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SOLVABILITY OF A FREE BOUNDARY PROBLEM OF MAGNETOHYDRODYNAMICS IN AN INFINITE TIME INTERVAL

Abstract. We prove global in time solvability of a free boundary problem governing the motion of a finite isolated mass of a viscous incompressible electrically conducting capillary liquid in vacuum, under the smallness assumptions on initial data. We assume that initial position of a free boundary is close to a sphere. We show that if $t \to \infty$, then the solution tends to zero exponentially and the free boundary tends to a sphere of the same radius, but, in general, the sphere may have a different center. The solution is obtained in Sobolev–Slobodetskii spaces $W_2^{2+l,1+l/2}$, $1/2 < l < 1$.

Dedicated to the memory of Professor M. Padula

§1. Introduction

We consider the free boundary problem governing the motion of a finite isolated mass of a viscous incompressible electrically conducting capillary liquid. It is assumed that the liquid is contained in a bounded variable domain Ω_{1t} whose boundary consists of two disjoint components: the free boundary Γ_t and the fixed surface Σ that ia also a boundary of the fixed domain D. Both Γ_t and Σ are homeomorphic to a sphere. The domain $\overline{D} \cup \Omega_{1t}$ is surrounded by a bounded vacuum region Ω_{2t}; we set $\Omega = \Omega_{1t} \cup \Gamma_t \cup \Omega_{2t}$; Ω is bounded by Σ and the exterior surface S, also homeomorphic to a sphere. It is assumed that S and Σ are independent of time, perfectly conducting closed surfaces such that $\Gamma_t \cap S = \varnothing$, $\Gamma_t \cap \Sigma = \varnothing$. The problem consists of determination of the variable domains Ω_{it}, $i = 1, 2$, together with the velocity vector field $\mathbf{v}(x,t)$, the pressure $p(x,t)$, $x \in \Omega_{1t}$, and the magnetic field $\mathbf{H}(x,t)$, $x \in \Omega_{1t} \cup \Omega_{2t}$, from the following system of

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equations, initial and boundary conditions:

$$
\begin{aligned}
&\mathbf{v}_t + (\mathbf{v}\cdot\nabla)\mathbf{v} - \nabla\cdot T(\mathbf{v},p) - \nabla\cdot T_M(\mathbf{H}) = 0, \quad \nabla\cdot\mathbf{v}(x,t) = 0,\\
&\mu_1\mathbf{H}_t + \alpha^{-1}\mathrm{rot}\,\mathrm{rot}\,\mathbf{H} - \mu_1\mathrm{rot}\,(\mathbf{v}\times\mathbf{H}) = 0,\ \nabla\cdot\mathbf{H}(x,t) = 0,\ x\in\Omega_{1t},\ t>0,\\
&\mathrm{rot}\,\mathbf{H} = 0, \quad \nabla\cdot\mathbf{H}(x,t) = 0, \quad x\in\Omega_{2t}, \quad t>0,\\
&\mathbf{v}(x,t) = 0, \quad x\in\Sigma, \quad t>0, \qquad (1.1)\\
&\big(T(\mathbf{v},p) + [T_M(\mathbf{H})]\big)\mathbf{n} = \sigma\mathbf{n}\mathcal{H}, \quad \mathbf{V}_n = \mathbf{v}\cdot\mathbf{n},\\
&[\mu\mathbf{H}\cdot\mathbf{n}] = 0, \quad [\mathbf{H}_\tau] = 0, \quad x\in\Gamma_t, \quad t>0,\\
&\mathbf{H}(x,t)\cdot\mathbf{n}(x) = 0, \quad x\in S, \quad t>0,\\
&\mathbf{H}(x,t)\cdot\mathbf{n}(x) = 0, \quad \mathrm{rot}_\tau\mathbf{H} = 0, \quad x\in\Sigma, \quad t>0,\\
&\mathbf{v}(x,0) = \mathbf{v}_0(x), \quad x\in\Omega_{10}, \qquad \mathbf{H}(x,0) = \mathbf{H}_0(x), \quad x\in\Omega_{10}\cup\Omega_{20}.
\end{aligned}
$$

Henceforth, we assume that the density of the fluid is equal to one and use the following notation:

ν, α, σ are positive constants (the kinematic viscosity, conductivity, coefficient of the surface tension);

$\mathcal{H}$ is the doubled mean curvature of Γ_t,

$T(\mathbf{v},p) = -pI + \nu S(\mathbf{v})$ is the viscous stress tensor,

$S(\mathbf{v}) = \nabla\mathbf{v} + (\nabla\mathbf{v})^T = \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right)_{i,j=1,2,3}$ is the doubled rate-of-strain tensor,

$\mathbf{V}_n$ is the velocity of evolution of the surface Γ_t in the direction of the exterior normal $\mathbf{n}$ to Γ_t,

μ (magnetic permeability) is a piecewise constant function (which is equal to $\mu_i > 0$ in Ω_{it}, $i = 1, 2$),

$T_M(\mathbf{H}) = \mu(\mathbf{H}\otimes\mathbf{H} - \frac{1}{2}I|\mathbf{H}|^2)$ is the magnetic stress tensor.

Ω_{10} is a given initial configuration of the liquid,

$\partial\Omega_{10} = \Sigma\cup\Gamma_0$,

$[u] = u^{(1)} - u^{(2)}$ – jump of $u(x)$ on Γ_t, $u^{(i)} = u|_{x\in\overline{\Omega}_{it}}$.

The problem similar to (1.1) but without a rigid domain D is studied in the paper [1], where local (in time) solvability of this problem is proved for a closed surface Γ_0 of arbitrary shape (but such that Ω_{01} and $\Omega = \overline{\Omega}_{10}\cup\Omega_{20}$ are simply connected) and arbitrary initial data $\mathbf{v}_0(x)$, $\mathbf{H}_0(x)$, satisfying natural compatibility conditions and regularity assumptions. In the present paper, we prove the unique solvability of problem (1.1) in an infinite time interval $t > 0$ under the additional smallness assumptions for the initial data and in the case when the surface Γ_0 is close to the sphere

$S_{R_0}(0) = \{|y| = R_0\}$ with R_0 defined by

$$|D| + |\Omega_{10}| = \frac{4}{3}\pi R_0^3\,,$$

where $|D|$ is the volume of D. We also show that when $t \to \infty$, then the velocity, pressure, and the magnetic field tend to zero exponentially and Γ_t tends to a sphere of the same radius but, in general, of a different center.

We assume that Γ_0 can be regarded as the normal perturbation of S_{R_0}:

$$\Gamma_0 = \big\{x = y + \mathbf{N}(y)\rho_0(y), \quad y \in S_{R_0}\big\}\,,$$

where $\mathbf{N}(y) = \frac{y}{|y|}$ is the exterior normal to S_{R_0} and ρ_0 is a given small function. Moreover, let Γ_t be given by the equation of a similar form, i.e.,

$$\Gamma_t = \big\{x = y + \mathbf{N}(y)\rho(y,t) + \xi(t), \quad y \in S_{R_0}\big\},$$

where $\rho(y,t)$ is an unknown function and

$$\xi(t) = \frac{1}{|\Omega_0|}\int\limits_{\Omega_t} x\,dx$$

is also unknown (this is the barycenter point of the domain $\Omega_t = \overline{D} \cup \Omega_{1t}$ filled with the liquid of the density 1). We assume that $\xi(0) = 0$, i.e., $\int\limits_{\Omega_0} x\,dx = 0$. The derivative of $\xi(t)$ is given by the relation

$$\xi'(t) = \frac{1}{|\Omega_0|}\frac{d}{dt}\int\limits_{\Omega_t} x\,dx = \frac{1}{|\Omega_0|}\int\limits_{\Omega_t} \mathbf{v}(x,t)dx = \frac{1}{|\Omega_0|}\int\limits_{\Omega_{1t}} \mathbf{v}(x,t)\,dx,$$

provided that $\mathbf{v}(x,t) = 0$ for $x \in D$.

We assume that

$$|\xi(t)| + |\rho(y,t)| < d_0,$$

then the free boundary Γ_t is located in the layer

$$0 < R_0 - d_0 \leqslant |y| \leqslant R_0 + d_0.$$

This assumption is valid for sufficiently small initial data (see Remark 3.1).

In order to reduce problem (1.1) to a problem in a fixed domain, we construct the mapping of $\Omega = \Omega_{1t} \cup \Gamma_t \cup \Omega_{2t}$ on $\Omega = \mathcal{F}_1 \cup S_{R_0} \cup \mathcal{F}_2$, where $\mathcal{F}_1$ is the domain bounded by Σ and S_{R_0} and $\mathcal{F}_2 = \Omega \setminus \overline{\mathcal{F}}_1$; $\partial\mathcal{F}_2 = S \cup S_{R_0}$. We take a smooth nonnegative cut-off function $\chi(y)$ equal to 1, when y belongs to the layer $R_0 - d_0 \leqslant |y| \leqslant R_0 + d_0$ and vanishing outside the layer $R_0 - 2d_0 \leqslant |y| \leqslant R_0 + 2d_0$ (for small d_0 this layer is contained in Ω) and we define the mapping

$$x = y + \mathbf{N}^*(y)\rho^*(y,t) + \chi(y)\xi(t) \equiv e_{\rho,\xi}, \quad y \in \Omega, \tag{1.2}$$

where $\mathbf{N}^*(y)$ and $\rho^*(y,t)$ are sufficiently regular extensions of $\mathbf{N}$ and ρ from S_{R_0} into Ω, in particular, we assume that $\rho^*(y,t) = 0$ near S and Σ, and that the C^1-norm of ρ^* is small. In our problem, it is possible to set $\mathbf{N}^*(y) = \mathbf{N}(y) = \frac{y}{|y|}$. When ρ^* and $\xi(t)$ are sufficiently small, then (1.2) establishes one-to-one correspondence between $\mathcal{F}_i$ and Ω_{it}, $i = 1,2$. We denote by $\mathcal{L}(y,\rho^*,\xi)$ the Jacobi matrix of the transformation $x = e_{\rho,\xi}(y)$ and we set $L = \det \mathcal{L}$, $\widehat{\mathcal{L}} = L\mathcal{L}^{-1}$.

Transformation (1.2) transforms (1.1) to a nonlinear problem in the fixed domain $\Omega = \mathcal{F}_1 \cup S_{R_0} \cup \mathcal{F}_2$ with respect to unknown functions

$$\mathbf{u}(y,t) = \mathbf{v} \circ e_{\rho,\xi}, \quad q(y,t) = p \circ e_{\rho,\xi} - \frac{2\sigma}{R_0},$$

and

$$\mathbf{h}(y,t) = \widehat{\mathcal{L}}(y,\rho^*,\xi)(\mathbf{H} \circ e_{\rho,\xi}).$$

If we separate linear and nonlinear terms in all the equations, then our problem can be rewritten in the form

$$\begin{aligned}
&\mathbf{u}_t - \nu\nabla^2\mathbf{u} + \nabla q = \mathbf{l}_1(\mathbf{u},q,\mathbf{h},\rho), \quad \nabla\cdot\mathbf{u} = l_2(\mathbf{u},\rho), \quad y \in \mathcal{F}_1,\ t>0,\\
&\mathbf{u}(y,t)\big|_{y\in\Sigma} = 0,\\
&\nu\Pi_0 S(\mathbf{u})\mathbf{N} = \mathbf{l}_3(\mathbf{u},\rho),\\
&-q + \nu\mathbf{N}\cdot S(\mathbf{u})\mathbf{N}(y) + \sigma B_0\rho = l_4(\mathbf{u},\mathbf{h},\rho) + l_5(\rho),\\
&\rho_t - \mathbf{u}\cdot\mathbf{N}(y) + \frac{1}{|\Omega_0|}\int\limits_{\mathcal{F}_1}\mathbf{u}\,dy\cdot\mathbf{N}(y) = l_6(\mathbf{u},\rho), \quad y \in S_{R_0}, \quad t>0, \qquad (1.3)\\
&\mu_1\mathbf{h}_t + \alpha^{-1}\mathrm{rot}\,\mathrm{rot}\,\mathbf{h} = \mathbf{l}_7(\mathbf{h},\mathbf{u},\rho), \quad \nabla\cdot\mathbf{h} = 0, \quad y\in\mathcal{F}_1, \quad t>0,\\
&\mathrm{rot}\,\mathbf{h} = \mathrm{rot}\,\mathbf{l}_8(\mathbf{h},\rho), \quad \nabla\cdot\mathbf{h} = 0, \quad y\in\mathcal{F}_2,\\
&[\mu\mathbf{h}\cdot\mathbf{N}] = 0, \quad [\mathbf{h}_\tau] = \mathbf{l}_9(\mathbf{h},\rho), \quad y\in S_{R_0}, \quad t>0,\\
&\mathbf{h}(y,t)\cdot\mathbf{n}(y) = 0, \quad y\in S\cup\Sigma, \quad \mathrm{rot}_\tau\mathbf{h} = 0, \quad y\in\Sigma, \quad t>0,\\
&\mathbf{u}(y,0) = \mathbf{u}_0(y), \quad y\in\mathcal{F}_1, \quad \mathbf{h}(y,0) = \mathbf{h}_0(y), \quad y\in\mathcal{F}_1\cup\mathcal{F}_2,\\
&\rho(y,0) = \rho_0(y), \quad y\in S_{R_0},
\end{aligned}$$

where $\mathbf{l}_1 - \mathbf{l}_9$ are the nonlinear terms presented by the relations

$$
\begin{aligned}
&\mathbf{l}_1(\mathbf{u},q,\rho,\mathbf{h}) = \nu(\widetilde{\nabla}^2-\nabla^2)\mathbf{u}+(\nabla-\widetilde{\nabla})q+(\mathcal{L}^{-1}(\mathbf{N}^*(y)\rho_t^*\\
&\qquad +\chi(y)\xi^{'}(t))\cdot\nabla)\mathbf{u}-(\mathcal{L}^{-1}\mathbf{u}\cdot\nabla)\mathbf{u}+\widetilde{\nabla}\cdot T_M\Big(\frac{\mathcal{L}}{L}\mathbf{h}\Big),\\
&l_2(\mathbf{u},\rho) = (I-\widehat{\mathcal{L}}^T)\nabla\cdot\mathbf{u} = \nabla\cdot(I-\widehat{\mathcal{L}})\mathbf{u},\quad y\in\mathcal{F}_1,\\
&\mathbf{l}_3(\mathbf{u},\rho) = \nu\Pi_0(\Pi_0 S(\mathbf{u})\mathbf{N})(y)-\Pi\widetilde{S}(\mathbf{u})\mathbf{n}(e_{\rho,\xi}(y)),\\
&l_4(\mathbf{u},\mathbf{h},\rho) = \nu(\mathbf{N}\cdot S(\mathbf{u})\mathbf{N}-\mathbf{n}\cdot\widetilde{S}(\mathbf{u})\mathbf{n})-\Big[T_M\Big(\frac{\mathcal{L}}{L}\mathbf{h}\Big)\Big]\mathbf{n},\\
&l_5(\rho) = -\sigma\int_0^1(1-s)\frac{d^2}{ds^2}\mathcal{L}^{-T}(y,s\rho)\nabla\cdot\frac{\mathcal{L}^T(y,s\rho)\mathbf{N}}{|\mathcal{L}^T(y,s\rho)\mathbf{N}|}\,ds, \qquad (1.4)\\
&l_6(\mathbf{u},\rho) = \Big(\mathbf{u}-\frac{1}{|\Omega_0|}\int_{\mathcal{F}_1}\mathbf{u}\,dz\Big)\cdot\Big(\frac{\mathbf{n}_{e_{\rho,\xi}}}{n_r}-\mathbf{N}\Big)\\
&\quad -\frac{1}{|\Omega_0|}\int_{\mathcal{F}_1}\mathbf{u}(L-1)\,dz\cdot\frac{\mathbf{n}(e_{\rho,\xi})}{n_r},\quad y\in S_{R_0},\\
&\mathbf{l}_7(\mathbf{h},\rho) = \alpha^{-1}\mathrm{rot}\Big(\mathrm{rot}\,\mathbf{h}-\frac{1}{L}\mathcal{L}^T\mathcal{L}\,\mathrm{rot}\,\frac{1}{L}\mathcal{L}^T\mathcal{L}\,\mathbf{h}\Big)+\frac{1}{L}\widehat{\mathcal{L}}_t\mathcal{L}\mathbf{h}\\
&\quad +\widehat{\mathcal{L}}(\mathcal{L}^{-1}(\mathbf{N}^*\rho_t^*+\chi(y)\xi^{'}(t))\cdot\nabla)\frac{1}{L}\mathcal{L}\mathbf{h}+\mu_1\mathrm{rot}\,(\mathcal{L}^{-1}\mathbf{u}\times\mathbf{h}),\quad y\in\mathcal{F}_1,\\
&\mathbf{l}_8(\mathbf{h},\rho) = \Big(I-\frac{1}{L}\mathcal{L}^T\mathcal{L}\Big)\mathbf{h},\quad y\in\mathcal{F}_2,\\
&l_9(\mathbf{h},\rho) = \Big(\frac{\widehat{\mathcal{L}}\widehat{\mathcal{L}}^T\mathbf{N}}{|\widehat{\mathcal{L}}^T\mathbf{N}|^2}-\mathbf{N}\Big)[\mathbf{h}\cdot\mathbf{N}],\quad y\in S_{R_0},
\end{aligned}
$$

where $\widetilde{\nabla}$ is the transformed gradient ∇_x, i.e., $\widetilde{\nabla}=\mathcal{L}^{-T}\nabla$, $\nabla=\nabla_y$, $\widetilde{S}(\mathbf{u})=\mathcal{L}^{-T}\nabla\otimes\mathbf{u}+(\mathcal{L}^{-T}\nabla\otimes\mathbf{u})^T$; the sign $(\cdot)^T$ means transposition;

$$
\Pi\mathbf{f}=\mathbf{f}-\mathbf{n}(\mathbf{n}\cdot\mathbf{f}),\quad \Pi_0\mathbf{f}=\mathbf{f}-\mathbf{N}(\mathbf{f}\cdot\mathbf{N}),\quad B_0\rho=-\frac{1}{R_0^2}(\Delta_{S_1}\rho+2\rho),
$$

Δ_{S_1} is the Laplacean on the unit sphere S_1. The expression $-\sigma B_0\rho$ is the first variation of $\sigma(\mathcal{H}+\frac{2}{R_0})$ with respect to ρ, and $l_5(\rho)$ is a nonlinear remainder. The nonlinear terms l_1-l_5, l_7-l_9 are calculated in the same way as it is done in [1], and we omit the details. Let us show that the nonlinear term in the dynamic boundary condition has the form l_6. Since

$V_n = \mathbf{n}\cdot\mathbf{x}_t = \mathbf{n}\cdot(\xi'(t)+\rho_t\mathbf{N})$, this condition is equivalent to

$$\rho_t n_r - \mathbf{v}\cdot\mathbf{n} + \xi'(t)\cdot\mathbf{n} = \rho_t n_r - \mathbf{v}\cdot\mathbf{n} + \frac{1}{|\Omega_0|}\int\limits_{\Omega_{1t}} \mathbf{v}(x,t)\,dx\cdot\mathbf{n} = 0,$$

i.e.,

$$\rho_t - \mathbf{u}\cdot\mathbf{N} + \frac{1}{|\Omega_0|}\int\limits_{\mathcal{F}_1}\mathbf{u}(y,t)\,dy\cdot\mathbf{N} = \big(\mathbf{u} - \frac{1}{|\Omega_0|}\int\limits_{\mathcal{F}_1}\mathbf{u}(y,t)\,dy\big)\cdot\Big(\frac{\mathbf{n}(e_{\rho,\xi})}{n_r} - \mathbf{N}\Big)$$
$$- \frac{1}{|\Omega_0|}\int\limits_{\mathcal{F}_1}\mathbf{u}\big(L-1\big)\,dy\cdot\frac{\mathbf{n}(e_{\rho,\xi})}{n_r} \equiv l_6,$$

where

$$\mathbf{n}(e_{\rho,\xi}) = \frac{\widehat{\mathcal{L}}^T\mathbf{N}(y)}{|\widehat{\mathcal{L}}^T\mathbf{N}(y)|}.$$

The assumptions

$$|\Omega_0| = \frac{4}{3}\pi R_0^3, \quad \text{and} \quad \int\limits_{\Omega_0} x_i\,dx = 0$$

lead to the following conditions for ρ_0:

$$\int\limits_{S_1}((R_0+\rho_0)^3 - R_0^3)\,dS = 0, \qquad \int\limits_{S_1} y_i((R_0+\rho_0)^4 - R_0^4)\,dS = 0, \quad i=1,2,3.$$

Thus,

$$\int\limits_{S_1}\rho_0(R_0y)\,dS = -\frac{1}{R_0}\int\limits_{S_1}\rho_0^2\,dS - \frac{1}{3R_0^2}\int\limits_{S_1}\rho_0^3\,dS, \tag{1.5}$$
$$\int\limits_{S_1} y_i\rho_0(R_0y)\,dS = -\frac{3}{2R_0}\int\limits_{S_1}y_i\rho_0^2\,dS - \frac{1}{R_0^2}\int\limits_{S_1}y_i\rho_0^3\,dS - \frac{1}{4R_0^3}\int\limits_{S_1}y_i\rho_0^4\,dS.$$

Natural compatibility conditions in the problem (1.3) are as follows:

$$\begin{gathered}\nabla\cdot\mathbf{u}_0 = l_2(\mathbf{u}_0,\rho_0), \quad y\in\mathcal{F}_1,\\ \nu\Pi_0 S(\mathbf{u}_0)\mathbf{N}(y) = \mathbf{l}_3(\mathbf{u}_0,\rho_0), \quad y\in S_{R_0}, \quad \mathbf{u}_0 = 0, \quad y\in\Sigma,\end{gathered} \tag{1.6}$$

and

$$\operatorname{rot}\mathbf{h}_0^{(2)} = \operatorname{rot}\mathbf{l}_8(\mathbf{h}_0^{(2)},\rho_0), \quad y\in\mathcal{F}_2, \quad [\mathbf{h}_{0\tau}] = \mathbf{l}_9(\mathbf{h}_0,\rho_0), \quad y\in S_{R_0},$$
$$\nabla\cdot\mathbf{h}_0^{(1)} = 0, \quad y\in\mathcal{F}_1, \quad \nabla\cdot\mathbf{h}_0^{(2)} = 0, \quad y\in\mathcal{F}_2, \tag{1.7}$$
$$[\mu\mathbf{h}_0\cdot\mathbf{N}] = 0, \quad y\in S_{R_0}, \quad \mathbf{h}_0\cdot\mathbf{n} = 0, \quad y\in\Sigma\cup S, \quad \operatorname{rot}_\tau\mathbf{h}_0 = 0, \quad y\in\Sigma.$$

To state the main result of the paper, we recall some definitions. By the Sobolev space $W_2^l(\Omega)$, $\Omega\in\mathbb{R}^n$ with noninteger $l>0$ we mean the space of functions $u(x)$, $x\in\Omega$ with the finite norm

$$\| u \|^2_{W_2^l(\Omega)} = \| u \|^2_{W_2^{[l]}(\Omega)} + \sum_{|\alpha|=[l]}\int_\Omega\int_\Omega \left|D^\alpha u(x) - D^\alpha u(y)\right|^2 \frac{dxdy}{|x-y|^{n+2(l-[l])}},$$

where

$$\| u \|^2_{W_2^{[l]}(\Omega)} = \sum_{0\leqslant|\alpha|\leqslant[l]}\int_\Omega \left|D^\alpha u(x)\right|^2 dx$$

is the norm in the space $W_2^{[l]}(\Omega)$. The anisotropic Sobolev–Slobodetskii space $W_2^{l,l/2}(Q_T)$ in the cylindrical domain $Q_T=\Omega\times(0,T)$ can be defined as the space $L_2((0,T),W_2^l(\Omega))\cap L_2(\Omega,W_2^{l/2}(0,T))$ with the norm

$$\| u \|^2_{W_2^{l,l/2}(Q_T)} = \int_0^T \| u(\cdot,t) \|^2_{W_2^l(\Omega)}\, dt + \int_\Omega \| u(x,\cdot) \|^2_{W_2^{l/2}(0,T)}\, dx. \tag{1.8}$$

There exists many other equivalent norms in $W_2^{l,l/2}(Q_T)$. Sobolev spaces of functions defined on the smooth surfaces are introduced in a standard way, with the help of local maps and partition of unity. We also find it convenient to use the spaces $W_2^{l,0}(Q_T) = L_2((0,T),W_2^l(\Omega))$ and $W_2^{0,l/2}(Q_T) = L_2((\Omega,W_2^{l/2}(0,T))$. The squares of norms in these spaces coincides with the first and the second term in (1.8), respectively.

In the present paper we prove global solvability of the problem (1.3) in anisotropic Sobolev–Slobodetskii spaces.

Theorem 1.1. *Let*

$$\mathbf{u}_0\in W_2^{1+l}(\mathcal{F}_1), \quad \rho_0\in W_2^{2+l}(S_{R_0}), \quad \mathbf{h}_0^{(i)}\in W_2^{1+l}(\mathcal{F}_i)$$

with a certain $l \in (1/2, 1)$*, the compatibility conditions* (1.6), (1.7)*, conditions* (1.5)*, and the following smallness condition*

$$\|\mathbf{u}_0\|_{W_2^{l+1}(\mathcal{F}_1)} + \|\rho_0\|_{W_2^{l+2}(S_{R_0})} + \sum_{i=1}^{2} \|\mathbf{h}_0^{(i)}\|_{W_2^{l+1}(\mathcal{F}_i)} \leqslant \epsilon \ll 1 \qquad (1.9)$$

be satisfied. Then problem (1.3) *has a unique solution with the following regularity properties:*

$$\mathbf{u} \in W_2^{2+l,1+l/2}(Q_\infty^1), \quad \nabla q \in W_2^{l,l/2}(Q_\infty^1),$$

$$q \in W_2^{l+1/2,0}(G_\infty) \cap W_2^{l/2}(0,\infty; W_2^{1/2}(S_{R_0})),$$

$$\rho \in W_2^{l+5/2,0}(G_\infty) \cap W_2^{l/2}(0,+\infty; W_2^{5/2}(S_{R_0})), \quad \rho_t \in W_2^{l+3/2,l/2+3/4}(G_\infty),$$

$$\mathbf{h}^{(i)} \in W_2^{2+l,1+l/2}(Q_\infty^i), \quad \mathbf{h}_t^{(i)} \in W_2^{l/2}(0,+\infty, W_2^{-1/2}(S_{R_0}),$$

where

$$Q_\infty^i = \mathcal{F}_i \times (0,\infty), \quad G_\infty = S_{R_0} \times (0,\infty), \quad \mathbf{h}^{(i)} = \mathbf{h}|_{x\in\mathcal{F}_i}, \quad i = 1,2.$$

The solution satisfies the estimate

$$\begin{aligned}
&\|e^{at}\mathbf{u}\|_{W_2^{2+l,1+l/2}(Q_\infty^1)} + \|e^{at}\nabla q\|_{W_2^{l,l/2}(Q_\infty^1)} + \|e^{at}q\|_{W_2^{l+1/2,0}(G_\infty)} \\
&\quad + \|e^{at}q\|_{W_2^{l/2}(0,\infty;W_2^{1/2}(S_{R_0}))} + \|e^{at}\rho\|_{W_2^{l+5/2,0}(G_\infty)} \\
&\quad + \|e^{at}\rho\|_{W_2^{l/2}(0,\infty;W_2^{5/2}(S_{R_0}))} + \|e^{at}\rho_t\|_{W_2^{l+3/2,l/2+3/4}(G_\infty)} \\
&\quad + \sum_{i=1}^{2}\Big(\|e^{at}\mathbf{h}^{(i)}\|_{W_2^{2+l,1+l/2}(Q_\infty^i)} + \|e^{at}\mathbf{h}_t^{(i)}\|_{W^{l/2}(0,\infty;W_2^{-1/2}(S_{R_0}))}\Big) \\
&\quad \leqslant c\bigg(\|\mathbf{u}_0\|_{W_2^{1+l}(\mathcal{F}_1)} + \|\rho_0\|_{W_2^{2+l}(S_{R_0})} + \sum_{i=1}^{2}\|\mathbf{h}_0^{(i)}\|_{W_2^{1+l}(\mathcal{F}_i)}\bigg), \qquad (1.10)
\end{aligned}$$

with a certain $a > 0$.

We have described the main relations of the problem and results of the paper, the outline of which is as follows. In Sec. 2, we present analysis of the linear problems corresponding to (1.3), and in Sec. 3, we prove the main result of the present paper – the solvability of the nonlinear problem (1.3) on the infinite time interval. Section 4 plays the role of appendix, we discuss there the estimates of the nonlinear terms. Detailed proofs will be given in subsequent publications.

§2. Linear problems

Omitting the nonlinear terms in (1.3), we arrive at two linear problems: one for the velocity, pressure and ρ and another for the magnetic field. We present the existence theorems for both problems in anisotropic Sobolev–Slobodetskii spaces. First, we consider the nonhomogeneous linear problem for $\mathbf{v}, p, \rho$, namely,

$$\begin{aligned}
&\mathbf{v}_t - \nu\nabla^2\mathbf{v} + \nabla p = \mathbf{f}(y,t), \quad \nabla\cdot\mathbf{v} = f_1(y,t) = \nabla\cdot\mathbf{F}(y,t), \quad y\in\mathcal{F}_1,\\
&\Pi_0 S(\mathbf{v})\mathbf{N} = \Pi_0\mathbf{d}(y,t),\\
&-p + \nu\mathbf{N}\cdot S(\mathbf{v})\mathbf{N} + \sigma B_0\rho = \mathbf{d}\cdot\mathbf{N},\\
&\rho_t - \Big(\mathbf{v} - |\Omega_0|^{-1}\int_{\mathcal{F}_1}\mathbf{v}(y,t)\,dy\Big)\cdot\mathbf{N} = g(y,t), \quad y\in S_{R_0},\\
&\mathbf{v}(y,t) = 0, \quad y\in\Sigma,\\
&\mathbf{v}(y,0) = \mathbf{v}_0(y), \quad y\in\mathcal{F}_1, \quad \rho(x,0) = \rho_0(y), \quad y\in S_{R_0};
\end{aligned}\tag{2.1}$$

Theorem 2.1. *Assume that $l\in[0,3/2)$, $l\neq 1/2$ and the data of problem* (2.1) *possess the following regularity properties:*

$$\mathbf{f}\in W_2^{l,l/2}(Q_T^1), \quad f_1\in W_2^{l+1,0}(Q_T^1), \quad f_1(y,t) = \nabla\cdot\mathbf{F}(y,t),$$
$$\mathbf{F}\in W_2^{0,1+l/2}(Q_T^1), \quad \mathbf{d}\cdot\mathbf{N}\in W_2^{l+1/2,0}(G_T)\cap W_2^{l/2}(0,T;W_2^{1/2}(S_{R_0})),$$
$$\mathbf{d}-\mathbf{N}(\mathbf{d}\cdot\mathbf{N})\in W_2^{l+1/2,l/2+1/4}(G_T), \quad g\in W_2^{l+3/2,l/2+3/4}(G_T),$$

$\mathbf{v}_0\in W_2^{l+1}(\mathcal{F}_1)$, $\rho_0\in W_2^{l+2}(S_{R_0})$, *where*

$$T<\infty, \quad Q_T^1 = \mathcal{F}_1\times(0,T), \quad G_T = S_{R_0}\times(0,T).$$

Moreover, let the compatibility conditions

$$\nabla\cdot\mathbf{v}_0(y) = f_1(y,0), \ \ y\in\mathcal{F}_1, \ \ \Pi_0 S(\mathbf{v}_0)\mathbf{N} = \Pi_0\mathbf{d}(y,0), \ \ y\in S_{R_0}, \ \ \mathbf{v}_0\big|_\Sigma = 0$$

be satisfied. Then, the problem (2.1) *has a unique solution* $\mathbf{v}, p, \rho$ *such that*

$$\mathbf{v}\in W_2^{l+2,l/2+1}(Q_T^1), \quad \nabla p\in W_2^{l,l/2}(Q_T^1),$$
$$p\in W_2^{l+1/2,0}(G_T)\cap W_2^{l/2}(0,T;W_2^{1/2}(S_{R_0})),$$
$$\rho_t\in W_2^{l+3/2,l/2+3/4}(G_T), \quad \rho\in W_2^{l+5/2,0}(G_T)\cap W_2^{l/2}(0,T;W_2^{5/2}(S_{R_0})),$$

and the solution satisfies the inequalities

$$
\begin{aligned}
&\|\mathbf{v}\|_{W_2^{l+2,l/2+1}(Q_T^1)} + \|\nabla p\|_{W_2^{l,l/2}(Q_T^1)} + \|p\|_{W_2^{l+1/2,0}(G_T)} \\
&\quad + \|p\|_{W_2^{l/2}(0,T;W_2^{1/2}(S_{R_0}))} + \|\rho\|_{W_2^{l+5/2,0}(G_T)} + \|\rho\|_{W_2^{l/2}(0,T;W_2^{5/2}(S_{R_0}))} \\
&\quad + \|\rho_t\|_{W_2^{l+3/2,l/2+3/4}(G_T)} \\
&\quad \leqslant c(T)\Big(\|\mathbf{f}\|_{W_2^{l,l/2}(Q_T^1)} + \|f_1\|_{W_2^{l+1,0}(Q_T^1)} + \|\mathbf{F}\|_{W_2^{0,1+l/2}(Q_T^1)} \\
&\quad + \|\Pi_0\mathbf{d}\|_{W_2^{l+1/2,l/2+1/4}(G_T)} + \|\mathbf{d}\cdot\mathbf{N}\|_{W_2^{l+1/2,0}(G_T)} \\
&\quad + \|\mathbf{d}\cdot\mathbf{N}\|_{W_2^{l/2}(0,T;W_2^{1/2}(S_{R_0}))} + \|g\|_{W_2^{l+3/2,l/2+3/4}(G_T)} \\
&\quad + \|\mathbf{v}_0\|_{W_2^{l+1}(\mathcal{F}_1)} + \|\rho_0\|_{W_2^{l+2}(S_{R_0})}\Big)
\end{aligned}
\tag{2.2}
$$

and

$$
\begin{aligned}
&\|\mathbf{v}\|_{W_2^{l+2,l/2+1}(Q_T^1)} + \|\nabla p\|_{W_2^{l,l/2}(Q_T^1)} + \|p\|_{W_2^{l+1/2,0}(G_T)} \\
&\quad + \|p\|_{W_2^{l/2}(0,T;W_2^{1/2}(S_{R_0}))} + \|\rho\|_{W_2^{l+5/2,0}(G_T)} + \|\rho\|_{W_2^{l/2}(0,T;W_2^{5/2}(S_{R_0}))} \\
&\quad + \|\rho_t\|_{W_2^{l+3/2,l/2+3/4}(G_T)} \leqslant C\Big(\| \mathbf{v} \|_{L_2(Q_T^1)} + \|\mathbf{f}\|_{W_2^{l,l/2}(Q_T^1)} \\
&\quad + \|f_1\|_{W_2^{l+1,0}(Q_T^1)} + \|\mathbf{F}\|_{W_2^{0,1+l/2}(Q_T^1)} + \|\Pi_0\mathbf{d}\|_{W_2^{l+1/2,l/2+1/4}(G_T)} \\
&\quad + \|\mathbf{d}\cdot\mathbf{N}\|_{W_2^{l+1/2,0}(G_T)} + \|\mathbf{d}\cdot\mathbf{N}\|_{W_2^{l/2}(0,T;W_2^{1/2}(S_{R_0}))} \\
&\quad + \|g\|_{W_2^{l+3/2,l/2+3/4}(G_T)} + \|\mathbf{v}_0\|_{W_2^{l+1}(\mathcal{F}_1)} + \|\rho_0\|_{W_2^{l+2}(S_{R_0})}\Big)
\end{aligned}
\tag{2.3}
$$

with the constant C independent of T.

This theorem can be proved with the help of the same scheme as Theorem1.1 in [2]. The problem (2.1) differs from the problem studied in [2] by the presence of an additional weak nonlocal term

$$|\Omega_0|^{-1}\int\limits_{\mathcal{F}_1} \mathbf{v}(y,t)\,dy\cdot\mathbf{N},$$

and by the geometry of the domain $\mathcal{F}_1$, the boundary of which has one more component Σ with the no-slip condition $\mathbf{v}=0$. Hence, the proof of (2.2), (2.3) requires local estimates of the solution near Σ in the spaces $W_2^{2+l,1+l/2}$. Such estimates are obtained in the paper [13]. Using the Schauder localization method and the results of [13] and [2], it is possible to obtain inequality (2.3). After this (2.2) is established in a standard way with the help of interpolation inequalities and the Gronwall lemma. We omit the details.

Now, we consider the problem

$$\begin{aligned}
&\mathbf{v}_t - \nu\nabla^2\mathbf{v} + \nabla p = 0, \quad \nabla\cdot\mathbf{v} = 0, \quad y \in \mathcal{F}_1,\\
&\nu\Pi_0 S(\mathbf{v})\mathbf{N} = 0,\\
&-p + \nu\mathbf{N}\cdot S(\mathbf{v})\mathbf{N} + \sigma B_0\rho = 0,\\
&\rho_t = \Big(\mathbf{v} - |\Omega_0|^{-1}\int_{\mathcal{F}_1}\mathbf{v}(y,t)\,dy\Big)\cdot\mathbf{N}, \quad y \in S_{R_0},\\
&\mathbf{v}(y,t) = 0, \quad y \in \Sigma,\\
&\mathbf{v}(y,0) = \mathbf{v}_0(y), \quad y \in \mathcal{F}_1, \quad \rho(x,0) = \rho_0(y), \quad y \in S_{R_0},
\end{aligned} \tag{2.4}$$

assuming that the initial data $\mathbf{v}_0$, ρ_0 satisfy only natural compatibility conditions

$$\nabla\cdot\mathbf{v}_0(y) = 0, \quad y \in \mathcal{F}_1, \quad \Pi_0 S(\mathbf{v}_0)\mathbf{N}(y) = 0, \quad y \in S_{R_0}, \quad \mathbf{v}_0\big|_\Sigma = 0, \tag{2.5}$$

and the orthogonality conditions

$$\int_{S_{R_0}} \rho_0(y)\,dS = 0, \qquad \int_{S_{R_0}} y_i\rho_0(y)\,dS = 0, \quad i = 1,2,3, \tag{2.6}$$

obtained by linearization of (1.5). It is easy to see that then the solution of (2.4) satisfies the same orthogonality conditions at any time $t > 0$:

$$\int_{S_{R_0}} \rho(y,t)\,dS = 0, \qquad \int_{S_{R_0}} y_i\rho(y,t)\,dS = 0, \quad i = 1,2,3. \tag{2.7}$$

This fact follows from the relations

$$\begin{aligned}
&\frac{d}{dt}\int_{S_{R_0}}\rho(y,t)\,dS = \int_{S_{R_0}}\mathbf{v}\cdot\mathbf{N}\,dS - \frac{1}{|\Omega_0|}\int_{\mathcal{F}_1}\mathbf{v}\,dy\cdot\int_{S_{R_0}}\mathbf{N}(y)\,dS = \int_{\mathcal{F}_1}\nabla\cdot\mathbf{v}\,dy = 0,\\
&\frac{d}{dt}\int_{S_{R_0}} y_i\rho(y,t)\,dS = \int_{\mathcal{F}_1}\nabla\cdot(\mathbf{v}y_i)\,dy - \frac{1}{|\Omega_0|}\int_{\mathcal{F}_1}\mathbf{v}\,dy\cdot\int_{S_{R_0}}\mathbf{N}(y)y_i\,dS\\
&\qquad = \int_{\mathcal{F}_1} v_i(y,t)\,dy - \frac{1}{|\Omega_0|}\int_{\mathcal{F}_1} v_i\,dy\,\frac{4\pi R_0^3}{3} = 0.
\end{aligned} \tag{2.8}$$

Now, we prove that under these assumptions the solution of (2.4) is decaying exponentially as $t \to \infty$.

Theorem 2.2. *For arbitrary* $\mathbf{v}_0 \in W_2^{1+l}(\mathcal{F}_1)$ *and* $\rho_0 \in W_2^{2+l}(S_{R_0})$, *where* $l \in [0, 3/2)$, *which satisfy the conditions* (2.5) *and* (2.6), *the problem* (2.4) *has a unique solution, and*

$$\begin{aligned}
&\|e^{at}\mathbf{v}\|_{W_2^{l+2,l/2+1}(Q_T^1)} + \|e^{at}\nabla p\|_{W_2^{l,l/2}(Q_T^1)} + \|e^{at}p\|_{W_2^{l+1/2,0}(G_T)}\\
&\quad + \|e^{at}p\|_{W_2^{l/2}(0,T;W_2^{1/2}(S_{R_0}))} + \|e^{at}\rho\|_{W_2^{l+5/2,0}(G_T)}\\
&\quad + \|e^{at}\rho\|_{W_2^{l/2}(0,T;W_2^{5/2}(S_{R_0}))} + \|e^{at}\rho_t\|_{W_2^{l+3/2,l/2+3/4}(G_T)}\\
&\quad + \sup_{t<T}\|e^{at}\mathbf{v}(\cdot,t)\|_{W_2^{l+1}(\mathcal{F}_1)} + \sup_{t<T}\|e^{at}\rho(\cdot,t)\|_{W_2^{l+2}(S_{R_0})}\\
&\quad \leqslant c\big(\|\mathbf{v}_0\|_{W_2^{l+1}(\mathcal{F}_1)} + \|\rho_0\|_{W_2^{l+2}(S_{R_0})}\big),
\end{aligned} \tag{2.9}$$

where $a > 0$; *the constant* c *is independent of* T.

Proof. To obtain the energy estimate, we multiply the first equation in (2.4) by $\mathbf{v}$, integrate over $\mathcal{F}_1$, and integrate by parts. We arrive at the relation

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\parallel \mathbf{v}(\cdot,t)\parallel^2_{L_2(\mathcal{F}_1)} + \frac{\nu}{2}\parallel S(\mathbf{v})\parallel^2_{L_2(\mathcal{F}_1)}&\\
+\int\limits_{\partial\mathcal{F}_1}\big(-\nu S(\mathbf{v})\mathbf{N}\cdot\mathbf{v} + p\mathbf{v}\cdot\mathbf{N}\big)\,ds = 0.&
\end{aligned} \tag{2.10}$$

Due to the boundary conditions in (2.4), the surface integral takes the form

$$\int\limits_{S_{R_0}} \sigma\rho_t B_0\rho\,ds + \sigma\int\limits_{S_{R_0}} B_0\rho\frac{1}{|\Omega_0|}\int\limits_{\mathcal{F}_1}\mathbf{v}(y,t)\,dy\cdot\mathbf{N}\,ds, \tag{2.11}$$

where

$$B_0\rho = -\frac{1}{R_0^2}\left(\Delta_{S_{R_0}}\rho + 2\rho\right).$$

The second term of the right-hand side in (2.11) is equal to zero due to the condition $B_0 N_i = 0$, while the first term equals

$$-\frac{\sigma}{R_0^2}\int\limits_{S_1}\big(\Delta_{S_1}\rho + 2\rho\big)\rho_t\,ds = \frac{\sigma}{2R_0^2}\frac{d}{dt}\int\limits_{S_1}\big(|\nabla_\omega\rho|^2 - 2\rho^2\big)\,ds = \frac{1}{2}\frac{d}{dt}M(t).$$

As a result, (2.10) reads

$$\frac{1}{2}\frac{d}{dt}\Big(\| \mathbf{v}(\cdot,t) \|^2_{L_2(\mathcal{F}_1)} + M(t)\Big) + \frac{\nu}{2} \| S(\mathbf{v}) \|^2_{L_2(\mathcal{F}_1)} = 0. \tag{2.12}$$

It can be shown in the same way as in [4, 5] that (2.7) imply positive definiteness of $M(t)$. Indeed, due to (2.8), ρ is orthogonal to the first and the second eigenfunctions of Laplace–Beltarmi operator Δ_{S_1}. Consequently, $\rho = \sum\limits_{n=2}^{+\infty} Y_n$, where Y_n are linear combinations of eigenfunctions, corresponding to the eigenvalues $\lambda_n = n(n+1)$, $n \geqslant 2$. We see that

$$M(t) = \frac{\sigma}{R_0^2}\int\limits_{S_1} \sum_{n=2}^{+\infty}\Big(n(n+1)Y_n - 2Y_n\Big)\sum_{n=2}^{+\infty} Y_n\, ds$$
$$\geqslant \frac{\sigma}{2R_0^2}\int\limits_{S_1} |\nabla_\omega \rho|^2 d\omega + \frac{\sigma}{R_0^2}\int\limits_{S_1} \rho^2 d\omega.$$

Hence,

$$M(t) \geqslant C \| \rho(\cdot,t) \|^2_{W_2^1(S_{R_0})}\,. \tag{2.13}$$

There is no dissipative term for ρ in (2.12). To add this term, we use the so-called "free energy" method (see, for example, [6–8]).

Lemma 2.1 ([6–8]). *For any function $\rho \in W_2^{1/2,0}(G_T)$ such that $\rho_t \in L_2(G_T)$, and*

$$\int\limits_{S_{R_0}} \rho(y,t)\, ds = 0,$$

there exists a vector field

$$\mathbf{w}(\cdot,t) \in W_2^1(\mathcal{F}_1), \quad \mathbf{w}_t(\cdot,t) \in L_2(\mathcal{F}_1),$$

which is a solution to the following problem

$$\nabla\cdot\mathbf{w} = 0, \qquad y \in \mathcal{F}_1, \qquad t > 0,$$
$$\mathbf{w}\big|_\Sigma = 0, \qquad \mathbf{w}\cdot\mathbf{N}\big|_{S_{R_0}} = \rho,$$

and satisfies the estimates

$$\begin{aligned}
&\| \mathbf{w}(\cdot,t) \|_{W_2^1(\mathcal{F}_1)} \leqslant c \| \rho(\cdot,t) \|_{W_2^{1/2}(S_{R_0})},\\
&\| \mathbf{w}(\cdot,t) \|_{L_2(\mathcal{F}_1)} \leqslant c \| \rho(\cdot,t) \|_{L_2(S_{R_0})},\\
&\| \mathbf{w}_t(\cdot,t) \|_{L_2(\mathcal{F}_1)} \leqslant c\big(\| \rho_t(\cdot,t) \|_{L_2(S_{R_0})} + \| \rho(\cdot,t) \|_{W_2^{1/2}(S_{R_0})} \big).
\end{aligned} \tag{2.14}$$

If we multiply the first equation in (2.4) by $\mathbf{w}$ and integrate over $\mathcal{F}_1$, we arrive at

$$\int_{\mathcal{F}_1} \mathbf{v}_t \cdot \mathbf{w}\, dx - \nu \int_{\mathcal{F}_1} \nabla^2 \mathbf{v} \cdot \mathbf{w}\, dx + \int_{\mathcal{F}_1} \nabla p \cdot \mathbf{w}\, dx = 0.$$

Now, we integrate by parts, take into account the boundary conditions in (2.4) and obtain:

$$\frac{d}{dt} \int_{\mathcal{F}_1} \mathbf{v} \cdot \mathbf{w}\, dx + \nu \int_{\mathcal{F}_1} S(\mathbf{v}) : S(\mathbf{w})\, dx - \int_{\mathcal{F}_1} \mathbf{v} \cdot \mathbf{w}_t\, dx + \int_{S_{R_0}} \sigma B_0 \rho \mathbf{w} \cdot \mathbf{N}\, ds = 0. \quad (2.15)$$

Due to the condition $\mathbf{w} \cdot \mathbf{N}\Big|_{S_{R_0}} = \rho$, the surface integral in (2.15) can be written in the form

$$\frac{\sigma}{R_0^2} \int_{S_1} \big(|\nabla_\omega \rho|^2 - 2\rho^2 \big)\, ds = M(t).$$

We multiply (2.15) by a small positive number γ and add it to (2.12), which gives

$$\frac{1}{2}\frac{d}{dt}\Big(E(t) + \gamma E_1(t) + M(t) \Big) + D(t) + \gamma D_1(t) + \gamma M(t) = 0, \qquad (2.16)$$

where

$$E(t) = \| \mathbf{v}(\cdot, t) \|^2_{L_2(\mathcal{F}_1)}, \quad E_1(t) = 2 \int_{\mathcal{F}_1} \mathbf{v} \cdot \mathbf{w}\, dx,$$

$$D(t) = \frac{\nu}{2} \| S(\mathbf{v}) \|^2_{L_2(\mathcal{F}_1)}, \quad D_1(t) = \nu \int_{\mathcal{F}_1} S(\mathbf{v}) : S(\mathbf{w})\, dx - \int_{\mathcal{F}_1} \mathbf{v} \cdot \mathbf{w}_t\, dx,$$

$$M(t) = \frac{\sigma}{R_0^2} \int_{S_1} \big(|\nabla_\omega \rho(\cdot, t)|^2 - 2\rho^2(\cdot, t) \big)\, ds.$$

Making use of $(2.14)_2$, we estimate $E_1(t)$ in the following way

$$\big|E_1(t)\big| \leqslant 2 \| \mathbf{v} \|_{L_2(\mathcal{F}_1)} \| \mathbf{w} \|_{L_2(\mathcal{F}_1)} \leqslant 2 \| \mathbf{v} \|_{L_2(\mathcal{F}_1)} \| \rho \|_{L_2(S_{R_0})} \leqslant \big(\| \mathbf{v} \|^2_{L_2(\mathcal{F}_1)} + \| \rho \|^2_{L_2(S_{R_0})} \big).$$

For the sufficiently small γ, it leads to

$$E(t) + \gamma E_1(t) + M(t) \geqslant c\Big(\| \mathbf{v}(\cdot,t) \|^2_{L_2(\mathcal{F}_1)} + \| \rho(\cdot,t) \|^2_{W^1_2(S_{R_0})} \Big). \quad (2.17)$$

Similarly, in view of $(2.14)_1$, $(2.14)_3$, and the boundary condition $(2.4)_4$, we have

$$\begin{aligned} \big|D_1(t)\big| &\leqslant \| \mathbf{v}(\cdot,t) \|_{W^1_2(\mathcal{F}_1)} \Big(\| \mathbf{w}(\cdot,t) \|_{W^1_2(\mathcal{F}_1)} + \| \mathbf{w}_t(\cdot,t) \|_{L_2(\mathcal{F}_1)} \Big) \\ &\leqslant c \| \mathbf{v}(\cdot,t) \|_{W^1_2(\mathcal{F}_1)} \Big(\| \rho(\cdot,t) \|_{W^{1/2}_2(S_{R_0})} + \| \rho_t(\cdot,t) \|_{L_2(S_{R_0})} \Big) \\ &\leqslant c \| \mathbf{v}(\cdot,t) \|_{W^1_2(\mathcal{F}_1)} \Big(\| \rho(\cdot,t) \|_{W^{1/2}_2(S_{R_0})} + \| \mathbf{v}(\cdot,t) \|_{L_2(S_{R_0})} \Big). \end{aligned} \quad (2.18)$$

By the Korn inequality,

$$\| \mathbf{v}(\cdot,t) \|_{W^1_2(\mathcal{F}_1)} \leqslant \| S(\mathbf{v}(\cdot,t)) \|_{L_2(\mathcal{F}_1)},$$

so that for sufficiently small γ, (2.13) and (2.18) imply

$$\begin{aligned} D(t) &+ \gamma D_1(t) + \gamma M(t) \\ &\geqslant c\Big(\| \mathbf{v}(\cdot,t) \|^2_{W^1_2(\mathcal{F}_1)} + \gamma \| \rho(\cdot,t) \|^2_{W^1_2(S_{R_0})} \Big). \end{aligned} \quad (2.19)$$

As a consequence of (2.16), (2.17), and (2.19), we obtain the exponential decay in L_2 norms:

$$\begin{aligned} \| \mathbf{v}(\cdot,t) \|^2_{L_2(\mathcal{F}_1)} &+ \| \rho(\cdot,t) \|^2_{W^1_2(S_{R_0})} \\ &\leqslant Ce^{-\beta t}\Big(\| \mathbf{v}_0 \|^2_{L_2(\mathcal{F}_1)} + \| \rho_0 \|^2_{W^1_2(S_{R_0})} \Big), \quad \beta > 0. \end{aligned} \quad (2.20)$$

Now we pass to estimate (2.9). Let us introduce the functions

$$\widetilde{\mathbf{v}} = e^{at}\mathbf{v}, \quad \widetilde{p} = e^{at}p, \quad \widetilde{\rho} = e^{at}\rho, \quad a > 0.$$

If $\mathbf{v}$, p, ρ is a solution to problem (2.4), then $\widetilde{\mathbf{v}}$, $\widetilde{p}$, $\widetilde{\rho}$ satisfy the following relations:

$$\begin{aligned} &\widetilde{\mathbf{v}}_t - \nu\nabla^2\widetilde{\mathbf{v}} + \nabla\widetilde{p} = a\widetilde{\mathbf{v}}, \quad \nabla\cdot\widetilde{\mathbf{v}} = 0, \quad y \in \mathcal{F}_1, \\ &\Pi_0 S(\widetilde{\mathbf{v}})\mathbf{N} = 0, \\ &-\widetilde{p} + \nu\mathbf{N}\cdot S(\widetilde{\mathbf{v}})\mathbf{N} + \sigma B_0\widetilde{\rho} = 0, \\ &\widetilde{\rho}_t = \Big(\widetilde{\mathbf{v}} - |\Omega_0|^{-1}\int_{\mathcal{F}_1} \widetilde{\mathbf{v}}(y,t)\,dy\Big)\cdot\mathbf{N} + a\widetilde{\rho}, \quad y \in S_{R_0}, \\ &\widetilde{\mathbf{v}}(y,t) = 0, \quad y \in \Sigma, \\ &\widetilde{\mathbf{v}}(y,0) = \mathbf{v}_0(y), \quad y \in \mathcal{F}_1, \quad \widetilde{\rho}(y,0) = \rho_0(y), \quad y \in S_{R_0}. \end{aligned} \quad (2.21)$$

In view of (2.3), we have

$$\begin{aligned}
&\|\widetilde{\mathbf{v}}\|_{W_2^{l+2,l/2+1}(Q_T^1)} + \|\nabla\widetilde{p}\|_{W_2^{l,l/2}(Q_T^1)} + \|\widetilde{p}\|_{W_2^{l+1/2,0}(G_T)}\\
&+ \|\widetilde{p}\|_{W_2^{l/2}(0,T;W_2^{1/2}(S_{R_0}))} + \|\widetilde{\rho}\|_{W_2^{l+5/2,0}(G_T)}\\
&+ \|\widetilde{\rho}\|_{W_2^{l/2}(0,T;W_2^{5/2}(S_{R_0}))} + \|\widetilde{\rho}_t\|_{W_2^{l+3/2,l/2+3/4}(G_T)}\\
&\leqslant C\Big(\| \widetilde{\mathbf{v}} \|_{L_2(Q_T^1)} + a\|\widetilde{\mathbf{v}}\|_{W_2^{l,l/2}(Q_T^1)} + a\|\widetilde{\rho}\|_{W_2^{l+3/2,l/2+3/4}(G_T)}\\
&+ \|\mathbf{v}_0\|_{W_2^{l+1}(\mathcal{F}_1)} + \|\rho_0\|_{W_2^{l+2}(S_{R_0})}\Big),
\end{aligned} \tag{2.22}$$

where the constant C is independent of T. We apply interpolation inequalities to the right-hand side of (2.22), and estimate the norms

$$\| \widetilde{\mathbf{v}}(\cdot,t) \|_{L_2(\mathcal{F}_1)}, \quad \| \widetilde{\rho}(\cdot,t) \|_{W_2^1(S_{R_0})}$$

by means of (2.20). For $a < \beta$ this gives (2.9). □

Now, we consider the main linear problem for the magnetic field:

$$\begin{aligned}
&\mu_1\mathbf{H}_t + \alpha^{-1}\mathrm{rot}\,\mathrm{rot}\,\mathbf{H} = \mu_1\mathbf{G}(y,t), \quad \nabla\cdot\mathbf{H} = 0, \quad y \in \mathcal{F}_1,\\
&\mathrm{rot}\,\mathbf{H} = 0, \quad \nabla\cdot\mathbf{H} = 0, \quad y \in \mathcal{F}_2,\\
&[\mu\mathbf{H}\cdot\mathbf{N}] = 0, \quad [\mathbf{H}_\tau] = 0, \quad y \in S_{R_0},\\
&\mathbf{H}\cdot\mathbf{n} = 0, \quad y \in S\cup\Sigma, \quad \mathrm{rot}_\tau\mathbf{H} = 0, \quad y \in \Sigma,\\
&\mathbf{H}(y,0) = \mathbf{H}_0(y), \quad y \in \mathcal{F}_1\cup\mathcal{F}_2.
\end{aligned} \tag{2.23}$$

Also we need to consider two auxiliary problems, namely,

$$\begin{aligned}
&\mathrm{rot}\,\mathbf{H}(x) = \boldsymbol{\xi}(x), \quad \nabla\cdot\mathbf{H} = 0, \quad x \in \mathcal{F}_1\cup\mathcal{F}_2,\\
&\mathbf{H}\cdot\mathbf{n} = 0, \quad x \in S\cup\Sigma, \quad [\mu\mathbf{H}\cdot\mathbf{N}]\big|_{S_{R_0}} = 0, \quad [\mathbf{H}_\tau]\big|_{S_{R_0}} = 0,
\end{aligned} \tag{2.24}$$

and

$$\begin{aligned}
&\mathrm{rot}\,\boldsymbol{\phi}(x) = \mathbf{g}(x), \quad \nabla\cdot\boldsymbol{\phi} = 0, \quad x \in \mathcal{F}_1,\\
&\boldsymbol{\phi}\cdot\mathbf{N} = 0, \quad x \in S_{R_0}, \quad \boldsymbol{\phi}_\tau = 0, \quad x \in \Sigma.
\end{aligned} \tag{2.25}$$

Lemma 2.2. *For arbitrary $\boldsymbol{\xi} \in W_2^{l+1}(\mathcal{F}_i)$, $i = 1,2$ satisfying the conditions*

$$\nabla\cdot\boldsymbol{\xi}(x) = 0, \quad x \in \mathcal{F}_i, \quad [\boldsymbol{\xi}\cdot\mathbf{N}|]_{S_{R_0}} = 0, \quad \boldsymbol{\xi}\cdot n\big|_S = 0;$$

the problem (2.24) *has a unique solution $\mathbf{H} \in W_2^{2+l}(\mathcal{F}_i)$, $i = 1,2$, and this solution satisfies the inequality*

$$\sum_{i=1}^{2} \|\mathbf{H}\|_{W_2^{2+l}(\mathcal{F}_i)} \leqslant c\sum_{i=1}^{2} \|\boldsymbol{\xi}\|_{W_2^{1+l}(\mathcal{F}_i)}. \tag{2.26}$$

Proof. *Uniqueness.* If $\mathbf{H}$ is a solution of (2.24) with $\boldsymbol{\xi} = 0$, then $\mathbf{H} = \nabla\varphi(x)$, where φ is a single-valued harmonic function satisfying the relations

$$\nabla^2\varphi(x) = 0, \quad x \in \mathcal{F}_1 \cup \mathcal{F}_2, \quad \frac{\partial\varphi}{\partial n}\Big|_{S\cup\Sigma} = 0,$$
$$[\varphi] = 0, \quad \left[\mu\frac{\partial\varphi}{\partial n}\right] = 0, \quad x \in S_{R_0}.$$

Hence, $\varphi = \text{const}$, $\mathbf{H} = 0$.

Existence. We seek the solution of (2.24) in the form $\mathbf{H} = \mathbf{H}_1(x) + \nabla U(x)$, where

$$\begin{aligned}
&\mathbf{H}_1(x) = \frac{1}{4\pi}\mathrm{rot}\left(\int_\Omega \frac{\boldsymbol{\xi}(y)\,dy}{|x-y|} + \int_D \frac{\nabla W(y)\,dy}{|x-y|}\right),\\
&\nabla^2 U(x) = 0, \quad x \in \mathcal{F}_1 \cup \mathcal{F}_2, \quad \frac{\partial U}{\partial n}\Big|_{S\cup\Sigma} = -\mathbf{H}_1\cdot\mathbf{n}\big|_{S\cup\Sigma},\\
&[U(x)] = 0, \quad \left[\mu\frac{\partial U(x)}{\partial n}\right] = -[\mu]\mathbf{H}_1\cdot\mathbf{N}, \quad x \in S_{R_0},
\end{aligned} \tag{2.27}$$

and W is a solution of the Neumann problem

$$\nabla^2 W(x) = 0, \quad x \in D, \quad \frac{\partial W}{\partial n} = \boldsymbol{\xi}\cdot\mathbf{n}, \quad x \in \Sigma.$$

This problem is solvable since $\int_\Sigma \boldsymbol{\xi}\cdot\mathbf{n}\,ds = \int_S \boldsymbol{\xi}\cdot\mathbf{n}\,ds = 0$. It is clear that

$$\|\nabla W\|_{W_2^{1+l}(D)} \leqslant c\|\boldsymbol{\xi}\|_{W_2^{1/2+l}(\Sigma)} \leqslant c\|\boldsymbol{\xi}\|_{W_2^{1+l}(\mathcal{F}_1)}.$$

Equations (2.24) are verified by straightforward calculations, and (2.26) follows from the known estimates of the volume potentials and of the solution of the elliptic problem for U. □

Let $\mathcal{H}^r(\Omega)$ denote the space of vector fields $\boldsymbol{\psi} \in W_2^r(\mathcal{F}_i)$, $i = 1, 2$, such that

$$\begin{aligned}
&\nabla\cdot\boldsymbol{\psi}(x) = 0, \quad x \in \mathcal{F}_1 \cup \mathcal{F}_2, \quad \mathrm{rot}\,\boldsymbol{\psi}(x) = 0, \quad x \in \mathcal{F}_2,\\
&\boldsymbol{\psi}\cdot\mathbf{n}\big|_{\Sigma\cup S} = 0, \quad \mathrm{rot}_\tau\boldsymbol{\psi}|_\Sigma = 0 \quad (\text{if} \quad \mathrm{r} > 3/2),\\
&[\mu\boldsymbol{\psi}\cdot\mathbf{N}] = 0, \quad [\boldsymbol{\psi}_\tau] = 0, \quad x \in S_{R_0}.
\end{aligned} \tag{2.28}$$

Corollary 2.1. *For arbitrary* $\mathbf{H} \in \mathcal{H}^{r+1}(\Omega)$, *we have*

$$c_1\sum_{i=1}^2 \|\mathbf{H}\|_{W_2^{r+1}(\mathcal{F}_i)} \leqslant c\|\mathrm{rot}\,\mathbf{H}\|_{W_2^r(\mathcal{F}_1)} \leqslant c_2\sum_{i=1}^2 \|\mathbf{H}\|_{W_2^{r+1}(\mathcal{F}_i)}. \tag{2.29}$$

Lemma 2.3. *For arbitrary* $\mathbf{g} \in W_2^r(\mathcal{F}_1)$ *such that*

$$\nabla \cdot \mathbf{g}(x) = 0, \quad x \in \mathcal{F}_1, \quad \mathbf{g} \cdot \mathbf{n}|_\Sigma = 0,$$

the problem (2.25) *has a unique solution* $\boldsymbol{\phi} \in W_2^{1+r}(\mathcal{F}_1)$, *and this solution satisfies the inequality*

$$\|\boldsymbol{\phi}\|_{W_2^{1+r}(\mathcal{F}_1)} \leqslant c\|\mathbf{g}\|_{W_2^r(\mathcal{F}_1)}. \tag{2.30}$$

Proof. *Uniqueness.* If $\boldsymbol{\phi}$ satisfies (2.25) with $\mathbf{g} = 0$, then $\boldsymbol{\phi} = \nabla\omega$,

$$\nabla^2\omega = 0, \quad x \in \mathcal{F}_1, \quad \frac{\partial\omega}{\partial n}\Big|_\Sigma = 0, \quad \omega\big|_{S_{R_0}} = \text{const}\,,$$

which implies $\omega = \text{const}$ and $\boldsymbol{\phi} = 0$.

Existence. The solution of (2.25) has the form $\boldsymbol{\phi} = \boldsymbol{\phi}_1(x) + \nabla U_1(x)$, where

$$\begin{aligned} &\boldsymbol{\phi}_1(x) = \frac{1}{4\pi}\text{rot}\Big(\int\limits_{\mathcal{F}_1}\frac{\mathbf{g}(y)\,dy}{|x-y|} + \int\limits_{\mathcal{F}_2}\frac{\nabla W_1(y)\,dy}{|x-y|}\Big), \\ &\nabla^2 U_1(x) = 0, \quad x \in \mathcal{F}_1, \quad \frac{\partial U_1}{\partial n}\Big|_{S_{R_0}} = -\boldsymbol{\phi}_1 \cdot \mathbf{n}\big|_{S_{R_0}}, \\ &U_1(x) = u_1(x), \quad x \in \Sigma, \end{aligned} \tag{2.31}$$

and W_1 is a solution of the Neumann problem

$$\nabla^2 W_1(x) = 0, \quad x \in \mathcal{F}_2, \quad \frac{\partial W}{\partial N} = \mathbf{g}\cdot\mathbf{N}, \quad x \in S_{R_0}, \quad \frac{\partial W}{\partial \mathbf{n}}\Big|_S = 0.$$

It is clear that

$$\|\nabla W_1\|_{W_2^{1+r}(D)} \leqslant c\|\mathbf{g}\|_{W_2^{1/2+r}(\Sigma)} \leqslant c\|\mathbf{g}\|_{W_2^{1+r}(\mathcal{F}_1)}.$$

The function u_1 is defined as follows: since $\text{rot}\,\boldsymbol{\phi}_1 \cdot \mathbf{n} = \mathbf{g}\cdot\mathbf{n} = 0$ on Σ, we have $\int\limits_{\Sigma'} \text{rot}\ \boldsymbol{\phi}_1 \cdot \mathbf{n}\,dS = 0$ for arbitrary $\Sigma' \subset \Sigma$. Hence, $\int\limits_{\ell} \boldsymbol{\phi}_1 \cdot d\ell = 0$ for arbitrary closed contour $\ell \subset \Sigma$, and this means that $\boldsymbol{\phi}_1\big|_\Sigma = -\nabla_\tau u_1(x)$ where u_1 is a certain single-valued function on Σ. It follows that $\boldsymbol{\phi}_\tau\big|_\Sigma = 0$.

It is straightforward to verify that $\boldsymbol{\phi} = \boldsymbol{\phi}_1 + \nabla U_1$ is a solution of (2.25) satisfying (2.30). Lemma is proved. □

Now we prove solvability of the problem (2.23).

Theorem 2.3. *For arbitrary* $\mathbf{H}_0 \in W_2^{l+1}(\mathcal{F}_i)$, $i=1,2$, *satisfying the compatibility conditions*

$$\begin{aligned} &\nabla\cdot\mathbf{H}_0(x)=0, \quad x\in\mathcal{F}_1\cup\mathcal{F}_2, \quad \operatorname{rot}\mathbf{H}_0(x)=0, \quad x\in\mathcal{F}_2,\\ &[\mu\mathbf{H}_0\cdot\mathbf{N}]=0, \quad [\mathbf{H}_{0\tau}]=0, \quad x\in S_{R_0},\\ &\mathbf{H}_0\cdot\mathbf{n}=0, \quad \operatorname{rot}_\tau\mathbf{H}_0=0, \quad x\in\Sigma,\\ &\mathbf{H}_0\cdot\mathbf{n}=0, \quad x\in S, \end{aligned} \tag{2.32}$$

and arbitrary divergence free $\mathbf{G}\in W_2^{l,l/2}(Q_T^1)$ *the problem* (2.23) *has a unique solution, and the inequality*

$$\begin{aligned} \sum_{i=1}^{2}\Big(\|e^{at}\mathbf{H}^{(i)}\|_{W_2^{l+2,l/2+1}(Q_T^i)}+\sup_{t<T}\|e^{at}\mathbf{H}^{(i)}(\cdot,t)\|_{W_2^{l+1}(\mathcal{F}_i)}\Big)\\ \leqslant c\bigg(\sum_{i=1}^{2}\|\mathbf{H}_0^{(i)}\|_{W_2^{l+1}(\mathcal{F}_i)}+\|e^{at}\mathbf{G}\|_{W_2^{l,l/2}(Q_T^1)}\bigg) \end{aligned} \tag{2.33}$$

is satisfied with a certain $a>0$ *and with the constant* c *independent of* T.

We confine ourselves to the case $l=0$, because the improvement of the regularity of the solution is made in the same way as in [1, Theorem 4].

Let $\boldsymbol{\psi}(\cdot,t)\in\mathcal{H}^1(\Omega)$. We note that the equations

$$\operatorname{rot}\boldsymbol{\psi}(x,t)=0, \quad \nabla\cdot\boldsymbol{\psi}(x,t)=0, \quad x\in\mathcal{F}_2$$

imply

$$\boldsymbol{\psi}(x,t)=\nabla\varphi(x,t), \quad x\in\mathcal{F}_2,$$

where φ is a solution of the problem

$$\begin{aligned} \nabla^2\varphi(x,t)=0, \quad x\in\mathcal{F}_2, \quad \frac{\partial\varphi(x,t)}{\partial n}\Big|_{x\in S}=0,\\ \mu_2\frac{\partial\varphi(x,t)}{\partial N}-\mu_1\mathbf{H}^{(1)}\cdot\mathbf{N}\Big|_{x\in S_{R_0}}=0. \end{aligned} \tag{2.34}$$

Let $\Phi(x,t)$, $x\in\mathcal{F}_1$, be the solution of

$$\nabla^2\Phi(x,t)=0, \quad x\in\mathcal{F}_1, \quad \Phi(x,t)-\varphi(x,t)\big|_{x\in S_{R_0}}=0, \quad \Phi\big|_{x\in\Sigma}=0.$$

Following the proof of Theorem 4 in [1], we multiply the first equation in (2.23) by $\boldsymbol{\psi}-\nabla\Phi$ and integrate over $\mathcal{F}_1$. Then we integrate by parts, using the boundary conditions for $\mathbf{H}$. This leads to

$$\int_0^T\int_{\mathcal{F}_1}\mu(\mathbf{H}_t-\mathbf{G})\cdot(\boldsymbol{\psi}-\nabla\Phi)\,dx\,dt+\alpha^{-1}\int_0^T\int_{\mathcal{F}_1}\operatorname{rot}\mathbf{H}\cdot\operatorname{rot}\boldsymbol{\psi}\,dx\,dt=0, \quad (2.35)$$

which is equivalent to

$$\int_0^T\int_{\Omega}\mu(\mathbf{H}_t-\mathbf{G}^*)\cdot\boldsymbol{\psi}\,dx\,dt+\alpha^{-1}\int_0^T\int_{\mathcal{F}_1}\operatorname{rot}\mathbf{H}\cdot\operatorname{rot}\boldsymbol{\psi}\,dx\,dt=0, \quad (2.36)$$

where $\mathbf{G}^*$ is an element of $L_2(0,T;\mathcal{H}^0(\Omega))$ defined by $\mathbf{G}^*(x,t)=\mathbf{G}(x,t)$ for $x\in\mathcal{F}_1$, $\mathbf{G}^*(x,t)=\nabla\varphi_G(x,t)$ for $x\in\mathcal{F}_2$, and φ_G is a solution of (2.34) with $\mathbf{G}$ instead of $\mathbf{H}$.

The existence of $\mathbf{H}(x,t)$ satisfying (2.36) and the initial condition $\mathbf{H}(x,0)=\mathbf{H}_0(x)$ can be proved by Galerkin's method (see [14]), and it is easily seen that

$$\|\mathbf{H}_t\|_{L_2(Q_T)}+\|\operatorname{rot}\mathbf{H}\|_{L_2(Q_T^1)}\leqslant c\Big(\|\mathbf{G}^*\|_{L_2(Q_T)}+\|\operatorname{rot}\mathbf{H}_0\|_{L_2(\mathcal{F}_1)}\Big),$$

or, in view of Corollary 2.1,

$$\|\mathbf{H}_t\|_{L_2(Q_T)}+\sum_{i=1}^2\|\mathbf{H}\|_{W_2^{1,0}(Q_T^i)}\leqslant c\Big(\|\mathbf{G}\|_{L_2(Q_T^1)}+\sum_{i=1}^2\|\mathbf{H}_0\|_{W_2^1(\mathcal{F}_i)}\Big). \quad (2.37)$$

To estimate the second derivatives of $\mathbf{H}$, we set $\boldsymbol{\xi}=\operatorname{rot}\mathbf{H}$ and we introduce $\boldsymbol{\phi}\in L_2(0,T;W_2^1(\mathcal{F}_1))$ as the solution of problem (2.25) with $\mathbf{g}=\alpha\mu_1(\mathbf{G}-\mathbf{H}_t)$. In view of (2.35), we have

$$\int_{\mathcal{F}_1}\operatorname{rot}\boldsymbol{\phi}\cdot(\boldsymbol{\psi}-\nabla\Phi)\,dx=\alpha\int_{\mathcal{F}_1}\mu_1(\mathbf{G}-\mathbf{H}_t)\cdot(\boldsymbol{\psi}-\nabla\Phi)\,dx=\int_{\mathcal{F}_1}\boldsymbol{\xi}\cdot\operatorname{rot}\boldsymbol{\psi}\,dx,$$

which implies

$$\int_0^T\int_{\mathcal{F}_1}(\boldsymbol{\xi}-\boldsymbol{\phi})\cdot\operatorname{rot}\boldsymbol{\psi}\,dx\,dt=0\quad\text{for any}\quad\boldsymbol{\psi}\in L_2\big(0,T;\mathcal{H}^1(\Omega)\big).$$

It follows that

$$\boldsymbol{\xi}-\boldsymbol{\phi}=\nabla s(x,t),\quad \nabla^2 s(x,t)=0,\quad x\in\mathcal{F}_1,$$
$$\frac{\partial s}{\partial N}\Big|_{x\in S_{R_0}}=0,\quad s(x,t)\big|_{x\in\Sigma}=\text{const},$$

which implies $\boldsymbol{\xi} - \boldsymbol{\phi} = 0$. Hence $\operatorname{rot}\mathbf{H} = \boldsymbol{\phi} \in W_2^{1,0}(\mathcal{F}_1)$ and, in view of (2.29),

$$\sum_{i=1}^{2} \|\mathbf{H}\|_{W_2^{2,0}(Q_T^i)} \leqslant c\|\operatorname{rot}\mathbf{H}\|_{W_2^{1,0}(Q_T^1)} \leqslant c\Big(\sum_{i=1}^{2} \|\mathbf{H}_0\|_{W_2^1(\mathcal{F}_i)} + \|\mathbf{G}\|_{L_2(Q_T)}\Big),$$

q.e.d.

From (2.35) it follows that $\mathbf{H}$ satisfies the first equation in (2.23) and the boundary condition $\operatorname{rot}_\tau \mathbf{H}|_\Sigma = 0$. Moreover, (2.23) can be written in the form of the Cauchy problem

$$\mathbf{H}_t + \mathcal{A}\mathbf{H} = \mathbf{G}, \quad \mathbf{H}\Big|_{t=0} = \mathbf{H}_0 \tag{2.38}$$

in the space $\mathcal{H}^0(\Omega)$ with a positive self-adjoint operator $\mathcal{A}$ defined on the space $\mathcal{H}^2$ as follows:

$$\mathcal{A}\mathbf{H} = P_{\mathcal{H}^0}\mu^{-1}\operatorname{rot}\mathcal{E}\alpha^{-1}\operatorname{rot}\mathbf{H},$$

where $P_{\mathcal{H}^0}$ is the orthogonal projection on $\mathcal{H}^0(\Omega)$ in the space $L_2(\Omega)$ supplied with the scalar product $\int_\Omega \mu\mathbf{H}_1 \cdot \mathbf{H}_2\,dx$ and $\mathcal{E}$ is an extension operator from $\mathcal{F}_1$ into Ω defined on the space of the divergence free vector fields $\mathbf{u}(x)$ with $\mathbf{u}\cdot\mathbf{N}|_{S_{R_0}} = 0$, $\mathbf{u}_\tau|_\Sigma = 0$ and such that

$$(\mathcal{E}\mathbf{u})_\tau|_S = 0, \quad \|\mathcal{E}\mathbf{u}\|_{W_2^{l+1}(\Omega)} \leqslant c\|\mathbf{u}\|_{W_2^{l+1}(\mathcal{F}_1)}.$$

The characteristic property of $\mathcal{A}$ is

$$\int_\Omega \mu\mathcal{A}\mathbf{H}\cdot\mathbf{h}\,dx = \alpha^{-1}\int_{\mathcal{F}_1} \operatorname{rot}\mathbf{H}\cdot\operatorname{rot}\mathbf{h}\,dx \quad \text{for all} \quad \mathbf{h},\mathbf{H} \in \mathcal{H}^2.$$

The spectrum of $\mathcal{A}$ consists of a countable number of real negative eigenvalues with the accumulation point at $-\infty$. This guarantees the weighted estimate (2.33) (see details in [12]).

§3. Proof of Theorem 1.1.

First, we find sufficiently small functions $\mathbf{u}_0'' = \mathbf{u}''(y,0)$, $\rho_0'' = \rho''(y,0)$, satisfying the relations

$$\int_{S_1} \rho_0''(R_0 y)\,dS = -\frac{1}{R_0}\int_{S_1} \rho_0^2(R_0 y)\,dS - \frac{1}{3R_0^2}\int_{S_1} \rho_0^3(R_0 y)\,dS,$$

$$\int_{S_1} y_i \rho_0''(R_0 y)\,dS = -\frac{3}{2R_0}\int_{S_1} y_i\rho_0^2(R_0y)\,dS - \frac{1}{R_0^2}\int_{S_1} y_i\rho_0^3(R_0y)\,dS$$
$$-\frac{1}{4R_0^3}\int_{S_1} y_i\rho_0^4(R_0y)\,dS, \quad i=1,2,3,$$
$$\nabla\cdot\mathbf{u}_0''(x) = l_2(\mathbf{u}_0,\rho_0), \quad y\in\mathcal{F}_1, \tag{3.1}$$
$$\nu\Pi_0 S(\mathbf{u}_0'')\mathbf{N}(y) = \mathbf{l}_3(\mathbf{u}_0,\rho_0), \quad y\in S_{R_0}, \quad \mathbf{u}_0''=0, \quad y\in\Sigma,$$

Below, we use the following result.

Proposition 3.1 ([9]). *For arbitrary number p, vector $\mathbf{m}$, function $g\in W_2^l(\mathcal{F}_1)$, and vector field $\mathbf{b}\in W_2^{l-1/2}(S_{R_0})$ there exist functions $r\in W_2^{2+l}(S_{R_0})$, and $\mathbf{u}\in W_2^{1+l}(\mathcal{F}_1)$ such that*

$$\int_{S_1} r(y)\,dS = p, \quad \int_{S_1} r(y)y_i\,dS = m_i, \quad i=1,2,3,$$

$$\nabla\cdot\mathbf{u}(y) = g(y), \quad y\in\mathcal{F}_1, \quad \nu\Pi_0 S(\mathbf{u})\mathbf{N}(y) = \mathbf{b}(y), \quad y\in S_{R_0}, \quad \mathbf{u}\Big|_{\Sigma} = 0,$$

and the following estimate

$$\|r\|_{W_2^{2+l}(S_{R_0})} + \|\mathbf{u}\|_{W_2^{1+l}(\mathcal{F}_1)} \leqslant c\Big(|p| + |\mathbf{m}| + \|g\|_{W_2^l(\mathcal{F}_1)} + \|\mathbf{b}\|_{W_2^{l-1/2}(S_{R_0})}\Big)$$

holds.

We apply Proposition 3.1 to $g = l_2(\mathbf{u}_0,\rho_0)$, $\mathbf{b} = \mathbf{l}_3(\mathbf{u}_0,\rho_0)$. Then, we conclude that there exist $\mathbf{u}_0''$ and ρ_0'' satisfying (3.1) and

$$\begin{aligned}
&\|\rho_0''\|_{W_2^{2+l}(S_{R_0})} + \|\mathbf{u}_0''\|_{W_2^{1+l}(\mathcal{F}_1)}\\
&\leqslant c\big(\|l_2(\mathbf{u}_0,\rho_0)\|_{W_2^l(\mathcal{F}_1)} + \|\mathbf{l}_3(\mathbf{u}_0,\rho_0)\|_{W_2^{l-1/2}(S_{R_0})} + \|\rho_0\|^2_{L_2(S_{R_0})}\big)\\
&\leqslant c\big(\|\rho_0\|_{W_2^{2+l}(S_{R_0})} + \|\mathbf{u}_0\|_{W_2^{l+l}(\mathcal{F}_1)}\big)^2.
\end{aligned} \tag{3.2}$$

Now we find $\mathbf{h}_0'' = \mathbf{h}''(y,0)$ such that

$$\begin{aligned}
&\operatorname{rot}\mathbf{h}_0'' = \operatorname{rot}\mathbf{l}_8(\mathbf{h}_0^{(2)},\rho_0), \quad y\in\mathcal{F}_2, \quad [\mathbf{h}_{0\tau}''] = \mathbf{l}_9(\mathbf{h}_0,\rho_0), \quad y\in S_{R_0},\\
&\nabla\cdot\mathbf{h}_0'' = 0, \quad y\in\mathcal{F}_1\cup\mathcal{F}_2, \quad [\mu\mathbf{h}_0''\cdot\mathbf{N}] = 0, \quad y\in S_{R_0},\\
&\mathbf{h}_0''\cdot\mathbf{n} = 0, \quad y\in S\cup\Sigma, \quad \operatorname{rot}_\tau\mathbf{h}_0'' = 0, \quad y\in\Sigma.
\end{aligned} \tag{3.3}$$

By Theorems 2 and 5 in [1], $\mathbf{h}_0''$ exists and the following estimate holds:

$$\sum_{i=1}^{2}\|\mathbf{h}_0''\|_{W_2^{l+1}(\mathcal{F}_i)} \leqslant c\Big(\|\mathrm{rot}\,\mathbf{l}_8(\mathbf{h}_0,\rho_0)\|_{W_2^l(\mathcal{F}_i)} + \|\mathbf{l}_9\|_{W_2^{l+1/2}(S_{R_0})}\Big)$$
$$\leqslant c\bigg(\sum_{i=1}^{2}\|\mathbf{h}_0^{(i)}\|_{W_2^{l+1}(\mathcal{F}_1)} + \|\rho_0\|_{W_2^{l+2}(S_{R_0})}\bigg)^2. \qquad (3.4)$$

Due to (3.2), (3.4), and condition (1.9), we see that the functions $\mathbf{u}_0''(x)$, $\rho_0''(x)$, $\mathbf{h}_0''(x)$ have the order ε^2, while

$$\mathbf{u}_0'(y) = \mathbf{u}_0(y) - \mathbf{u}_0''(y), \quad \rho_0'(y) = \rho_0(y) - \rho_0''(y), \quad \mathbf{h}_0'(y) = \mathbf{h}_0(y) - \mathbf{h}_0''(y)$$

satisfy the homogeneous conditions (1.6), (1.7). We define $\mathbf{u}', q', \rho', \mathbf{h}'$ as solutions of the homogeneous linear problems (2.4), (2.23) with initial data $(\mathbf{u}_0', \rho_0', \mathbf{h}_0')$. It is clear, that the compatibility conditions take place. By Theorems 2.2 and 2.3, such solutions exist and admit exponential decay in time, in particular, there holds the inequalities

$$\begin{aligned}
&\|\mathbf{u}'(\cdot,t)\|_{W_2^{1+l}(\mathcal{F}_1)} + \|\rho'(\cdot,t)\|_{W_2^{2+l}(S_{R_0})}\\
&\leqslant ce^{-at}\big(\|\mathbf{u}_0'\|_{W_2^{1+l}(\mathcal{F}_1)} + \|\rho_0'\|_{W_2^{2+l}(S_{R_0})}\big)\\
&\leqslant c_1e^{-at}\big(\|\mathbf{u}_0\|_{W_2^{1+l}(\mathcal{F}_1)} + \|\rho_0\|_{W_2^{2+l}(S_{R_0})}\big),\\
&\sum_{i=1}^{2}\|\mathbf{h}'(\cdot,t)\|_{W_2^{1+l}(\mathcal{F}_i)} \leqslant c_2e^{-at}\sum_{i=1}^{2}\|\mathbf{h}_0\|_{W_2^{1+l}(\mathcal{F}_i)}, \quad a>0,
\end{aligned} \qquad (3.5)$$

with the constants c_1, c_2 independent of t.

In this section, we use the notation

$$Y(t) = \|\mathbf{u}(\cdot,t)\|_{W_2^{1+l}(\mathcal{F}_1)} + \|\rho(\cdot,t)\|_{W_2^{2+l}(S_{R_0})} + \sum_{i=1}^{2}\|\mathbf{h}(\cdot,t)\|_{W_2^{1+l}(\mathcal{F}_i)},$$

$$\begin{aligned}
X_{[a,b]}(\mathbf{u},q,\rho,\mathbf{h}) &= \|\mathbf{u}\|_{W_2^{2+l,1+l/2}(\mathcal{F}_1\times(a,b))} + \|\nabla q\|_{W_2^{l,l/2}(\mathcal{F}_1\times(a,b))}\\
&+\|q\|_{W_2^{l+1/2,0}(S_{R_0}\times(a,b))} + \|\rho\|_{W_2^{l+5/2,0}(S_{R_0}\times(a,b))}\\
&+\|\rho_t\|_{W_2^{l+3/2,l/2+3/4}(S_{R_0}\times(a,b))}\\
&+\sum_{i=1}^{2}\Big(\|\mathbf{h}^{(i)}\|_{W_2^{2+l,1+l/2}(\mathcal{F}_i\times(a,b))} + \|\mathbf{h}_t^{(i)}\|_{W_2^{l/2}(a,b,W_2^{-1/2}(S_{R_0}))}\Big)
\end{aligned}$$

and denote by $Y^{'}(t)$, $X^{'}_{[a,b]}$, $Y^{''}(t)$, $X^{''}_{[a,b]}$ the same expressions for the functions $\mathbf{u}^{'}$, $\rho^{'}$, $q^{'}$, $\mathbf{h}^{'}$ or $\mathbf{u}^{''}$, $\rho^{''}$, $q^{''}$, $\mathbf{h}^{''}$, respectively. The inequalities (3.5) imply

$$Y^{'}(t) \leqslant c_3 e^{-at} Y(0), \quad c_3 = \max\{c_1, c_2\}.$$

Let T be so large that

$$Y^{'}(T) \leqslant \frac{1}{8} Y(0). \tag{3.6}$$

We are seeking the solution of problem (1.3) in the form

$$\mathbf{u} = \mathbf{u}^{'} + \mathbf{u}^{''}, \quad q = q^{'} + q^{''}, \quad \rho = \rho^{'} + \rho^{''}, \quad \mathbf{h} = \mathbf{h}^{'} + \mathbf{h}^{''}.$$

Then, for the functions $(\mathbf{u}^{''}, q^{''}, \rho^{''}, \mathbf{h}^{''})$ we obtain the following nonlinear problem:

$$\begin{aligned}
&\mathbf{u}^{''}_t - \nu\nabla^2\mathbf{u}^{''} + \nabla q^{''} = \mathbf{l}_1(\mathbf{u}' + \mathbf{u}^{''}, q' + q^{''}, \mathbf{h}' + \mathbf{h}^{''}, \rho' + \rho^{''}),\\
&\nabla\cdot\mathbf{u}^{''} = l_2(\mathbf{u}' + \mathbf{u}^{''}, \rho' + \rho^{''}) \ \text{ in } \ \mathcal{F}_1, \quad \mathbf{u}''(y,t)\Big|_{y\in\Sigma} = 0,\\
&\nu\Pi_0 S(\mathbf{u}^{''})\mathbf{N} = \mathbf{l}_3(\mathbf{u}' + \mathbf{u}^{''}, \rho' + \rho^{''}),\\
&-q^{''} + \nu\mathbf{N}\cdot S(\mathbf{u}^{''})\mathbf{N}(y) + \sigma B_0\rho^{''}\\
&= l_4(\mathbf{u}' + \mathbf{u}^{''}, \mathbf{h}^{'} + \mathbf{h}^{''}, \rho' + \rho^{''}) + l_5(\rho' + \rho^{''}),\\
&\rho^{''}_t - \mathbf{u}^{''}\cdot\mathbf{N}(y) + |\Omega_0|^{-1}\int\limits_{\mathcal{F}_1}\mathbf{u}^{''}\,dz\cdot\mathbf{N}(y) = l_6(\mathbf{u}' + \mathbf{u}^{''}, \rho' + \rho^{''}), \ \text{ on } \ S_{R_0},\\
&\mu_1\mathbf{h}^{''}_t + \alpha^{-1}\mathrm{rot}\,\mathrm{rot}\,\mathbf{h}^{''} = \mathbf{l}_7(\mathbf{h}' + \mathbf{h}^{''}, \mathbf{u}' + \mathbf{u}^{''}, \rho' + \rho^{''}), \qquad (3.7)\\
&\nabla\cdot\mathbf{h}^{''} = 0, \quad in \quad \mathcal{F}_1,\\
&\mathrm{rot}\,\mathbf{h}^{''} = \mathrm{rot}\,\mathbf{l}_8(\mathbf{h}' + \mathbf{h}^{''}, \rho' + \rho^{''}), \quad \nabla\cdot\mathbf{h}^{''} = 0, \quad in \quad \mathcal{F}_2,\\
&[\mu\mathbf{h}^{''}\cdot\mathbf{N}] = 0, \quad [\mathbf{h}^{''}_\tau] = \mathbf{l}_9(\mathbf{h}' + \mathbf{h}^{''}, \rho' + \rho^{''}), \quad on \quad S_{R_0},\\
&\mathbf{h}^{''}(y,t)\cdot\mathbf{n}(y) = 0, \quad on \quad S\cup\Sigma, \quad \mathrm{rot}_\tau\mathbf{h}^{''} = 0, \quad on \quad \Sigma,\\
&\mathbf{u}^{''}(y,0) = \mathbf{u}^{''}_0(y), \quad y\in\mathcal{F}_1, \quad \mathbf{h}^{''}(y,0) = \mathbf{h}^{''}_0(y), \qquad y\in\mathcal{F}_1\cup\mathcal{F}_2,\\
&\rho^{''}(y,0) = \rho^{''}_0(y), \quad y\in S_{R_0}.
\end{aligned}$$

As the initial data in problem (3.7) are of the order ε^2, this problem can be solved for $t\in[0,T]$, provided that ε is sufficiently small.

Theorem 3.1. *Let all the assumptions of Theorem* 1.1 *be fulfilled. For a given* $T > 0$, *there exists such* $\varepsilon > 0$ *that if the initial data satisfy conditions*

(3.2), (3.4), (1.9) *with this* ε, *then problem* (3.7) *is uniquely solvable on the time interval* $(0,T)$ *and for the solution the following estimate*

$$X_{[0,T]}\left(\mathbf{u}^{''},q^{''},\rho^{''},\mathbf{h}^{''}\right)+\sup_{t<T}Y^{''}(t)\leqslant c(T)Y^{''}(0)\leqslant c_4(T)\varepsilon Y(0) \qquad (3.8)$$

holds.

Proof. The proof of this theorem can be carried out by the successive approximations method.

For the first approximation $(\mathbf{u}_1^{''},q_1^{''},\rho_1^{''},\mathbf{h}_1^{''})$ we take extensions of initial data constructed in the following way: we put $q_1^{''}=0$ and assume that $\mathbf{u}_1^{''}$, $\rho_1^{''}$ satisfy the initial conditions

$$\mathbf{u}_1^{''}\big|_{t=0}=\mathbf{u}_0^{''},\quad \rho_1^{''}\big|_{t=0}=\rho_0^{''},$$

and the inequalities

$$\begin{aligned}\|\,\mathbf{u}_1^{''}\,\|_{W_2^{2+l,1+l/2}(Q_T^1)}&\leqslant c\,\|\,\mathbf{u}_0^{''}\,\|_{W_2^{1+l}(\mathcal{F}_1)},\\ \|\,\rho_1^{''}\,\|_{W_2^{5/2+l,0}(G_T)}&+\|\,\rho_1^{''}\,\|_{W_2^{l/2}(0,T,W_2^{5/2})(S_{R_0})}\\ &+\|\,\rho_{1t}^{''}\,\|_{W_2^{l+3/2,l/2+3/4}(G_T)}\leqslant\|\,\rho_0^{''}\,\|_{W_2^{2+l}(S_{R_0})}\,.\end{aligned} \qquad (3.9)$$

The existence of $\mathbf{u}_1^{''}$, $\rho_1^{''}$ with such properties follows from inverse trace theorems in Sobolev–Slobodetskii spaces and Proposition 4.1 in [2]. For $\mathbf{h}_1^{''}$ we take a divergence free vector field satisfying the initial condition

$$\mathbf{h}_1^{''}\big|_{t=0}=\mathbf{h}_0^{''},\quad y\in\mathcal{F}_1\cup\mathcal{F}_2,$$

and the estimate

$$\begin{aligned}\sum_{i=1}^{2}\Big(\|\,(\mathbf{h}_1^{''})^{(i)}\,\|_{W_2^{2+l,1+l/2}(Q_T^i)}+\|\,(\mathbf{h}_{1t}^{''})^{(i)}\,\|_{W_2^{l/2}(0,T;W_2^{-1/2}(S_{R_0}))}\Big)\\ \leqslant\sum_{i=1}^{2}\|\,\mathbf{h}_0^{''}\,\|_{W_2^{1+l}(\mathcal{F}_i)}\leqslant c\varepsilon\sum_{i=1}^{2}\|\,\mathbf{h}_0^{(i)}\,\|_{W_2^{1+l}(\mathcal{F}_i)}\,.\end{aligned} \qquad (3.10)$$

The construction of this vector field is presented in [9].

For an arbitrary function $\rho(y,t)$, $y\in S_{R_0}$, we define a linear extension operator E with the following properties:

$$\operatorname{supp}E\rho\subset\Omega,\qquad \frac{\partial E\rho}{\partial n}\Big|_{S_{R_0}}=0,$$

$$\| E\rho(\cdot,t) \|_{W_2^{r+1/2}(\Omega)} \leqslant c \| \rho \|_{W_2^r(S_{R_0})} \qquad r \in (0, l+5/2],$$
$$\left\| \frac{\partial}{\partial t} E\rho(\cdot,t) \right\|_{W_2^{r+1/2}(\Omega)} \leqslant c \|\rho_t\|_{W_2^r(S_{R_0})} \qquad r \in (0, l+3/2],$$

and we set $\rho_1^{''*}(y,t) = E\rho_1^{''}$, $\rho^{'*}(y,t) = E\rho^{'}$.

Approximations $\mathbf{u}_{m+1}^{''}$, $q_{m+1}^{''}$, $\rho_{m+1}^{''}$, $\mathbf{h}_{m+1}^{''}$ for $m \geqslant 1$ can be found step by step from the following linear system:

$$\begin{aligned}
&\frac{\partial}{\partial t}\mathbf{u}_{m+1}^{''}(y,t) - \nu\nabla^2\mathbf{u}_{m+1}^{''} + \nabla q_{m+1}^{''} \\
&= \mathbf{l}_1(\mathbf{u}' + \mathbf{u}_m^{''}, q' + q_m^{''}, \mathbf{h}' + \mathbf{h}_m^{''}, \rho' + \rho_m^{''}), \\
&\nabla \cdot \mathbf{u}_{m+1}^{''} = l_2(\mathbf{u}' + \mathbf{u}_m^{''}, \rho' + \rho_m^{''}) \ \text{ in } \ \mathcal{F}_1, \ (\rho^{'} + \rho_m^{''})^* = E\rho^{'} + E\rho_m^{''}, \\
&\nu\Pi_0 S(\mathbf{u}_{m+1}^{''})\mathbf{N} = \mathbf{l}_3(\mathbf{u}' + \mathbf{u}_m^{''}, \rho' + \rho_m^{''}), \\
&- q_{m+1}^{''} + \nu\mathbf{N}\cdot S(\mathbf{u}_{m+1}^{''})\mathbf{N}(y) + \sigma B_0\rho_{m+1}^{''} \\
&= l_4(\mathbf{u}' + \mathbf{u}_m^{''}, \mathbf{h}^{'} + \mathbf{h}_m^{''}, \rho' + \rho_m^{''}) + l_5(\rho' + \rho_m^{''}), \\
&\frac{\partial}{\partial t}\rho_{m+1}^{''} - \mathbf{u}_{m+1}^{''}\cdot\mathbf{N}(y) + |\Omega_0|^{-1}\int_{\mathcal{F}_1}\mathbf{u}_{m+1}^{''}\,dz\cdot\mathbf{N}(y) \\
&= l_6(\mathbf{u}' + \mathbf{u}_m^{''}, \rho' + \rho_m^{''}) \ \text{ on } \ S_{R_0}, \quad \mathbf{u}_{m+1}^{''}\Big|_{y\in\Sigma} = 0, \\
&\mu_1\frac{\partial}{\partial t}\mathbf{h}_{m+1}^{''} + \alpha^{-1}\operatorname{rot}\operatorname{rot}\mathbf{h}_{m+1}^{''} = \mathbf{l}_7(\mathbf{h}' + \mathbf{h}_m^{''}, \mathbf{u}' + \mathbf{u}_m^{''}, \rho' + \rho_m^{''}), \\
&\nabla\cdot\mathbf{h}_{m+1}^{''} = 0, \ \text{ in } \mathcal{F}_1, \\
&\operatorname{rot}\mathbf{h}_{m+1}^{''} = \operatorname{rot}\mathbf{l}_8(\mathbf{h}' + \mathbf{h}_m^{''}, \rho' + \rho_m^{''}), \quad \nabla\cdot\mathbf{h}_{m+1}^{''} = 0, \quad \textit{in} \quad \mathcal{F}_2, \\
&[\mu\mathbf{h}_{m+1}^{''}\cdot\mathbf{N}] = 0, \quad [\mathbf{h}_{m+1,\tau}^{''}] = \mathbf{l}_9(\mathbf{h}' + \mathbf{h}_m^{''}, \rho' + \rho_m^{''}) \ \text{ on } \ S_{R_0}, \\
&\mathbf{h}_{m+1}^{''}(y,t)\cdot\mathbf{n}(y) = 0 \ \text{ on } \ S\cup\Sigma, \quad \operatorname{rot}_\tau\mathbf{h}_{m+1}^{''} = 0, \quad \textit{on} \quad \Sigma, \\
&\mathbf{u}_{m+1}^{''}(y,0) = \mathbf{u}_0^{''}(y), \quad y\in\mathcal{F}_1, \quad \mathbf{h}_{m+1}^{''}(y,0) = \mathbf{h}_0^{''}(y) \ \text{ in } \ \mathcal{F}_1\cup\mathcal{F}_2, \\
&\rho_{m+1}^{''}(y,0) = \rho_0^{''}(y) \ \text{ on } \ S_{R_0}.
\end{aligned} \tag{3.11}$$

Due to (3.2), (3.4), (3.9), (3.10), and assumption (1.9), the first approximation satisfies the estimate

$$X_{[0,T]}\left(\mathbf{u}_1^{''}, q_1^{''}, \rho_1^{''}, \mathbf{h}_1^{''}\right) + \sup_{t<T} Y_1^{''}(t) \leqslant C_1(T)\varepsilon Y(0). \tag{3.12}$$

We intend to prove similar result for the $(m+1)$th approximation. To this end, we estimate the nonlinear terms.

Lemma 3.1. *Let*

$$\| \rho(\cdot,t) \|_{W_2^{2+l}(S_{R_0})} \leqslant \delta_1 < 1,$$
$$\| \mathbf{u}(\cdot,t) \|_{W_2^{1+l}(\mathcal{F}_1)} \leqslant \delta_2 < 1, \quad t < T, \tag{3.13}$$

and let the functions $\mathbf{u}, q, \rho, \mathbf{h}$ *have finite norm*

$$X_{[0,T]}(\mathbf{u}, q, \rho, \mathbf{h}) + \sup_{t<T} Y(t),$$

then the sum of the norms

$$\begin{aligned}
Z[(\mathbf{u}, q, \rho, \mathbf{h})](T) &= \| \mathbf{l}_1(\mathbf{u}, q, \rho, \mathbf{h}) \|_{W_2^{l,l/2}(Q_T^1)} + \| l_2(\mathbf{u}, \rho) \|_{W_2^{1+l,0}(Q_T^1)} \\
&+ \sup_{t<T} \| l_2(\mathbf{u}, \rho) \|_{W_2^l(\mathcal{F}_1)} + \Big\| \frac{\partial}{\partial t}\big((I - \widehat{\mathcal{L}}^T)\mathbf{u}\big) \Big\|_{W_2^{0,l/2}(Q_T^1)} \\
&+ \| \mathbf{l}_3(\mathbf{u}, \rho) \|_{W_2^{l+1/2,l/2+1/4}(G_T)} + \| l_4(\mathbf{u}, \rho, \mathbf{h}) \|_{W_2^{l+1/2,0}(G_T)} \\
&+ \| l_4(\mathbf{u}, \rho, \mathbf{h}) \|_{W_2^{l/2}(0,T,W_2^{1/2}(S_{R_0}))} + \| l_5(\rho) \|_{W_2^{l+1/2,0}(G_T)} \\
&+ \| l_5(\rho) \|_{W_2^{l/2}(0,T,W_2^{1/2}(S_{R_0}))} + \| l_6(\mathbf{u}, \rho) \|_{W_2^{l+3/2,l/1+3/4}(G_T)} \\
&+ \| \mathbf{l}_7(\mathbf{u}, \rho, \mathbf{h}) \|_{W_2^{l,l/2}(Q_T^1)} + \| \operatorname{rot} \mathbf{l}_8(\rho, \mathbf{h}) \|_{W_2^{1+l,0}(Q_T^2)} \\
&+ \sup_{t<T} \| \operatorname{rot} \mathbf{l}_8(\rho, \mathbf{h}) \|_{W_2^l(\mathcal{F}_2)} + \Big\| \frac{\partial}{\partial t}\mathbf{l}_8(\rho, \mathbf{h}) \Big\|_{W_2^{0,l/2}(Q_T^2)} \\
&+ \| \mathbf{l}_9(\rho, \mathbf{h}) \|_{W_2^{l+3/2,0}(G_T)} + \sup_{t<T} \| \mathbf{l}_9(\rho, \mathbf{h}) \|_{W_2^{l+1/2}(S_{R_0})} \\
&+ \sum_{i=1}^{2} \left(\Big\| \frac{\partial}{\partial t}\mathbf{A}^{(i)}(\rho, \mathbf{h}) \Big\|_{W_2^{0,l/2}(Q_T^i)} \Big\| \frac{\partial}{\partial t}\mathbf{A}^{(i)}(\rho, \mathbf{h}) \Big\|_{W_2^{l/2}(0,T,W_2^{-1/2}(S_{R_0}))} \right),
\end{aligned}$$

$$Q_T^i = \mathcal{F}_i \times (0,T), \quad G_T = S_{R_0} \times (0,T)$$

satisfies the inequality

$$Z[(\mathbf{u}, q, \rho, \mathbf{h})](T) \leqslant C(T)\Big[\Big(X_{[0,T]} + \sup_{t<T} Y(t)\Big)^2 + \Big(X_{[0,T]} + \sup_{t<T} Y(t)\Big)^3\Big]. \tag{3.14}$$

The main ideas of the proof of (3.14) are outlined in Sec. 4.

From Theorems 2.1, 2.3, and Lemma 2.2, it can be deduced that the problem (3.11) has a unique solution $\mathbf{u}''_{m+1}$, q''_{m+1}, ρ''_{m+1}, $\mathbf{h}''_{m+1}$, and in accordance with (2.2), (2.33) (with $a = 0$), and (3.14), we have

$$X''_{m+1}[0,T] + \sup_{t<T} Y''_{m+1}(t) \leqslant C(T)\Big(Z\big[\mathbf{u}_m, q_m, \rho_m, \mathbf{h}_m\big] + Y''(0)\Big)$$

$$\leqslant C(T)\Big(\big(X''_m[0,T] + \sup_{t<T} Y''_m(t)\big)^2 + \big(X'[0,T] + \sup_{t<T} Y'(t)\big)^2$$

$$+\big(X''_m[0,T] + \sup_{t<T} Y''_m(t)\big)^3 + \big(X'[0,T] + \sup_{t<T} Y'(t)\big)^3 + Y''(0)\Big) \tag{3.15}$$

$$\leqslant C_2(T)\Big[(X''_m[0,T]+\sup_{t<T} Y''_m(t))^2+(X''_m[0,T]+\sup_{t<T} Y''_m(t))^3\Big]+C_3(T)\varepsilon Y(0).$$

For $m = 1$, (3.12) and (3.15) imply

$$X''_2[0,T] + \sup_{t<T} Y''_2(t) \leqslant C_2(T)\left(C_1(T)\varepsilon Y(0)\right)^2 + C_3(T)\varepsilon Y(0). \tag{3.16}$$

We choose ε in such a way that the right-hand side of (3.16) does not exceed $2C_3(T)\varepsilon Y(0)$. It is clear that if the estimate

$$X''_m[0,T] + \sup_{t<T} Y''_m(t) \leqslant 2C_3(T)\varepsilon Y(0) \tag{3.17}$$

holds for the m-th approximation, it holds also for the $m+1$-th approximation, provided ε is sufficiently small. And if conditions (3.13) are valid for the m-th approximation then they are valid for the $m+1$-th approximation.

In order to prove the convergence of the sequence $(\mathbf{u}''_m, q''_m, \rho''_m, \mathbf{h}''_m)$, we introduce the differences

$$\mathbf{k}_{m+1} = \mathbf{h}''_{m+1} - \mathbf{h}''_m, \quad \mathbf{w}_{m+1} = \mathbf{u}''_{m+1} - \mathbf{u}''_m,$$
$$r_{m+1} = \rho''_{m+1} - \rho''_m, \quad s_{m+1} = q''_{m+1} - q''_m.$$

They satisfy the relations

$$\frac{\partial}{\partial t}\mathbf{w}_{m+1}(y,t) - \nu\nabla^2\mathbf{w}_{m+1} + \nabla s_{m+1}$$
$$= \mathbf{l}_1(\mathbf{u}' + \mathbf{u}''_m, q' + q''_m, \mathbf{h}' + \mathbf{h}''_m, \rho' + \rho''_m)$$
$$- \mathbf{l}_1(\mathbf{u}' + \mathbf{u}''_{m-1}, q' + q''_{m-1}, \mathbf{h}' + \mathbf{h}''_{m-1}, \rho' + \rho''_{m-1}),$$
$$\nabla\cdot\mathbf{w}_{m+1} = l_2(\mathbf{u}' + \mathbf{u}''_m, \rho' + \rho''_m) - l_2(\mathbf{u}' + \mathbf{u}''_{m-1}, \rho' + \rho''_{m-1}) \quad \text{in } \mathcal{F}_1,$$
$$\nu\Pi_0 S(\mathbf{w}_{m+1})\mathbf{N} = \mathbf{l}_3(\mathbf{u}' + \mathbf{u}''_m, \rho' + \rho''_m) - \mathbf{l}_3(\mathbf{u}' + \mathbf{u}''_{m-1}, \rho' + \rho''_{m-1}),$$

$$
\begin{aligned}
&- s_{m+1} + \nu \mathbf{N} \cdot S(\mathbf{w}_{m+1})\mathbf{N}(y) + \sigma B_0 r_{m+1} \\
&= l_4(\mathbf{u}' + \mathbf{u}''_m, \mathbf{h}' + \mathbf{h}''_m, \rho' + \rho''_m) + l_5(\rho' + \rho''_m) \\
&- l_4(\mathbf{u}' + \mathbf{u}''_{m-1}, \mathbf{h}' + \mathbf{h}''_{m-1}, \rho' + \rho''_{m-1}) - l_5(\rho' + \rho''_{m-1}), \qquad (3.18) \\
&\frac{\partial}{\partial t} r_{m+1} - \mathbf{w}_{m+1} \cdot \mathbf{N}(y) + |\Omega_0|^{-1} \int_{\mathcal{F}_1} \mathbf{w}_{m+1} dz \cdot \mathbf{N}(y) \\
&= l_6(\mathbf{u}' + \mathbf{u}''_m, \rho' + \rho''_m) - l_6(\mathbf{u}' + \mathbf{u}''_{m-1}, \rho' + \rho''_{m-1}) \quad \text{on} \quad S_{R_0}, \\
&\mathbf{w}_{m+1}\big|_{y \in \Sigma} = 0, \\
&\mu_1 \frac{\partial}{\partial t} \mathbf{k}_{m+1} + \alpha^{-1} \mathrm{rot}\, \mathrm{rot}\, \mathbf{k}_{m+1} = \mathbf{l}_7(\mathbf{h}' + \mathbf{h}''_m, \mathbf{u}' + \mathbf{u}''_m, \rho' + \rho''_m) \\
&- \mathbf{l}_7(\mathbf{h}' + \mathbf{h}''_{m-1}, \mathbf{u}' + \mathbf{u}''_{m-1}, \rho' + \rho''_{m-1}), \quad \nabla \cdot \mathbf{k}_{m+1} = 0 \ \text{ in } \ \mathcal{F}_1, \\
&\mathrm{rot}\, \mathbf{k}_{m+1} = \mathrm{rot}\, \mathbf{l}_8(\mathbf{h}' + \mathbf{h}''_m, \rho' + \rho''_m) - \mathrm{rot}\, \mathbf{l}_8(\mathbf{h}' + \mathbf{h}''_{m-1}, \rho' + \rho''_{m-1}), \\
&\nabla \cdot \mathbf{k}_{m+1} = 0 \ \text{ in } \ \mathcal{F}_2, \\
&[\mu \mathbf{k}_{m+1} \cdot \mathbf{N}] = 0, \quad [\mathbf{k}_{m+1,\tau}] = \mathbf{l}_9(\mathbf{h}' + \mathbf{h}''_m, \rho' + \rho''_m) \\
&- \mathbf{l}_9(\mathbf{h}' + \mathbf{h}''_{m-1}, \rho' + \rho''_{m-1}) \ \text{ on } \ S_{R_0}, \\
&\mathbf{k}_{m+1}(y,t) \cdot \mathbf{n}(y) = 0, \quad \textit{on} \quad S \cup \Sigma, \quad \mathrm{rot}_{\,\tau} \mathbf{k}_{m+1} = 0 \ \text{ on } \ \Sigma, \\
&\mathbf{w}_{m+1}(y,0) = 0, \quad y \in \mathcal{F}_1, \quad \mathbf{k}_{m+1}(y,0) = 0 \ \text{ in } \ \mathcal{F}_1 \cup \mathcal{F}_2, \\
&r_{m+1}(y,0) = 0 \ \text{ on } \ S_{R_0}.
\end{aligned}
$$

Estimates of the differences of nonlinear terms in the right-hand side of (3.18) can be obtained by the same technics as estimates of the nonlinear terms (see Sec. 4). Precisely, the following result holds.

Lemma 3.2. *Let all the assumptions of Lemma* 3.1 *be fulfilled, and let* (3.17) *hold for* $m \in \mathbf{N}$. *Then*

$$
\begin{aligned}
&Z_T[(\mathbf{u}' + \mathbf{u}''_m, q' + q''_m, \mathbf{h}' + \mathbf{h}''_m, \rho' + \rho''_m) \\
&- (\mathbf{u}' + \mathbf{u}''_{m-1}, q' + q''_{m-1}, \mathbf{h}' + \mathbf{h}''_{m-1}, \rho' + \rho''_{m-1})] \\
&\leqslant c\theta(\varepsilon, T) X \left(\mathbf{u}''_m - \mathbf{u}''_{m-1}, q''_m - q''_{m-1}, \mathbf{h}''_m - \mathbf{h}''_{m-1}, \rho''_m - \rho''_{m-1} \right) \qquad (3.19) \\
&= c\theta(\varepsilon, T) X \left(\mathbf{w}_m, s_m, \mathbf{k}_m, r_m \right),
\end{aligned}
$$

where the function $\theta(\varepsilon, T)$ *is small for small* ε.

Since for solutions to the linear problems inequalities (2.2) and (2.33) are valid, (3.19) implies

$$X_T[\mathbf{w}_{m+1}, s_{m+1}, \mathbf{k}_{m+1}, r_{m+1}] \leqslant c\theta(\varepsilon, T) X_T[\mathbf{w}_m, s_m, \mathbf{k}_m, r_m]. \qquad (3.20)$$

For sufficiently small $\theta(\varepsilon, T)$, (3.20) guarantees the convergence of the sequence $(\mathbf{u}_m^{''}, q_m^{''}, \rho_m^{''}, \mathbf{h}_m^{''})$ to the solution of problem (3.7) (see [1, Sec. 3]). Passing to the limit in (3.17), we obtain (3.8). Uniqueness of the solution follows from the above estimates applied to the difference of two solutions of (3.7). □

Now we fix ε so small that on the right-hand side of (3.8) $c_4(T)\varepsilon < \frac{1}{8}$. Taking a sum of (3.6) and (3.8), we obtain

$$Y(T) \leqslant \frac{1}{4} Y(0). \qquad (3.21)$$

Then, we consider $Y(T)$ as initial data at $t = T$ and repeat the above scheme on $[T, 2T]$. On this step conservation of volume still holds (it means that the first of conditions (1.5) for $\rho(y, T)$ takes place), while the barycenter is located at the point $\xi(T)$, which not necessarily coincides with the origin. We have

$$\int\limits_{\Omega_T} x_i \, dx = \xi_i(T) \frac{4}{3} \pi R_0^3 = \xi_i(T) \int\limits_{\Omega_T} dx,$$

i.e., $\int\limits_{\Omega_T} (x_i - \xi_i(T)) \, dx = 0$, $i = 1, 2, 3$. We pass to the spherical coordinates with the center at the point $\xi(T)$: $x_i = \nu_i r + \xi_i(T)$, and see that linear part of the second condition (1.5) for $\rho(y, T)$ has the same form as for ρ_0, precisely, $\int\limits_{S_1} y_i \rho(R_0 y, T) \, dS = 0$. Consequently, we can use all the results of Sec. 2 on the time interval $[T, 2T]$. Repeating these arguments on time intervals $[iT, (i+1)T]$, $i = 1, \dots, k-1$, we have

$$Y(kT) \leqslant \frac{1}{4} Y((k-1)T) \leqslant \cdots \leqslant \frac{1}{4^k} Y(0), \quad k \in \mathrm{N}. \qquad (3.22)$$

It means the exponential decay for $Y(t)$. Let us prove estimate (1.10). Due to (2.9), (2.33), the following inequality

$$X_{[0,T]}\big(e^{at}\mathbf{u}^{'}, e^{at}q^{'}, e^{at}\rho^{'}, e^{at}\mathbf{h}^{'}\big) \leqslant c_5 Y(0) \qquad (3.23)$$

holds with a certain $a>0$ and a constant c_5 independent of t. Under the assumption $c_4(T)\varepsilon \leqslant \frac{1}{8}$ in (3.8), we have

$$X_{[0,T]}\big(\mathbf{u}^{''}, q^{''}, \rho^{''}, \mathbf{h}^{''}\big) \leqslant \frac{1}{8} Y(0),$$

and, if we take $a \leqslant \frac{1}{T}$, it means that

$$X_{[0,T]}\big(e^{at}\mathbf{u}^{''}, e^{at}q^{''}, e^{at}\rho^{''}, e^{at}\mathbf{h}^{''}\big) \leqslant Y(0). \tag{3.24}$$

Taking a sum of (3.23) and (3.24), we arrive at

$$X_{[0,T]}\big(e^{at}\mathbf{u}, e^{at}q, e^{at}\rho, e^{at}\mathbf{h}\big) \leqslant c_6 Y(0), \tag{3.25}$$

where the constant c_6 is independent of T. On the second step we can apply estimates (2.9), (2.33), replacing t by $t-T$, and obtain

$$X_{[T,2T]}\big(e^{a(t-T)}\mathbf{u}^{'}, e^{a(t-T)}q^{'}, e^{a(t-T)}\rho^{'}, e^{a(t-T)}\mathbf{h}^{'}\big) \leqslant c_5 Y^{'}(T) \leqslant \frac{c_5}{4} Y(0).$$

Consequently, for $a \leqslant \frac{1}{T}$,

$$X_{[T,2T]}\big(e^{at}\mathbf{u}^{'}, e^{at}q^{'}, e^{at}\rho^{'}, e^{at}\mathbf{h}^{'}\big) \leqslant \frac{e^{aT}}{4} c_5 Y(0) \leqslant c_5 Y(0). \tag{3.26}$$

By (3.22), we see that

$$X_{[T,2T]}\big(\mathbf{u}^{''}, q^{''}, \rho^{''}, \mathbf{h}^{''}\big) \leqslant \frac{1}{8} Y(T) \leqslant \frac{1}{4^2} Y(0). \tag{3.27}$$

Inequalities (3.26), (3.27), give us (3.25) on $[T,2T]$, namely

$$X_{[T,2T]}\big(e^{at}\mathbf{u}, e^{at}q, e^{at}\rho, e^{at}\mathbf{h}\big) \leqslant c_6 Y(0),$$

where the constant c_6 is independent of T. Let us consider the interval $[kT,(k+1)T]$, $k=2,3,\dots$. Now we use weighted estimates (2.9), (2.33) for linear problems, replacing t by $t-kT$. For $a<T$, we arrive at

$$X_{[kT,(k+1)T]}\big(e^{at}\mathbf{u}^{'}, e^{at}q^{'}, e^{at}\rho^{'}, e^{at}\mathbf{h}^{'}\big) \leqslant e^{akT} Y^{'}(kT) \leqslant \frac{e^k}{4^k} Y(0) \leqslant Y(0).$$

Taking into account also (3.8), (3.22), we obtain (3.25) on $[kT,(k+1)T]$. Consequently,

$$X_{[0,nT]} \leqslant \sum_{k=0}^{n-1} X_{[kT,(k+1)T]}\big(e^{at}\mathbf{u}, e^{at}q, e^{at}\rho, e^{at}\mathbf{h}\big) \leqslant n c_6 Y(0), \tag{3.28}$$

with a certain $a \leqslant \frac{1}{T}$, and constant c_6 independent on T. For n tends to infinity, (3.28) implies the exponential decay in corresponding Sobolev norms for $\mathbf{u}, q, \rho, \mathbf{h}$ with a certain power $a_1 < a$.

Making use of coordinate transform (1.2), we obtain the solution $\mathbf{v}$, p, $\mathbf{H}$, ρ to problem (1.1), which is defined for $t > 0$ and tends to zero exponentially as t tends to infinity.

Estimate (3.22) shows that

$$\|\mathbf{u}(\cdot,t)\|_{L_2(\mathcal{F}_1)} \leqslant e^{-bt}\Big(\|\mathbf{u}_0\|_{W_2^{l+1}(\mathcal{F}_1)} + \|\rho_0\|_{W_2^{l+2}(S_{R_0})} + \sum_{i=1}^{2} \|\mathbf{h}_0^{(i)}\|_{W_2^{l+1}(\mathcal{F}_i)}\Big) \leqslant e^{-bt}\epsilon, \quad b > 0.$$

Consequently,

$$|\xi(+\infty)| = \frac{1}{|\Omega_0|}\Big|\int_0^{+\infty} dt \int_{\Omega_{1t}} \mathbf{v}(x,t)\,dx\Big| \leqslant \frac{1}{|\Omega_0|}\int_0^{+\infty} dt \int_{\mathcal{F}_1} |\mathbf{u}(y,t)||L|\,dy$$
$$\leqslant c\int_0^{+\infty} \|\mathbf{u}(\cdot,t)\|_{L_2(\mathcal{F}_1)}\,dt \leqslant c\int_0^{+\infty} \varepsilon e^{-bt}\,dt \leqslant C\varepsilon. \quad (3.29)$$

It means that $\xi(t)$ is uniformly bounded for any $t > 0$: $|\xi(t)| \leqslant C\epsilon$ with the constant C independent of t, which justifies the above arguments.

Remark 3.1. To be sure that our free boundary not intersect the fixed boundaries Σ and S, we should assume that at the initial moment of time

$$\operatorname{dist}\{\Sigma, \Gamma_0\} \geqslant 2d_0, \quad \operatorname{dist}\{S, \Gamma_0\} \geqslant 2d_0, \quad d_0 > C\varepsilon,$$

and also $\operatorname{dist}\{\xi(0), \Sigma\} \geqslant \delta > C\varepsilon$.

§4. Estimates of nonlinear terms

In connection with the proof of local solvability of a free boundary problem for the Navier–Stokes equation, nonlinear terms similar to $\mathbf{l}_1, l_2, \mathbf{l}_3$ are estimated in [10], under the assumption that T is sufficiently small. Similar estimates for the nonlinear terms depending of the magnetic field are done in [1]. In [9], it is shown how to treat the nonlinear terms when T is not small any more, on the contrary, $T > 1$, while the initial data are sufficiently small. The method used in [1, 9, 10] based on estimates for the product of two functions in Sobolev norms and the following result:

Proposition 4.1 ([1, Proposition 3]). *Let*

$$\mathbf{R}(x,t) = \left(\frac{\partial N_i^* \rho^*}{\partial y_j}\right)_{i,j=1,2,3}, \quad \rho^* = E\rho,$$

and let the function ρ satisfies the condition

$$\sup_{t<T} \|\rho(\cdot,t)\|_{W_2^{2+l}(S_{R_0})} \leqslant \delta < 1.$$

Then, for the arbitrary smooth function $f(\mathbf{R})$ defined for $|\mathbf{R}| \leqslant c\|\rho\|_{W_2^{2+l}}$, the inequalities

$$\| f(\mathbf{R}) \|_{W_2^{1+l}(\mathcal{F}_i)} \leqslant c, \tag{4.1}$$

$$\| \mathbf{R}f(\mathbf{R}) \|_{W_2^r(\mathcal{F}_i)} \leqslant c \| \mathbf{R} \|_{W_2^r(\mathcal{F}_i)} \leqslant c \| \rho \|_{W_2^{r+1/2}(S_{R_0})}, \quad r \in [0, 1+l], \quad i = 1,2.$$

$$\| \mathbf{R}f(\mathbf{R}) \|_{W_2^{1+l}(\mathcal{F}_i)} \leqslant c \| \rho \|_{W_2^{5/2+l}(S_{R_0})} \tag{4.2}$$

are valid.

In the present paper, we make the coordinate transformation (1.2), which has the additional term $\chi(y)\xi(t)$ in comparison with the one used in [1,9]. Consequently, the Jacobi matrix has the entries

$$\delta_i^j + \frac{\partial}{\partial y_j}\left(N_i \rho^*\right) + \frac{\partial}{\partial y_j}\chi(y)\xi_i(t).$$

As a result, all the nonlinear terms in our case are slightly different from the terms in [1,9]. Under the assumptions that the cut-off function $\chi(y)$ is a smooth function with uniformly bounded derivatives and

$$\sup_{t<T} |\xi(t)| \leqslant \delta < 1, \tag{4.3}$$

inequality (4.1) is evidently valid for $\mathbf{R} = \left(\frac{\partial}{\partial y_j}\chi(y)\xi_i(t)\right)_{i,j=1,2,3}$. Under the second of conditions (3.13), we have

$$\sup_{t<T} |\xi(t)| \leqslant \frac{3}{4\pi R_0^3} \int_0^T d\tau \int_{\mathcal{F}_1} |\mathbf{u}(y,\tau)||L|\, dy \leqslant c(|\mathcal{F}_1|)T\delta_2,$$

and condition (4.3) holds if $\delta_2 < \frac{1}{Tc(|\mathcal{F}_1|)} < 1$. This makes it possible to apply in our case the method suggested in [9].

To illustrate the proof, we estimate the term $(\nabla - \widetilde{\nabla})q$ in $\mathbf{l}_1$. In our case $l \in (1/2, 1)$, consequently $l < n/2$, and product of two functions we estimate by (4.6) in [9]:

$$\| uv \|_{W_2^l(\mathcal{F}_1)} \leqslant c \| u \|_{W_2^l(\mathcal{F}_1)} \| v \|_{W_2^s(\mathcal{F}_1)}, \quad s > n/2.$$

We obtain:

$$\| (\nabla - \widetilde{\nabla})q \|_{W_2^{l,0}(Q_T^1)} \leqslant \sup_{t<T} \| I - \mathcal{L}^{-T} \|_{W_2^{3/2+\eta}(\mathcal{F}_1)} \| \nabla q \|_{W_2^{l,0}(Q_1^T)}$$
$$\leqslant c \left(\sup_{t<T} \| \rho^*(\cdot, t) \|_{W_2^{l+3/2}(\mathcal{F}_1)} + \sup_{t<T} |\xi(t)| \right) \| \nabla q \|_{W_2^{l,0}(Q_1^T)},$$
$$\eta \in (0, l - 1/2). \quad (4.4)$$

In $W_2^{l/2}(0, T)$, where $T > 1$, we use the norm

$$\left(\| f \|^2_{L_2(0,T)} + \int_0^1 \frac{dh}{h^{1+l}} \int_h^T |\triangle_t(-h) f(t)|^2 \, dt \right)^{1/2},$$

where $\triangle_t(-h)f(t) = f(t-h) - f(t)$. This norm is equivalent to the standard one. We have

$$\| (\nabla - \widetilde{\nabla})q \|_{W_2^{0,l/2}(Q_T^1)} \leqslant \| (I - \mathcal{L}^{-T}) \nabla q \|_{L_2(Q_T^1)}$$

$$+ \left(\int_0^1 \frac{dh}{h^{1+l}} \int_h^T \| (I - \mathcal{L}^{-T}) \triangle_t(-h) \nabla q \|^2_{L_2(\mathcal{F}_1)} \, dt \right)^{1/2}$$

$$+ \left(\int_0^1 \frac{dh}{h^{1+l}} \int_h^T \| (\triangle_t(-h) \mathcal{L}^{-T}) \nabla q \|^2_{L_2(\mathcal{F}_1)} \, dt \right)^{1/2}. \quad (4.5)$$

To estimate the right-hand side of (4.5), we use the inequalities

$$\| (I-\mathcal{L}^{-T})\nabla q \|_{L_2(\mathcal{F}_1)} \leqslant c\left(\sup_{\mathcal{F}_1} |\mathbf{R}(y,t)| + |\xi(t)|\right) \| \nabla q \|_{L_2(\mathcal{F}_1)},$$

$$\| (I-\mathcal{L}^{-T})\triangle_t(-h)\nabla q \|_{L_2(\mathcal{F}_1)}$$
$$\leqslant c\left(\sup_{\mathcal{F}_1} |\mathbf{R}(y,t)| + |\xi(t)|\right) \| \triangle_t(-h)\nabla q \|_{L_2(\mathcal{F}_1)},$$

$$\| (\triangle_t(-h)\mathcal{L}^{-T})\nabla q \|_{L_2(\mathcal{F}_1)} \leqslant c \| \nabla q \|_{W_2^l(\mathcal{F}_1)} \| \triangle_t(-h)\mathcal{L}^{-T} \|_{W_2^{3/2-l}(\mathcal{F}_1)}$$
$$\leqslant c \| \nabla q \|_{W_2^l(\mathcal{F}_1)} \left(\int_0^h \| \mathbf{R}_t(t-\tau) \|_{W_2^{3/2-l}(\mathcal{F}_1)} \, d\tau + \int_0^h (|\xi^{'}(t-\tau)| \, d\tau \right).$$

For the terms containing $\mathbf{R}$, we repeat the arguments given in [9, Sec. 4], and obtain

$$\| (\nabla - \widetilde{\nabla})q \|_{W_2^{0,l/2}(Q_T^1)}$$
$$\leqslant c\left(\sup_{t<T} \| \rho(\cdot,t) \|_{W_2^{l+3/2}(S_{R_0})} + \sup_{t<T} |\xi(t)| \right) \| \nabla q \|_{W_2^{0,l/2}(Q_T^1)}$$
$$+ c\left(\int_0^T \| \nabla q \|^2_{W_2^l(\mathcal{F}_1)} \, dt \int_0^t \| \rho_\tau \|^2_{W_2^{l+3/2}(S_{R_0})} \, d\tau \right)^{1/2}$$
$$+ c\left(\int_0^T \| \nabla q \|^2_{W_2^l(\mathcal{F}_1)} \, dt \int_0^{\min(t,1)} \frac{dh}{h^{1+l}} \left(\int_0^h |\xi^{'}(t-\tau)| \, d\tau \right)^2 \right)^{1/2}. \quad (4.6)$$

Since

$$\sup_{t<T} |\xi(t)| \leqslant \frac{1}{|\Omega_0|} \int_0^T d\tau \int_{\mathcal{F}_1} |\mathbf{u}(y,\tau)||L| \, dy \leqslant c(T) \| \mathbf{u} \|_{L_2(Q_T^1)}, \quad (4.7)$$

it remains to estimate the last term at the right-hand side of (4.6). As

$$\xi^{'}(t) = \frac{1}{|\Omega_0|} \int_{\mathcal{F}_1} \mathbf{u}(y,t) L \, dy, \quad (4.8)$$

and $l<1$, we have, using the Minkovskii inequality:

$$\int\limits_0^T \| \nabla q \|^2_{W_2^l(\mathcal{F}_1)} \, dt \int\limits_0^{\min(t,1)} \frac{dh}{h^{1+l}} \left(\int\limits_0^h |\xi^{'}(t-\tau)| \, d\tau \right)^2$$

$$\leqslant \int\limits_0^T \| \nabla q \|^2_{W_2^l(\mathcal{F}_1)} \, dt \left(\int\limits_0^{\min(1,t)} |\xi^{'}(t-\tau)| \, d\tau \left(\int\limits_\tau^{\min(1,t)} \frac{dh}{h^{1+l}} \right)^{1/2} \right)^2 \tag{4.9}$$

$$\leqslant c \int\limits_0^T \| \nabla q \|^2_{W_2^l(\mathcal{F}_1)} \, dt \int\limits_0^1 \| \mathbf{u}(\cdot,\tau) \|^2_{L_2(\mathcal{F}_1)} \, d\tau \leqslant cX^2[0,T].$$

Estimates (4.4)–(4.9) give us

$$\| (\nabla - \widetilde{\nabla}) q \|_{W_2^{l,l/2}(Q_T^1)} \leqslant cX^2[0,T].$$

In comparison with the previous papers, we have the additional terms in $\mathbf{l}_1$, l_6, and $\mathbf{l}_7$, for example,

$$\mathcal{L}^{-1}\xi'(t)\cdot\nabla\mathbf{u} \ \text{ in } \ \mathbf{l}_1 \ \text{ or } \ \int\limits_{\mathcal{F}_1} u \, dy \cdot (\mathbf{N} - \frac{\mathbf{n}}{n_r}) \ \text{ in } \ l_6.$$

Due to (4.7), (4.8) the estimate of these terms can be done in a similar way.

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