b \in L_n(B_R)$, $\|b\|_{L_n(B_R)} \leqslant \varepsilon_0$, $f \in L_q(B_R)$, then there exists a unique function $v \in W^2_q(B_R)$ satisfying (2.2). Moreover, $\|v\|_{W^2_q(B_R)} \leqslant C\|f\|_{L_q(B_R)}$.*

Proof. Denote by L_0 the Laplace operator of the Dirichlet problem,

$$L_0 = -\Delta : W^2_q \cap \overset{\circ}{W}{}^1_q \to L_q.$$

The operator $b \cdot \nabla L_0^{-1}$ is bounded in $L_q(B_R)$. Indeed, let $f \in L_q(B_R)$, $u = L_0^{-1} f \in W^2_q(B_R)$. Due to the imbedding theorem $W^2_q \subset W^1_{nq/(n-q)}$ we have

$$\|b \cdot \nabla u\|_{L_q} \leqslant \|b\|_{L_n} \|\nabla u\|_{L_{nq/(n-q)}} \leqslant C_0 \|b\|_{L_n} \|u\|_{W^2_q} \leqslant C_0 C_1 \|b\|_{L_n} \|f\|_{L_q},$$

where on the last step we used Theorem 2.1. If $\varepsilon_0 < (2C_0C_1)^{-1}$, then $\|b \cdot \nabla L_0^{-1}\|_{L_q \to L_q} \leqslant 1/2$. Now, we set

$$v = L_0^{-1} \left(I + b \cdot \nabla L_0^{-1}\right)^{-1} f.$$

Clearly,

$$-\Delta v + b \cdot \nabla v = f, \quad v \in W^2_q \cap \overset{\circ}{W}{}^1_q, \quad \text{and} \quad \|v\|_{W^2_q(B_R)} \leqslant 2C_1 \|f\|_{L_q(B_R)},$$

as $\left\| \left(I + b \cdot \nabla L_0^{-1}\right)^{-1} \right\|_{L_q \to L_q} \leqslant 2.$ □

2.2. Spaces H_1 and BMO. Let us recall a definition of the Hardy space $H_1(\mathbb{R}^n)$. Let $\Phi \in C_0^\infty(B_1)$, $\int\limits_{B_1} \Phi(x)\,dx = 1$. For $f \in L_1(\mathbb{R}^n)$ we set

$$(M_\Phi f)(x) = \sup_{t>0} \left| \frac{1}{t^n} \int\limits_{\mathbb{R}^n} \Phi\left(\frac{x-y}{t}\right) f(y)\,dy \right|,$$

and

$$H_1(\mathbb{R}^n) = \{f \in L_1(\mathbb{R}^n) : M_\Phi f \in L_1(\mathbb{R}^n)\}, \quad \|f\|_{H_1} = \|M_\Phi f\|_{L_1(\mathbb{R}^n)}.$$

The space H_1 does not depend on the choice of a function Φ, and the norms constructed with different functions Φ are equivalent. A detailed exposition of the theory of Hardy spaces can be found in [11]. The dual space to H_1 is the space $BMO(\mathbb{R}^n)$ (Bounded Mean Oscillation). Its definition read as follows: a function $f \in L_{1,loc}(\mathbb{R}^n)$ belong to BMO if and only if

$$\sup_{x \in \mathbb{R}^n} \sup_{R>0} \frac{1}{|B_R(x)|} \int\limits_{B_R(x)} \left| f(y) - f_{B_R(x)} \right| dy =: \|f\|_{BMO} < \infty.$$

Here $f_{B_R(x)} = \frac{1}{|B_R(x)|} \int\limits_{B_R(x)} f(y)\,dy$. The functional $\|\,.\,\|_{BMO}$ is a seminorm (it vanishes on the constants). We will use the following result.

Lemma 2.3 ([1], Theorem II.1.2)**.** *Let* $b \in L_p(\mathbb{R}^n)$, $1 < p < \infty$, $\operatorname{div} b = 0$, $\varphi \in W^1_{p'}(\mathbb{R}^n)$. *Then* $b \cdot \nabla\varphi \in H_1(\mathbb{R}^n)$,

$$\|b \cdot \nabla\varphi\|_{H_1} \leqslant C\|b\|_{L_p}\|\nabla\varphi\|_{L_{p'}}.$$

Now, we can establish the following estimate.

Lemma 2.4. *Let* $n = 2$, $b \in L_2(B_R)$, $\operatorname{div} b = 0$. *Then*

$$\left|\int\limits_{B_R} b \cdot \nabla\varphi\,\psi\,dx\right| \leqslant C\|b\|_{L_2(B_R)}\|\nabla\varphi\|_{L_2(B_R)}\|\nabla\psi\|_{L_2(B_R)}$$
$$\forall\ \varphi \in \mathring{W}^1_2(B_R),\ \ \psi \in C_0^\infty(B_R). \tag{2.3}$$

Proof. First, as $\operatorname{div} b = 0$, we can represent the function b as $(b_1, b_2) = (\partial_2\omega, -\partial_1\omega)$ with $\omega \in W^1_2(B_R)$. We extend the function ω into the whole plane, and denote this extension by $\tilde\omega$,

$$\tilde\omega \in W^1_2(\mathbb{R}^2), \quad \tilde\omega|_{B_R} = \omega, \quad \|\tilde\omega\|_{W^1_2(\mathbb{R}^2)} \leqslant C\|\omega\|_{W^1_2(B_R)}.$$

Let us define a vector-function $\tilde b = (\partial_2\tilde\omega, -\partial_1\tilde\omega)$. Clearly,

$$\tilde b \in L_2(\mathbb{R}^2), \quad \|\tilde b\|_{L_2(\mathbb{R}^2)} \leqslant C\|b\|_{L_2(B_R)}, \quad \tilde b\Big|_{B_R} = b, \quad \operatorname{div}\tilde b = 0.$$

Therefore, by Lemma 2.3, $\tilde b \cdot \nabla\varphi \in H_1(\mathbb{R}^2)$ and

$$\|\tilde b \cdot \nabla\varphi\|_{H_1} \leqslant C\|b\|_{L_2(B_R)}\|\nabla\varphi\|_{L_2(B_R)}.$$

On the other hand, it is well known, that the space $W^1_2(\mathbb{R}^2)$ is imbedded in $BMO(\mathbb{R}^2)$, and the estimate

$$\|\psi\|_{BMO(\mathbb{R}^2)} \leqslant C\|\nabla\psi\|_{L_2(\mathbb{R}^2)}$$

holds (it is a simple consequence of the Poincaré inequality, see for example [2]).

Finally, the integral of a product of an H_1-function and a bounded BMO-function can be estimated by the product of the corresponding norms (see [11]),

$$\left|\int_{\mathbb{R}^2} \tilde b\cdot\nabla\varphi\,\psi\,dx\right|$$
$$\leqslant C\|\tilde b\cdot\nabla\varphi\|_{H_1}\|\psi\|_{BMO}\leqslant C\|b\|_{L_2(B_R)}\|\nabla\varphi\|_{L_2(B_R)}\|\nabla\psi\|_{L_2(B_R)}. \qquad \square$$

Remark 2.5. Lemma 2.4 is borrowed from the paper [6]. In this paper a detailed investigation of the boundedness of the integral in the left hand side of (2.3) under different conditions on b, φ, ψ is done. We gave the proof of (2.3) in our particular case for the convenience of a reader.

2.3. Uniqueness of weak solution.

Lemma 2.6. *Let $b\in L_2(B_R)$, $\operatorname{div} b = 0$. Then the solution to the problem (2.2) is unique in the space $\overset{\circ}{W}{}^1_2(B_R)$.*

Proof. Let u solve the homogeneous problem

$$-\Delta u + b\cdot\nabla u = 0,\quad u\in \overset{\circ}{W}{}^1_2(B_R). \tag{2.4}$$

Choose a sequence $\psi_n\in C_0^\infty(B_R)$ such that $\psi_n\to u$ in $W_2^1(B_R)$. Then

$$\int_{B_R}|\nabla u|^2dx\leqslant\int_{B_R}\nabla u\cdot\nabla\psi_n dx+\|\nabla u\|_{L_2(B_R)}\|\nabla u-\nabla\psi_n\|_{L_2(B_R)}.$$

The second term tends to 0 when $n\to\infty$. For the first term we have

$$\int_{B_R}\nabla u\cdot\nabla\psi_n dx = -\int_{B_R} b\cdot\nabla u\psi_n dx = \int_{B_R} b\cdot\nabla(\psi_n-u)\psi_n dx,$$

where we used (2.4) and the equality $\int\limits_{B_R} b\cdot\nabla\psi_n\psi_n dx = 0$ which is due to the divergence-free condition. By virtue of Lemma 2.4,

$$\left|\int_{B_R} b\cdot\nabla(\psi_n-u)\psi_n dx\right|\leqslant C\|b\|_{L_2(B_R)}\|\nabla\psi_n-\nabla u\|_{L_2(B_R)}\|\nabla\psi_n\|_{L_2(B_R)}\underset{n\to\infty}{\longrightarrow}0.$$

So, $\|\nabla u\|^2_{L_2(B_R)}=0$, and $u\equiv 0$. $\square$

Remark 2.7. Example 1 shows that the uniqueness of weak solution can be violated in the case $\operatorname{div} b\neq 0$.

2.4. Proof of Theorem 1.2. The statement of the Theorem is local. Therefore, without loss of generality, we can assume that the norm $\|b\|_{L_2(B_R)}$ is arbitrarily small. Let $u \in W_2^1(B_R)$ be a solution to the equation (1.1), and let $\zeta \in C_0^\infty(B_R)$, $\zeta|_{B_r} \equiv 1$. Then

$$-\Delta(\zeta u) + b \cdot \nabla(\zeta u) = -\Delta\zeta u - 2\nabla\zeta \cdot \nabla u + b \cdot \nabla\zeta u \in L_q(B_R) \quad \forall\, q < 2.$$

Thus, the function (ζu) solves the problem (2.2) with the right hand side in L_q. By virtue of Lemma 2.2, such a problem has a solution from $W_q^2(B_R)$. On the other hand, the solution is unique due to Lemma 2.6. So, $u \in W_q^2(B_r)$ for all $q < 2$.

Remark 2.8. Proof of Theorem 1.5 can be done similarly. The existence of strong solution is due to Lemma 2.2. The uniqueness of weak solution is given by

Lemma 2.9. *Let $n \geqslant 3$. There is a number $\varepsilon_1 = \varepsilon_1(n)$ such that a solution to the problem 2.2 is unique in $\overset{\circ}{W}{}_2^1(B_R)$ if $b \in L_n(B_R)$, $\|b\|_{L_n(B_R)} \leqslant \varepsilon_1$.*

Proof. Let u be a solution to the problem (2.2) with $f = 0$. Using the Hölder inequality and the imbedding Theorem $W_2^1 \subset L_{2n/(n-2)}$ we have

$$\int_{B_R} |\nabla u|^2 dx = -\int_{B_R} b \cdot \nabla u u\, dx \leqslant \|b\|_{L_n(B_R)} \|\nabla u\|_{L_2(B_R)} \|u\|_{L_{\frac{2n}{n-2}}(B_R)}$$
$$\leqslant C_0 \|b\|_{L_n(B_R)} \|\nabla u\|^2_{L_2(B_R)}. \tag{2.5}$$

If $\varepsilon_1 < 1/C_0$, then $\|\nabla u\|_{L_2(B_R)} = 0$. $\square$

Now, multiplying a solution to the equation (1.1) by a cut-off function, we get the relation

$$u \in W_q^1(B_R),\ 2 \leqslant q < n \quad \Longrightarrow \quad u \in W_q^2(B_r) \subset W^1_{\frac{nq}{n-q}}(B_r), \quad \forall\, r < R.$$

Iterating this relation $\left[\frac{n+1}{2}\right]$ times we obtain $u \in W_q^2(B_r)$ for all $q < n$ and $r < R$.

§3. Proof of Theorem 1.6

The proof of Theorem 1.9 (with $p = 1$) in [8] is based on the theory of the stochastic processes. We prove Theorem 1.6 and Theorem 1.9 following the general scheme of [8], but without using the probability theory.

3.1. Coefficient b. Let $n \geqslant 3$, let Ω be a cylinder in $\mathbb{R}^n$,

$$\Omega = \{x \in \mathbb{R}^n : \rho < 1, z \in (-1,1)\},$$

where $\rho = \sqrt{x_1^2 + \cdots + x_{n-1}^2}$, $z = x_n$. We will use the auxiliary parameters $\mu \in (1,2)$, $\varepsilon \in (0,1/2)$ and a function $\eta \in C^\infty(\mathbb{R})$, $\eta(t) = 0$ if $t \leqslant 1/2$, $\eta(t) = 1$ if $t \geqslant 1$. Introduce the function

$$H_\varepsilon(x) = \rho^{n-1} z^{-\mu} \eta(z/\varepsilon)\eta(z/\rho) \tag{3.1}$$

if $x_n \geqslant 0$, and $H_\varepsilon(x_1, \dots, x_{n-1}, x_n) = -H_\varepsilon(x_1, \dots, x_{n-1}, -x_n)$ if $x_n < 0$. It is clear that $H_\varepsilon \in C^\infty(\overline{\Omega})$ if the dimension n is odd, and $\rho^{-1}H_\varepsilon \in C^\infty(\overline{\Omega})$ if n is even. We define the function b_ε as follows

$$b_\varepsilon(x) = K\rho^{1-n}\left(x_1\partial_z H_\varepsilon, \dots, x_{n-1}\partial_z H_\varepsilon, -\rho\partial_\rho H_\varepsilon\right).$$

In cylindrical coordinates it means that

$$(b_\varepsilon)_\rho = K\rho^{2-n}\partial_z H_\varepsilon, \quad (b_\varepsilon)_z = -K\rho^{2-n}\partial_\rho H_\varepsilon, \tag{3.2}$$

and all other components are zero. Here K is a large constant, which we choose later (see Lemma 3.4 below); it does not depend on ε.

Lemma 3.1. *The function b_ε possesses the following properties:*

- $b_\varepsilon \in C^\infty(\overline{\Omega})$*;*
- $\operatorname{div} b_\varepsilon = 0$*;*
- *we have*

  $$(b_\varepsilon)_\rho = -\mu K\rho z^{-1-\mu}, \quad (b_\varepsilon)_z = -(n-1)Kz^{-\mu}$$

  on the set

  $$\Omega_\varepsilon := \{x \in \Omega : \rho < z, \varepsilon < z < 1\} \tag{3.3}$$

  (it is a truncated cone in the upper half of the cylinder Ω);
- $b_\varepsilon \in L_p(\Omega)$ *for $p < n/\mu$, and the norms $\|b_\varepsilon\|_{L_p}$ are uniformly bounded with respect to ε.*

Proof. The first three properties follows directly from the construction. Let us verify the last one. For postitve z we have

$$|\nabla H_\varepsilon(x)| \leqslant C\rho^{n-1}z^{-\mu}\left(\frac{1}{\rho} + \frac{1}{z} + \frac{1}{\varepsilon}\chi_{[1/2,1]}\left(\frac{z}{\varepsilon}\right) + \frac{z}{\rho^2}\chi_{[1/2,1]}\left(\frac{z}{\rho}\right)\right)\chi_{[1/2,\infty)}\left(\frac{z}{\rho}\right), \tag{3.4}$$

where $\chi_{[1/2,1]}$ and $\chi_{[1/2,\infty)}$ are the characteristic functions of the interval $[1/2,1]$ and $[1/2,\infty)$ respectively. Next,

$$\frac{1}{\varepsilon}\chi_{[1/2,1]}\left(\frac{z}{\varepsilon}\right)\leqslant\frac{1}{z},\quad \frac{z}{\rho^2}\chi_{[1/2,1]}\left(\frac{z}{\rho}\right)\leqslant\frac{1}{\rho},$$

and

$$\frac{1}{z}\chi_{[1/2,\infty)}\left(\frac{z}{\rho}\right)\leqslant\frac{2}{\rho}\chi_{[1/2,\infty)}\left(\frac{z}{\rho}\right).$$

Therefore,

$$|\nabla H_\varepsilon(x)|\leqslant C\rho^{n-2}z^{-\mu}\chi_{[1/2,\infty)}\left(\frac{z}{\rho}\right)$$

and

$$|b_\varepsilon(x)|\leqslant CKz^{-\mu}\chi_{[1/2,\infty)}\left(\frac{z}{\rho}\right), \tag{3.5}$$

where the constant C depends on the function η only and does not depend on ε. The last inequality implies

$$\int\limits_\Omega |b_\varepsilon(x)|^p dx\leqslant CK^p\int\limits_0^1\rho^{n-2}d\rho\int\limits_{\rho/2}^\infty z^{-\mu p}dz<\infty,$$

because $n-\mu p>0$. □

3.2. Auxiliary function f.

Lemma 3.2. *There exists a function $f\equiv f_\varepsilon\in C^2[\varepsilon,1]$ which possesses the following properties*

1) $f(z)\geqslant 0$, $f'(z)\geqslant 0$;
2) $f(\varepsilon)=0$, $f(2\varepsilon)\geqslant c_1>0$, $f(1)=1$;
3) $f(z)\leqslant c_2f'(z)z^{2-\mu}$, $-f''(z)\leqslant c_3f'(z)z^{-\mu}$.

Here the positive constants c_1, c_2, c_3 depend on μ and do not depend on ε.

Remark 3.3. Such a function can not exist when $\mu=1$. Indeed, the conditions

$$f'(z)\geqslant 0,\quad f(2\varepsilon)\geqslant c_1\quad\text{and}\quad f(z)\leqslant c_2f'(z)z$$

imply that $f'(z)\geqslant c_1c_2^{-1}z^{-1}$ when $z\geqslant 2\varepsilon$. Therefore,

$$1-c_1\geqslant f(1)-f(2\varepsilon)\geqslant c\int\limits_{2\varepsilon}^1\frac{dz}{z}=c|\ln 2\varepsilon|,$$

and we have a contradiction.

Proof of Lemma 3.2. First, we define the function

$$h(t)=\begin{cases}\frac{1}{2}\left(\varepsilon^{-3}-\varepsilon^{\mu-4}\right)t^2-\left(2\varepsilon^{-2}-\varepsilon^{\mu-3}\right)t+\left(2\varepsilon^{-1}+\frac{2}{2-\mu}\varepsilon^{\mu-2}\right), & \varepsilon\leqslant t\leqslant 2\varepsilon,\\ \frac{2^{3-\mu}}{2-\mu}t^{\mu-2}, 2\varepsilon<t\leqslant 1.\end{cases}$$

Its derivative

$$h'(t)=\begin{cases}\left(\varepsilon^{-3}-\varepsilon^{\mu-4}\right)t-2\varepsilon^{-2}+\varepsilon^{\mu-3}, & \varepsilon\leqslant t\leqslant 2\varepsilon,\\ -2^{3-\mu}t^{\mu-3}, & 2\varepsilon<t\leqslant 1,\end{cases}$$

is continuous and negative everywhere. Therefore, the function $h\in C^1[\varepsilon,1]$ is decreasing.

Put $g(z)=\int\limits_{\varepsilon}^{z}h(t)\,dt$. The function g increases, $g\in C^2[\varepsilon,1]$ and $g(\varepsilon)=0$. We have

$$\begin{aligned}g(2\varepsilon)&=\left(\varepsilon^{-3}-\varepsilon^{\mu-4}\right)\frac{7\varepsilon^3}{6}-\left(2\varepsilon^{-2}-\varepsilon^{\mu-3}\right)\frac{3\varepsilon^2}{2}\\&+\left(2\varepsilon^{-1}+\frac{2}{2-\mu}\varepsilon^{\mu-2}\right)\varepsilon=\frac{1}{6}+\left(\frac{1}{3}+\frac{2}{2-\mu}\right)\varepsilon^{\mu-1}>\frac{1}{6},\end{aligned}\tag{3.6}$$

and

$$\begin{aligned}g(1)&=g(2\varepsilon)+\int\limits_{2\varepsilon}^{1}h(t)\,dt=g(2\varepsilon)+\frac{2^{3-\mu}}{(2-\mu)(\mu-1)}\left(1-(2\varepsilon)^{\mu-1}\right)\\&<\frac{1}{6}+\frac{2^{3-\mu}}{(2-\mu)(\mu-1)}+\frac{\varepsilon^{\mu-1}}{3}<\frac{1}{2}+\frac{2^{3-\mu}}{(2-\mu)(\mu-1)}=:d_\mu.\end{aligned}$$

Now, we define the function f as $f(z)=g(z)/g(1)$. It is immediate that the properties 1) and 2) are fulfilled; one can take $c_1=(6d_\mu)^{-1}$. Let us verify the property 3). It is sufficient to check the corresponding inequalities for the function g instead of function f. For $z\leqslant 2\varepsilon$ we have

$$g'(z)=h(z)\geqslant h(2\varepsilon)=\frac{2}{2-\mu}\varepsilon^{\mu-2},\quad g(z)\leqslant g(1)<d_\mu\leqslant Cg'(z)z^{2-\mu},$$

where $C=(2-\mu)d_\mu/2$. Further,

$$g'(z)z^{-\mu}\geqslant\frac{2}{2-\mu}\varepsilon^{\mu-2}(2\varepsilon)^{-\mu}=\frac{2^{1-\mu}}{2-\mu}\varepsilon^{-2},\quad g''(z)=h'(z)\geqslant h'(\varepsilon)=-\varepsilon^2,$$

therefore,

$$-g''(z)\leqslant(2-\mu)2^{\mu-1}g'(z)z^{-\mu}.$$

For $z > 2\varepsilon$ we have

$$g'(z) = \frac{2^{3-\mu}}{2-\mu} z^{\mu-2} \quad \Longrightarrow \quad g(z) \leqslant C g'(z) z^{2-\mu},$$

$C = (2-\mu)2^{\mu-3} d_\mu$. Finally,

$$-g''(z) = (2-\mu) g'(z) \leqslant (2-\mu) g'(z) z^{-\mu}.$$

□

3.3. Barrier function v. Let $f = f_\varepsilon$ be a function constructed in Lemma 3.2. Consider the function $v_\varepsilon(z) = f(z)\cos\frac{\pi\rho}{2z}$ on the set Ω_ε defined by (3.3). Clearly, $v_\varepsilon \in C^2(\overline{\Omega_\varepsilon})$,

$$v_\varepsilon \geqslant 0 \text{ in } \Omega_\varepsilon, \quad v_\varepsilon|_{z=\varepsilon} = 0, \quad v_\varepsilon|_{z=\rho} = 0, \quad v_\varepsilon|_{z=1} = \cos\frac{\pi\rho}{2}, \tag{3.7}$$

and

$$\begin{aligned}
\partial_\rho v_\varepsilon &= -\frac{\pi}{2z} f(z) \sin\frac{\pi\rho}{2z}, \quad \partial_\rho^2 v_\varepsilon = -\frac{\pi^2}{4z^2} f(z)\cos\frac{\pi\rho}{2z}, \\
\partial_z v_\varepsilon &= f'(z)\cos\frac{\pi\rho}{2z} + \frac{\pi\rho}{2z^2} f(z)\sin\frac{\pi\rho}{2z}, \\
\partial_z^2 v_\varepsilon &= f''(z)\cos\frac{\pi\rho}{2z} + \frac{\pi\rho}{z^2} f'(z)\sin\frac{\pi\rho}{2z} - \frac{\pi\rho}{z^3} f(z)\sin\frac{\pi\rho}{2z} - \frac{\pi^2\rho^2}{4z^4} f(z)\cos\frac{\pi\rho}{2z}.
\end{aligned}$$

Lemma 3.4. *Let the function b_ε be defined by formulas* (3.1), (3.2) *with*

$$K > \max\left(\frac{4n}{n-\mu-1}, \pi^2 c_2 + c_3\right),$$

where c_2 and c_3 are the constants from Lemma 3.2. *Then the inequality*

$$\Delta v_\varepsilon(x) - b_\varepsilon(x)\cdot\nabla v_\varepsilon(x) > 0$$

holds in Ω_ε.

Proof. We have

$$\begin{aligned}
\Delta v_\varepsilon &= \partial_\rho^2 v_\varepsilon + \frac{n-2}{\rho}\partial_\rho v_\varepsilon + \partial_z^2 v_\varepsilon \\
&= \left(-\frac{\pi^2}{4z^2} f(z) - \frac{\pi^2\rho^2}{4z^4} f(z) + f''(z)\right)\cos\frac{\pi\rho}{2z} \\
&+ \left(-\frac{(n-2)\pi}{2\rho z} f(z) - \frac{\pi\rho}{z^3} f(z) + \frac{\pi\rho}{z^2} f'(z)\right)\sin\frac{\pi\rho}{2z} \\
&\geqslant \left(-\frac{\pi^2}{2z^2} f(z) + f''(z)\right)\cos\frac{\pi\rho}{2z} - \frac{n\pi}{2\rho z} f(z)\sin\frac{\pi\rho}{2z},
\end{aligned}$$

where we used the inequalities $\rho \leqslant z$ in Ω_ε and $f'(z) > 0$.

Next,

$$-b_\varepsilon \cdot \nabla v_\varepsilon = (n-1)Kz^{-\mu} f'(z)\cos\frac{\pi\rho}{2z} + \frac{n-\mu-1}{2}K\pi\rho z^{-2-\mu} f(z)\sin\frac{\pi\rho}{2z}.$$

Taking into account Lemma 3.2, we get

$$\Delta v_\varepsilon - b_\varepsilon \cdot \nabla v_\varepsilon \geqslant \left((n-1)K - \frac{\pi^2}{2}c_2 - c_3\right) z^{-\mu} f'(z)\cos\frac{\pi\rho}{2z} \qquad (3.8)$$
$$+\left(\frac{n-\mu-1}{2}K\pi\rho z^{-2-\mu} - \frac{n\pi}{2\rho z}\right) f(z)\sin\frac{\pi\rho}{2z}.$$

If $0 < \rho \leqslant z/2$ then $\sin\frac{\pi\rho}{2z} \leqslant \frac{\pi\rho}{2z}$ and $\cos\frac{\pi\rho}{2z} \geqslant \frac{1}{\sqrt{2}}$, therefore

$$\frac{n\pi}{2\rho z} f(z)\sin\frac{\pi\rho}{2z} \leqslant \frac{n\pi^2}{4z^2} f(z) \leqslant \frac{n\pi^2}{4} c_2 f'(z) z^{-\mu} \leqslant \frac{n\pi^2\sqrt{2}}{4} c_2 f'(z) z^{-\mu}\cos\frac{\pi\rho}{2z},$$

where we have used Lemma 3.2 again. Thus, $\Delta v_\varepsilon(x) - b_\varepsilon(x)\cdot\nabla v_\varepsilon(x) > 0$ when $\rho \leqslant z/2$ due to the fact that

$$K > \pi^2 c_2 + c_3 \quad \Longrightarrow \quad (n-1)K > \left(\frac{1}{2} + \frac{n\sqrt{2}}{4}\right)\pi^2 c_2 + c_3.$$

If $z/2 < \rho < z$ then $4\rho^2 \geqslant z^2 \geqslant z^{1+\mu}$ and $\frac{n\pi}{2\rho z} \leqslant 2n\pi\rho z^{-2-\mu}$. Therefore, the last term in the right hand side of (3.8) is positive, as $(n-\mu-1)K > 4n$. □

Remark 3.5. This construction does not work for $n = 2$, because we have used the positiveness of the multiplier $(n-\mu-1)$ in (3.8), and $\mu > 1$.

3.4. Proof of Theorem 1.6 and Theorem 1.9.

Proof of Theorem 1.9. Let the sets Ω, Ω_ε and the function b_ε be defined as before. Then $b_\varepsilon \in C^\infty$, $\operatorname{div} b_\varepsilon = 0$ and the norms $\|b_\varepsilon\|_{L_p(\Omega)}$ are uniformly bounded with respect to ε. Let $u_\varepsilon \in W_2^1(\Omega)$ be the unique solution to the problem

$$\begin{cases} -\Delta u_\varepsilon + b_\varepsilon \cdot \nabla u_\varepsilon = 0 \quad \text{in } \Omega, \\ u_\varepsilon|_{z=\pm 1} = \pm\cos\frac{\pi\rho}{2}, \quad u_\varepsilon|_{\rho=1} = 0. \end{cases}$$

Evidently, $u_\varepsilon|_{\overline{B}_1} \in C^\infty(\overline{B}_1)$ and $\|u_\varepsilon\|_{L_\infty(B_1)} = 1$. The norms $\|u_\varepsilon\|_{W_2^1(B_{1/2})}$ are also uniformly bounded due to the Theorem 1.8. Next, it is clear that the function u_ε is odd,

$$u_\varepsilon(x_1, \dots, x_{n-1}, -x_n) = -u_\varepsilon(x_1, \dots, x_{n-1}, x_n).$$

Therefore, $u_\varepsilon|_{z=0} = 0$. By the maximum principle, $u_\varepsilon(x) \geqslant 0$ when $z \geqslant 0$. This means that $u_\varepsilon(x) \geqslant v_\varepsilon(x)$ on the boundary $\partial\Omega_\varepsilon$, where v_ε is the barrier function constructed in Section 3.3 (see (3.7)). Using the maximum principle for the set Ω_ε and the Lemma 3.2, we get

$$u_\varepsilon(0,\dots,0,z) \geqslant v_\varepsilon(0,\dots,0,z) = f_\varepsilon(z) \geqslant c_1 \quad \forall z \geqslant 2\varepsilon. \tag{3.9}$$

□

Proof of Theorem 1.6. Without loss of generality we can assume $p > n/2$.

We deduce Theorem 1.6 from the Theorem 1.9. Roughly speaking, we repeat here the argument of [8]. Put

$$H_0(x) = \rho^{n-1} z^{-\mu}\eta(z/\rho) \quad \text{when} \quad x_n \geqslant 0,$$
$$H_0(x_1,\dots,x_{n-1},x_n) = -H_0(x_1,\dots,x_{n-1},-x_n) \quad \text{when} \quad x_n < 0.$$

Let

$$(b_0)_\rho = K\rho^{2-n}\partial_z H_0, \quad (b_0)_z = -K\rho^{2-n}\partial_\rho H_0,$$

and all other components be zero. The constant K here is defined in Lemma 3.4. It is evident that $b_\varepsilon \to b_0$ a.e. as $\varepsilon \to 0$, and $|b_\varepsilon(x)| \leqslant CKz^{-\mu}\chi_{[1/2,\infty)}(z/\rho)$ due to (3.5). Therefore, the same estimate has place for the function b_0, $b_0 \in L_p$, and $b_\varepsilon \to b_0$ in L_p for all $p < n/\mu$. This yields also that $\operatorname{div} b_0 = 0$.

By virtue of the Theorem 1.1 and the inequality (3.5), the functions u_ε are uniformly bounded in $W^2_p(U)$, for all subdomains U with smooth boundaries such that $\overline{U} \subset \Omega \setminus \{0\}$. The imbedding $W^2_p(U) \subset C(\overline{U})$ is compact, therefore, there is a subsequence $\{u_{\varepsilon_k}\}$ which converges uniformly on $\overline{U}$. Furthermore, Theorem 1.8 implies that the sequence $\{u_{\varepsilon_k}\}$ is uniformly bounded in $W^1_2(B_{1/2})$. Without loss of generality one can assume that u_{ε_k} tends pointwise to a function u_0,

$$u_{\varepsilon_k}(x) \to u_0(x) \quad \forall x \neq 0,$$

and $u_{\varepsilon_k} \to u_0$ weakly in $W^1_2(B_{1/2})$. Clearly, $\|u_0\|_{L_\infty(B_{1/2})} \leqslant 1$.

We have for any $h \in C_0^\infty(B_{1/2})$

$$\int u_0\,(\Delta h + b_0\cdot\nabla h)\,dx = \lim_{k\to\infty}\int u_{\varepsilon_k}\,(\Delta h + b_{\varepsilon_k}\cdot\nabla h)\,dx = 0.$$

Thus, the equations (1.1) and (1.4) are fulfilled for u_0, b_0.

Finally, the function u_0 is odd,

$$u_0(x_1,\dots,x_{n-1},-x_n) = -u_0(x_1,\dots,x_{n-1},x_n),$$

but

$$u_0(0,\dots,0,z) \geqslant c_1, \quad \forall z > 0,$$

due to (3.9). Therefore, the function u_0 is discontinuous at the origin. □

§4. Comments and remarks

4.1. Case $n = 1$. We do not consider the one-dimensional case, because the equation $-u''(x) + b(x)u'(x) = 0$ admits an explicit solution

$$u(x) = C_1 \int_0^x \exp\left(\int_0^y b(t)\,dt\right) dy + C_2.$$

4.2. On Stampacchia's Theorem. It is announced in [10] that a solution to (1.2) under the conditions (1.3) and $b \in L_n$ must be bounded [10, Theorem 4.1], and therefore, Hölder continuous [10, Theorem 7.1] for all $n \geqslant 2$. These Theorems are proven in [10] for $n \geqslant 3$. However, for $n = 2$, both statement are false, see Examples 1 and 2 in §1. The reason is that the imbedding Theorem $W_2^1 \subset L_{2n/(n-2)}$ used in [10] has no place when $n = 2$.

4.3. Morrey space. Let us recall the definition of Morrey's spaces:

$$M_q^\alpha(\Omega) = \{f \in L_q(\Omega) : \|f\|_{M_q^\alpha} = \sup_{B_r(x)\subset\Omega} r^{-\alpha}\|f\|_{L_q(B_r(x))} < \infty\}.$$

The following result is proved in [7].

Theorem 4.1. *Let a satisfy* (1.3), $b \in M_q^{\frac{n}{q}-1}(B_R)$, $n/2 < q < n$, $\operatorname{div} b = 0$. *Let $u \in W_2^1(B_R)$ solve the equation* (1.2). *Then $u \in C^\alpha(B_R)$ with some $\alpha > 0$.*

The Hölder inequality implies that $L_p \subset M_q^{\frac{n}{q}-\frac{n}{p}}$, $1 \leqslant q \leqslant p$. Therefore, Theorem 1.6 shows that the power $(n/q - 1)$ in Theorem 4.1 is sharp.

4.4. Space $L_{2,\ln}$. The following result has place.

Theorem 4.2. *Let $n = 2$. Assume that the coefficient b satisfies the condition*

$$\int_{B_R} |b(x)|^2 \ln(1 + |b(x)|^2)\,dx < \infty. \tag{4.1}$$

Let $u \in W_2^1(B_R)$ be a solution to (1.1). *Then*

$$u \in \bigcap_{q<2} W_q^2(B_r) \subset \bigcap_{\alpha<1} C^\alpha(B_r) \quad \forall r < R.$$

Denote by $L_{2,\ln}(B_R)$ the space of measurable functions b (modulo functions vanishing on the set of full measure) satisfying (4.1) (clearly, $L_{2,\ln} \subset L_2$). It is the Orlicz space corresponding to the function $t^2 \ln(1+t^2)$. The theory of Orlicz spaces can be found for example in [4]. Recall some basic facts on such space. The quantity

$$\|b\|_{L_{2,\ln}(B_R)} = \inf\left\{k > 0 : \int\limits_{B_R} \left|\frac{b(x)}{k}\right|^2 \ln\left(1 + \left|\frac{b(x)}{k}\right|^2\right) dx \leqslant 1\right\}$$

is well defined for $b \in L_{2,\ln}(B_R)$. One can show that this functional is a norm, and that

$$\|b\|_{L_{2,\ln}(B_r)} \to 0 \quad \text{as} \quad r \to 0.$$

Lemma 4.3. *Let $n = 2$, $R \leqslant 1$, $b \in L_{2,\ln}(B_R)$, $\psi \in \mathring{W}_2^1(B_R)$. Then $b\psi \in L_2(B_R)$ and*

$$\|b\psi\|_{L_2(B_R)} \leqslant C_0 \|b\|_{L_{2,\ln}(B_R)} \|\nabla\psi\|_{L_2(B_R)},$$

where C_0 is an absolute constant.

Proof. Follows form the fact (see, for example, [3, Theorem 7.15]) that all functions from $\mathring{W}_2^1(B_R)$ satisfy the estimate

$$\int\limits_{B_R} \exp\left(\frac{|\psi(x)|^2}{a_1^2 \|\psi\|^2_{W_2^1(B_R)}}\right) dx \leqslant a_2 |B_R|$$

with two constants a_1, a_2, and the elementary inequality

$$\xi\eta \leqslant \xi \ln \xi + e^\eta, \quad \xi, \eta > 0. \qquad \square$$

Now, the proof of Theorem 4.2 is similar to the proof of Theorem 1.2. The uniqueness of weak solution (an analogue of Lemma 2.6) follows from the estimate

$$\begin{aligned} \left|\int\limits_{B_R} b \cdot \nabla u u \, dx\right| &\leqslant \|bu\|_{L_2(B_R)} \|\nabla u\|_{L_2(B_R)} \\ &\leqslant C_1 \|b\|_{L_{2,\ln}(B_R)} \|\nabla u\|^2_{L_2(B_R)} \quad \forall\, u \in \mathring{W}_2^1(B_R) \qquad (4.2) \end{aligned}$$

if the norm $\|b\|_{L_{2,\ln}(B_R)}$ is sufficiently small.

We borrowed the condition (4.1) from [7]. Under the conditions (1.3) and (4.1) it is proven in [7] that any solution to the equation (1.2) is Hölder continuous (see comments at the end of §2 in [7]). Note that the condition (4.1) can not be changed by the finiteness of the integral $\int\limits_{B_R} |b(x)|^2 \left(\ln(1+|b(x)|^2)\right)^\gamma dx$ with any $\gamma < 1$ (see Example 1).

4.5. Maximum principle. If the coefficient b satisfies the conditions of Theorems 1.2, 1.5 or 4.2, then a solution u to the equation (1.2) satisfies the maximum principle [7, Corollary 2.2 and comments at the end of §2]. Examples 2) and 4) in Section 1 show that the conditions imposed on b again can not be weakened.

4.6. Open questions. The following questions remain open.

- Let $n \geqslant 3$, $b \in L_p(B_R)$, $2 \leqslant p < n$, and $\operatorname{div} b = 0$. Whether a solution $u \in W_2^1(B_R)$ to equation (1.1) should be bounded in B_r, $r < R$?
- Let $n = 2$, $b \in L_2(B_R)$. Whether a solution $u \in W_2^1(B_R) \cap L_\infty(B_R)$ to equation (1.1) should be continuous?

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