Записки научных
семинаров ПОМИ
Том 410, 2013 г.

S. Repin

ESTIMATES OF DEVIATIONS FROM EXACT SOLUTION OF THE GENERALIZED OSEEN PROBLEM

Abstract. The paper is concerned with a generalized version of the stationary Oseen problem, which often arises in semidiscrete approximation methods used for quantitative analysis of Navier–Stokes equations. We derive a fully computable functional defined for admissible velocity, stress, and pressure fields and prove that this functional generates upper and lower bounds of the total error evaluated in the corresponding combined norm. Moreover, this functional vanishes if and only if its arguments coincide with the exact velocity, stress, and pressure. Therefore, minimization of it is equivalent to solving the Oseen problem.

Dedicated to 90th anniversary of O.A. Ladyzhenskaya

§1. Introduction

One of widely used linearizations of the Navier–Stokes system is the Oseen problem ([8,16,23]). In this paper, we consider a generalized version of this problem presented by the relations

$$\kappa u - \mathrm{Div}\sigma + \mathrm{Div}(\mathrm{a}\otimes u) = f \qquad \text{in } \Omega, \tag{1.1}$$

$$\sigma = \nu\nabla u - p\mathbb{1} \qquad \text{in } \Omega, \tag{1.2}$$

$$\mathrm{div} u = 0 \qquad \text{in } \Omega, \tag{1.3}$$

$$u = u_0 \qquad \text{on } \partial\Omega. \tag{1.4}$$

Here Ω is a connected bounded domain in $\mathbb{R}^d$ $(d = 2, 3)$ with Lipschitz boundary $\partial\Omega$, $\{u \otimes w\}_{ij} := u_i w_j$ is the diad product of vectors, $\mathbb{1}$ is the unit tensor, and $\nu > 0$ is the viscosity (which, in general, may be a bounded function of x). In (1.1)–(1.4), $\kappa \geqslant 0$ is a constant (or a nonnegative bounded function), a is a given solenoidal vector–valued function, u is the velocity (vector–valued function), p is the pressure (it is defined up to a constant

Key words and phrases: Oseen problem, a posteriori error estimates of the functional type, incompressible viscous fluids.

This work is supported by RFBR grants 11-01-00324-a and 11-01-00531-a.

and, therefore, for the sake of definiteness we assume that it has zero mean in Ω, i.e., $\{p\}_\Omega = 0$), and σ is the stress (tensor–valued function).

Our interest to this system is motivated by the fact that it is often used in quantitative analysis of the corresponding Navier–Stokes problem (see, e.g., [3, 10, 11, 18]) if the latter problem is solved by means of the so–called semidiscrete schemes (in which the derivative $\frac{\partial u}{\partial t}$ is replaced by the incremental relation for a small time interval Δt). This way leads to the iteration scheme ($k = 1, 2, 3, \ldots$)

$$\frac{u^{k+1} - u^k}{\Delta t} - \mathrm{Div}(\nu \nabla u^{k+1}) + \mathrm{Div}(u^k \otimes u^{k+1}) = f - \nabla p^{k+1}, \tag{1.5}$$

$$\mathrm{div} u^{k+1} = 0, \tag{1.6}$$

$$u^{k+1} = u_0 \qquad \text{on } \partial\Omega. \tag{1.7}$$

This system is equivalent to (1.1)–(1.4) if u^k is considered as a given function, $\kappa = \frac{1}{\Delta t}$, and u^{k+1} and p^{k+1} are the functions to be found. Also, generalized forms of the Oseen problem arise in models of viscous flow with polymerization (see, e.g., [5]).

We assume that

$$f \in L^2(\Omega, \mathbb{R}^d),\ u_0 \in V(\Omega) := H^1(\Omega, \mathbb{R}^d),\ \text{and} \int_{\partial\Omega} u_0 \cdot n\, ds = 0, \tag{1.8}$$

where n denotes the outward normal vector and

$$\mathrm{a} \in L^\infty(\Omega, \mathbb{R}^d), \qquad \mathrm{div\,a} = 0. \tag{1.9}$$

Henceforth, for the sake of convenience, we set $\alpha = \sqrt{\kappa}$ and use this function instead of κ. By $\overset{\circ}{J}{}^1_2(\Omega)$ we denote the closure of smooth solenoidal functions with compact supports in Ω with respect to the norm of V and introduce the space

$$H(\Omega, \mathrm{Div}) = \left\{\tau = \{\tau_{ij}\} \mid \tau_{ij} \in L^2(\Omega),\ \mathrm{div}\tau \in L^2(\Omega, \mathbb{R}^d)\right\}.$$

Instead of the standard norm of this space we use an equaivalent norm

$$\|\tau\|_{\mathrm{Div}} := \|\tau\| + C_{F\Omega}\|\mathrm{Div}\tau\|,$$

where $C_{F\Omega}$ is the constant in the Friedrics inequality for functions vanishing on $\partial\Omega$.

The generalized solution of the problem is defined as a function u such that

$$u \in \overset{\circ}{J}{}^1_2(\Omega) + u_0 := \left\{u = w + u_0\,, w \in \overset{\circ}{J}{}^1_2(\Omega)\right\}$$

and

$$\int_\Omega \big(\nu\nabla u : \nabla w + \alpha^2 u \cdot w - (\mathrm{a} \otimes u) : \nabla w\big) dx$$
$$= \int_\Omega f \cdot w dx, \ \forall w \in \overset{\circ}{J}{}^1_2(\Omega), \quad (1.10)$$

where $\cdot$ and $:$ denote the scalar products of vectors and tensors, respectively. It is known (see, e.g., Chapter 2 of the book [12]) that under the above made assumptions the generalized solution u exists and is unique.

If a wider set of trial functions is used, then the solution can be defined as the pair (u, p) such that $u \in \overset{\circ}{J}{}^1_2(\Omega) + u_0$,

$$p \in \widetilde{L}^2(\Omega) := \{q \in L_2(\Omega) \mid \{q\}_\Omega = 0\},$$

and

$$\int_\Omega \big(\nu\nabla u : \nabla w + \alpha^2 u \cdot w - (\mathrm{a} \otimes u) : \nabla w\big)\, dx$$
$$= \int_\Omega (f \cdot w + p\, \mathrm{div} w)\, dx, \quad \forall w \in V_0, \quad (1.11)$$

where $V_0(\Omega)$ denotes the subspace of $V(\Omega)$ containing functions vanishing on the boundary.

In this paper, we deduce directly computable and realistic bounds of the difference between the exact solution (u, σ, p) of the Oseen problem and functions (v, τ, q) considered as approximations. The difference is measured in terms of the norms of the spaces

$$\overset{\circ}{J}{}^1_2(\Omega) \times H(\Omega, \mathrm{Div}) \times \widetilde{L}^2(\Omega) \quad \text{and} \quad V(\Omega) \times H(\Omega, \mathrm{Div}) \times \widetilde{L}^2(\Omega),$$

which encompass errors in the velocity, stress, and pressure and are natural for such a problem. The corresponding functionals (we denote them $\mathrm{M}_i(v, \tau, q)$, $i = 1, 2, 3$) are derived in Section 2 for the velocity field and in Section 3 for the pressure and stress fields. The method is based upon transformations of integral identities and a modification of the technique earlier used for viscous flow problems in [9, 14, 20, 21]. As in the case of the Stokes problem, the functionals contain multipliers defined by the constants in the Friederichs–Poincáre and LBB inequalities. Such a functional can be viewed as new variational statement defined on variables presenting approximations of the velocity, pressure and stress. It possess three principal properties. First, it is nonnegative and vanish if and only if (v, τ, q) coincides with the exact solution (u, σ, p). Also, it is continuous (with respect to all variables) in the natural (energy) metric. Finally, in Section 4, we show that the functional produces two–sided bounds of the error (measured in the combined velocity–stress-pressure norm) if it is multiplied by constants, which depend only on the problem data (see inequalities (4.3)–(4.5)) and analyze the behavior of these bounds with respect to data. The main conclusion, which follows from these results is that *quantitative analysis of* (1.1)–(1.4) *is equivalent to minimization of* $\mathrm{M}(v, \tau, q)$.

§2. Estimates of deviations from u

2.1. Estimates for $v \in \overset{\circ}{J}{}^1_2(\Omega) + u_0$. In this section, we assume that a vector–valued function v compared with u is a solenoidal function. Our goal is to deduce computable estimates of the error $e := u - v$, measured in terms of the energy norm.

First, we note that the integral identity (1.10) implies the basic relation

$$\mathbf{m}(e) = \sup_{w \in \overset{\circ}{J}{}^1_2(\Omega)} \frac{\mathcal{R}_v(w)}{|\!|\!| w |\!|\!|}, \tag{2.1}$$

where $|\!|\!| w |\!|\!|^2 := \int\limits_\Omega (\nu |\nabla w|^2 + \alpha^2 w^2)\, dx$,

$$\mathcal{R}_v(w) := \int\limits_\Omega \big(f \cdot w - \nu \nabla v : \nabla w - \alpha^2 v \cdot w + (\mathrm{a} \otimes v) : \nabla w\big)\, dx$$

is the *residual functional* generated by the approximation v and

$$\mathbf{m}(e) := \sup_{w \in \overset{\circ}{J}{}^1_2(\Omega)} \frac{\int\limits_\Omega \big(\nu \nabla e : \nabla w + \alpha^2 e \cdot w - (\mathrm{a} \otimes e) : \nabla w\big)\, dx}{|\!|\!| w |\!|\!|}$$

is the *error measure.*

It is worth outlining that the relation (2.1) has the principal value: it connects a suitable measure of the deviation from exact solution with the norm of the residual functional. Similar relations hold for many other boundary value problems including nonlinear nonlinear problems (see an overview in [22] and Remark 2.1).

In order to rearrange the right–hand side of (2.1), we introduce a tensor–valued function $\tau \in L_2(\Omega, \mathbb{M}^{d\times d})$ and the linear continuous functional

$$\mathcal{L}_{\tau,v}(w) := \int\limits_\Omega (-\tau : \nabla w - \alpha^2 v \cdot w + (\mathrm{a} \otimes v) : \nabla w + f \cdot w) dx.$$

with the corresponding norm

$$|\mathcal{L}_{\tau,v}| := \sup_{w \in \overset{\circ}{J}{}^1_2(\Omega)} \frac{\int\limits_\Omega (-\tau : \nabla w - \alpha^2 v \cdot w + (\mathrm{a} \otimes v) : \nabla w + f \cdot w)\, dx}{\|w\|}.$$

Since w is a solenoidal field, we have

$$\begin{aligned} \sup_{w \in \overset{\circ}{J}{}^1_2(\Omega)} \frac{\mathcal{R}_v(w)}{\|w\|} &= \sup_{w \in \overset{\circ}{J}{}^1_2(\Omega)} \frac{\mathcal{L}_{v,\tau}(w) + \int\limits_\Omega (\tau - \nu\nabla v + q\mathbb{1}) : \nabla w\, dx}{\|w\|} \\ &\leqslant \sup_{w \in \overset{\circ}{J}{}^1_2(\Omega)} \frac{\mathcal{L}_{\tau,v}(w) + \|\nu^{-1/2}(\tau - \nu\nabla v + q\mathbb{1})\|\, \|\nu^{1/2}\nabla w\|}{\|w\|} \\ &\leqslant |\mathcal{L}_{\tau,v}| + \|\nu^{-1/2}(\tau - \nu\nabla v + q\mathbb{1})\|, \end{aligned} \tag{2.2}$$

where $q \in \widetilde{L}^2(\Omega)$. Here and later on all L^2 norms of scalar, vector, and tensor–valued functions defined in Ω are denoted by $\|\ \|$.

It is easy to show that the error measure $\mathbf{m}(e)$ is equivalent to a norm. Note that

$$\begin{aligned} \int\limits_\Omega (\mathrm{a} \otimes w) : \nabla w\, dx &= -\int\limits_\Omega \mathrm{Div}(\mathrm{a} \otimes w) \cdot w\, dx \\ &= -\int\limits_\Omega (\mathrm{a} \cdot \nabla w) \cdot w\, dx = -\frac{1}{2} \int\limits_\Omega \mathrm{a} \cdot \nabla(|w|^2)\, dx = 0. \end{aligned} \tag{2.3}$$

Therefore,

$$\mathbf{m}(e) := \sup_{w \in \overset{\circ}{J}{}_2^1(\Omega)} \frac{\int\limits_\Omega \big(\nu\nabla e : \nabla w + \alpha^2 e\cdot w - (\mathrm{a}\otimes e) : \nabla w\big)\, dx}{\|\!| w \|\!|} \geqslant \frac{\int\limits_\Omega \big(\nu\nabla e : \nabla e + \alpha^2 e\cdot e\big)\, dx}{\|\!| e \|\!|} = \|\!| e \|\!|.$$

Thus, (2.2) implies the basic estimate

$$\|\!| e \|\!| \leqslant |\!| \mathcal{L}_{\tau,v} |\!| + \|\nu^{-1/2}(\tau - \nu\nabla v + q\mathbb{1})\| =: \mathcal{M}(v,\tau,q), \tag{2.4}$$

which shows the principal structure of the error majorant related to the generalized Oseen problem. The functions v, τ, and q entering the majorant can be viewed as approximations of the exact velocity field u, stress field σ and pressure p. If $\mathcal{M}(v,\tau,q) = 0$, then

$$\int\limits_\Omega ((-\nu\nabla v + q\mathbb{1}) : \nabla w - \alpha^2 v\cdot w + (\mathrm{a}\otimes v) : \nabla w + f\cdot w)dx = 0 \quad \forall w \in \overset{\circ}{J}{}_2^1(\Omega).$$

This relation means (cf. (1.11)) that $v = u$, $q = p$, and $\tau = \sigma$. In other words, the majorant vanishes if and only if its arguments are the exact velocity, pressure, and stress fields.

Another important property of $\mathcal{M}(v,\tau,q)$ is that for any $v \in V_0 + u_0$, the estimate (2.4) is sharp (i.e., it has no irremovable gap between the left and right–hand sides). Indeed, for $\tau = \nu\nabla v - p\mathbb{1}$ and $q = p$, we have

$$\begin{aligned}
&\mathcal{M}(v,\tau,p)\\
&= \sup_{w\in W_0} \frac{\int_\Omega (-\nu\nabla v : \nabla w + (\mathrm{a}\otimes v) : \nabla w - \alpha^2 v\cdot w + p\mathbb{1} : \nabla w + f\cdot w)\, dx}{\|\!| w \|\!|}\\
&= \sup_{w\in W_0} \frac{\int_\Omega (\nu\nabla(u-v) : \nabla w + (\mathrm{a}\otimes(v-u)) : \nabla w + \alpha^2(u-v)\cdot w)\, dx}{\|\!| w \|\!|}.
\end{aligned}$$

We set $w = u - v$, use (2.3), and find that

$$\mathcal{M}(v,\tau,p) \leqslant \frac{\int_\Omega \big(\nu|\nabla(u-v)|^2 + \alpha^2|u-v|^2\big)\, dx}{\|\!| u-v \|\!|} = \|\!| u-v \|\!|.$$

In view of (2.4), this means that $\mathcal{M}(v,\tau,p)$ is equal to the error norm.

However, the majorant $\mathcal{M}(v,\tau,p)$ contains an incomputable term $|\!|\mathcal{L}_{\tau,v}|\!|$ (it is defined as supremum over an infinite amount of test functions). Therefore, our first goal is to deduce computable bounds of $|\!|\mathcal{L}_{\tau,v}|\!|$, which lead to practically valuable estimates. For this purpose, we impose additional regularity requirements on τ, namely, we assume that

$$\tau \in H(\Omega, \mathrm{Div}) := \left\{\tau \in L^2(\Omega, \mathbb{M}^{d\times d}) \mid \mathrm{Div}\tau := \nabla\cdot\tau \in L^2(\Omega, \mathbb{R}^d)\right\}.$$

Then, the majorant can be presented as the sum of two integrals.

Theorem 2.1. *For any* $v \in \overset{\circ}{J}{}^1_2(\Omega) + u_0$, $q \in \widetilde{L}^2(\Omega)$, *and* $\tau \in H(\Omega, \mathrm{Div})$, *the following estimate holds:*

$$|\!|\!|u-v|\!|\!| \leqslant C_{F\Omega}\left\|\mu^{1/2}\,\mathcal{R}(v,\tau)\right\| + \left\|\nu^{-1/2}(\tau - \nu\nabla v + q\mathbb{1})\right\| =: \mathrm{M}(v,\tau,q), \tag{2.5}$$

where

$$\mu(x) = \frac{1}{\nu(x) + C^2_{F\Omega}\alpha^2(x)}, \tag{2.6}$$

$$\mathcal{R}(v,\tau) = \mathrm{Div}\tau - \mathrm{a}\cdot\nabla v - \alpha^2 v + f,$$

and $C_{F\Omega}$ *is a constant in the inequality*

$$\|w\| \leqslant C_{F\Omega}\|\nabla w\| \qquad \forall w \in V_0(\Omega). \tag{2.7}$$

$\mathrm{M}(v,\tau,q) = 0$ *if and only if* $v = u$, $p = q$, *and* $\tau = \sigma$.

Proof. Let

$$\Omega_+ := \{x \in \Omega \mid \alpha(x) > 0\}, \quad \Omega_0 := \{x \in \Omega \mid \alpha(x) = 0\},$$

and the function $\phi(x)$ be such that $0 \leqslant \phi(x) \leqslant 1$ and $\phi(x) = 0$ in Ω_0. Integrating by parts, we represent $\mathcal{L}_{\tau,v}(w)$ in the form

$$\mathcal{L}_{\tau,v}(w) = \int_\Omega \mathcal{R}(v,\tau)\cdot w\,dx$$
$$= \int_{\Omega_+} \phi\,\mathcal{R}(v,\tau)\cdot w dx + \int_\Omega (1-\phi)\,\mathcal{R}(v,\tau)\cdot w dx.$$

We use such a decomposition in order to deduce the sharpest upper bound of this term, which is maximally efficient for all values of α and ν. By (2.9) we obtain

$$|\mathcal{L}_{\tau,v}(w)| \leqslant \left\| \frac{\phi}{\alpha}\,\mathcal{R}(v,\tau) \right\|_{\Omega_+} \|\alpha w\| + C_{F\Omega} \left\| \frac{1-\phi}{\nu^{1/2}}\,\mathcal{R}(v,\tau) \right\| \|\nu^{1/2}\nabla w\|$$

$$\leqslant \left(\left\| \frac{\phi}{\alpha}\,\mathcal{R}(v,\tau) \right\|^2_{\Omega_+} + C^2_{F\Omega} \left\| \frac{1-\phi}{\nu^{1/2}}\,\mathcal{R}(v,\tau) \right\|^2 \right)^{1/2} |\!|\!| w |\!|\!|. \quad (2.8)$$

Thus,

$$|\!|\!| \mathcal{L}_{\tau,v} |\!|\!|^2 \leqslant \left\| \frac{\phi}{\alpha}\,\mathcal{R}(v,\tau) \right\|^2_{\Omega_+} + C^2_{F\Omega} \left\| \frac{1-\phi}{\nu^{1/2}}\,\mathcal{R}(v,\tau) \right\|^2$$

$$= \int\limits_{\Omega_+} \left(\frac{\phi^2}{\alpha^2} + C^2_{F\Omega} \frac{(1-\phi)^2}{\nu} \right) |\,\mathcal{R}(v,\tau)|^2\,dx + C^2_{F\Omega} \int\limits_{\Omega_0} \frac{1}{\nu} |\,\mathcal{R}(v,\tau)|^2\,dx. \quad (2.9)$$

We select $\phi(x)$ that minimizes the integral over Ω_+ , i.e.,

$$\phi = \frac{C^2_{F\Omega}\alpha^2}{\nu + C^2_{F\Omega}\alpha^2} < 1.$$

Then,

$$|\!|\!| \mathcal{L}_{\tau,v} |\!|\!|^2 \leqslant C^2_{F\Omega} \left(\int\limits_{\Omega} \frac{1}{\nu + C^2_{F\Omega}\alpha^2} |\,\mathcal{R}(v,\tau)|^2\,dx + \int\limits_{\Omega_0} \frac{1}{\nu} |\,\mathcal{R}(v,\tau)|^2\,dx \right) \quad (2.10)$$

and (2.5) follows from (2.4).

Assume that $\mathrm{M}(v,\tau,q) = 0$. Then

$$\mathrm{Div}\tau - \mathrm{a}\cdot\nabla v - \alpha^2 v + f = 0$$

and

$$\tau = \nu\nabla v - q\mathbb{1}$$

almost everywhere in Ω. Since v is a solenoidal field satisfying the boundary conditions, we conclude that v, q, and τ coincide with the respective components of the exact solution (which is unique). □

Below we extend the estimate (2.5) to the set of non–solenoidal functions, which satisfy the boundary conditions, but before passing to this point it is necessary to add more comments on the relation (2.1), which admits wide generalizations.

Remark 2.1. Let $\mathcal{A}: V \to V'$ be a bounded operator defined on a Banach space V (or a certain subspace of it). Assume that there exists a unique generalized solution $u \in V$ of the problem

$$\langle \mathcal{A}u, w\rangle = \ell(w) \qquad \forall w \in V,$$

where $\ell \in V'$ is a bounded liner functional. Let $v \in V$ be a function compared with u. Then, $\mathcal{R}_v(w) := \ell(w) - \langle \mathcal{A}v, w\rangle$ is the residual functional generated by the function v. Since

$$\sup_{w\in V} \frac{\langle \mathcal{A}u - \mathcal{A}v, w\rangle}{\|w\|_V} = \sup_{w\in V} \frac{\ell(w) - \langle \mathcal{A}v, w\rangle}{\|w\|_V} =: |\!|\mathcal{R}_v|\!|,$$

we see that the measure

$$\mathbf{m}(u-v) := \sup_{w\in V} \frac{\langle \mathcal{A}u - \mathcal{A}v, w\rangle}{\|w\|_V}$$

is dictated by the norm of the residual functional. In other words, this is the right measure, which evaluates a posteriori information encompassed in $\mathcal{R}_v$ for $v \in V$. Therefore, it is natural to consider $\mathbf{m}(u-v)$ as a proper quantity estimating the distance between u and v. For linear elliptic and parabolic problems this measure is equivalent to the energy norm. However, for nonlinear problems the situation is more complicated. For example, for the Navier–Stokes problem the corresponding measure can be defined as follows:

$$\mathbf{m}(u-v) := \sup_{w\in V} \frac{\int\limits_\Omega (\nu\nabla(u-v) : \nabla w - (u\otimes u - v\otimes v) : \nabla w) dx}{\|\nabla w\|}.$$

This quantity is equivalent to the energy norm only if ν is suffiviently large. Two-sided computable bounds of $|\!|\mathcal{R}_v|\!|$ can be derived by known methods (see a systematic exposition in [22]). In view of the above discussed equivalence, they provide two–sided bounds of the measure $\mathbf{m}(u-v)$

2.2. Estimates of the distance to the set $\overset{\circ}{J}{}^1_2(\Omega) + u_0$. First, we recall principal results in the theory of functions, which were used by O. A. Ladyzhenskaya and V. A. Solonnikov for proving existence of a solution to the Stokes problem in domains with nonsmooth boundaries (see [12,13]). The first result concerns the possibility to extend a solenoidal field inside a domain in such a way that the extended function norm is subject to the trace norm on the boundary.

Lemma 2.1. *Let Ω be a bounded domain with Lipschitz continuous boundary. Then, a positive constant c_Ω exists, which depends only on the domain Ω such that for any vector-valued function b having $H^{1/2}(\partial\Omega)$ traces on the boundary and satisfying the condition $\int_{\partial\Omega} b \cdot n\,dx = 0$ one can find a function $\bar{u} \in V_0$ such that* $\mathrm{div}\bar{u} = 0$ *and* $\|\nabla\bar{u}\| \leqslant c_\Omega \|b\|_{1/2,\partial\Omega}$.

This Lemma implies the following proposition.

Lemma 2.2. *Let Ω be a bounded domain with Lipschitz continuous boundary. Then, a positive constant κ_Ω exists (which depends only on Ω) such that for any $f \in \widetilde{L}^2(\Omega)$ one can find a function $w_f \in V_0$ such that*

$$\mathrm{div} w_f = f \quad \text{and} \quad \|\nabla w_f\| \leqslant \kappa_\Omega \|f\|. \tag{2.11}$$

We refer to [2, 13] for the proof of this lemma, which has important corollaries. It implies the key relation in the mathematical theory of incompressible fluids, which is often called the Inf–Sup (or *Ladyzhenskaya–Babuška–Brezzi* (LBB)) condition. The latter condition reads as follows: there exists a positive constant c_Ω such that

$$\inf_{\substack{q \in \widetilde{L}^2(\Omega)\\ q \neq 0}} \sup_{\substack{w \in V_0\\ w \neq 0}} \frac{\int_\Omega q\,\mathrm{div} w\,dx}{\|q\|\,\|\nabla w\|} \geqslant c_\Omega. \tag{2.12}$$

By Lemma 2.1, it is easy to show that (2.12) holds with $c_\Omega = (\kappa_\Omega)^{-1}$. The LBB condition (2.12) and its discrete analogs are used for proving stability and convergence of numerical methods in various problems related to the theory of viscous incompressible fluids. In [1] and [6], this condition was introduced, proved, and used in order to justify the convergence of the so–called *mixed* methods, in which a boundary–value problem is reduced to a saddle–point problem for a certain Lagrangian. Also, (2.12) can be justified by the Nečas inequality (for domains with Lipschitz boundaries a simple proof is presented in [4]). Estimates of the constant c_Ω (or κ_Ω) for various domains are discussed in, e.g., [7, 15, 17].

Also, Lemma 2.2 implies an estimate of the distance between a function $\widehat{v} \in V_0 + u_0$ and the set $\overset{\circ}{J}{}^1_2(\Omega) + u_0$. Assume that Ω is divided into a collection of nonoverlapping Lipschitz subdomains Ω_i, $i = 1, 2, \ldots N$, and

$$\widehat{v} \in V^N_{\mathrm{div}}(\Omega) := \left\{ \widehat{w} \in V_0 + u_0 \mid \{\mathrm{div}\widehat{w}\}_{\Omega_i} = 0,\ i = 1, 2, \ldots, N \right\}. \tag{2.13}$$

Using Lemma 2.2, we can prove the following statement

Lemma 2.3. *For any function $\widehat{v} \in V^N_{\rm div}(\Omega)$ there exists $v \in \overset{\circ}{J}{}^1_2(\Omega) + u_0$ such that*

$$\|\nabla(\widehat{v} - v)\|^2 \leqslant \sum_{i=1}^{N} \kappa^2_{\Omega_i} \|{\rm div}\widehat{v}\|^2_{\Omega_i}. \tag{2.14}$$

Proof. Since $\int\limits_{\partial\Omega_i} \widehat{v} \cdot n\, ds = 0$, we know (see, e.g., [12]) that there exists a vector field $u^{(i)} \in H^1(\Omega_i, \mathbb{R}^d)$ solving the Stokes problem

$$\begin{aligned} -\Delta u^{(i)} + \nabla p^{(i)} &= 0 \qquad \text{in} \quad \Omega_i,\\ {\rm div} u^{(i)} &= 0 \qquad \text{in} \quad \Omega_i,\\ u^{(i)} &= \widehat{v} \qquad \text{on} \quad \partial\Omega_i. \end{aligned}$$

In all Ω_i, we define $w^{(i)} := \widehat{v} - u^{(i)}$. Note that $w_i \in H^1(\Omega_i, \mathbb{R}^d)$ and $w^{(i)} = 0$ on $\partial\Omega_i$ (in the sense of traces).

Now, we apply Lemma 2.2 and for any $w^{(i)}$ find $w^{(i)}_g \in H^1(\Omega_i, \mathbb{R}^d)$ such that $w^{(i)}_g = 0$ on $\partial\Omega_i$ and

$${\rm div} w^{(i)}_g = g := {\rm div} w^{(i)} = {\rm div}\widehat{v}, \tag{2.15}$$

$$\|\nabla w^{(i)}_g\|_{\Omega_i} \leqslant \kappa_{\Omega_i} \|g\|_{\Omega_i}. \tag{2.16}$$

Let w_g be the vector valued function that coincides with $w^{(i)}_g$ in each Ω_i. It is continuous and belongs to $V_0(\Omega)$. From (2.16), it follows that

$$\|\nabla w_g\|^2_\Omega \leqslant \sum_{i=1}^{N} \kappa^2_{\Omega_i} \|g\|^2_{\Omega_i}. \tag{2.17}$$

On the other hand,

$$\|\nabla w_g\|_\Omega = \|\nabla(w_g - \widehat{v} + \widehat{v})\|_\Omega,$$

where ${\rm div}(w_g - \widehat{v}) = 0$. We set $v = \widehat{v} - w_g$ and arrive at (2.14). □

Corollary 2.1. *Since $\widehat{v} = u_0$ on $\partial\Omega$ and $\int\limits_{\partial\Omega} u_0 \cdot n\, ds = 0$, we set $N = 1$ and conclude that for any function $\widehat{v} \in V_0 + u_0$ there exists $v \in \overset{\circ}{J}{}^1_2(\Omega) + u_0$ such that*

$$\|\nabla(\widehat{v} - v)\| \leqslant \kappa_\Omega \|{\rm div}\widehat{v}\|. \tag{2.18}$$

Remark 2.2. Finding κ_Ω (or a close upper bound of this quantity) for an arbitrary domain Ω is a very difficult problem. However, for some special (simple) domains, estimates of κ_Ω are known (see, e.g., [7, 15, 17]).

Lemma 2.3 suggests a way to overcome this difficulty (at least for a certain class of domains). Indeed, if Ω is decomposed into "simple" subdomains Ω_i and $\widehat{v} \in V^N_{\rm div}(\Omega)$, then an upper bound of the distance between $\widehat{v}$ and the set of solenoidal fields $\overset{\circ}{J}{}^1_2(\Omega) + u_0$ can be explicitly computed by the estimate (2.14).

2.3. Estimates of $\|\widehat{v} - u\|$ for $\widehat{v} \in V_0 + u_0$. Assume that $\widehat{v} \in V_0 + u_0$ ("hat" over v is added to outline that $\operatorname{div}\widehat{v} \neq 0$). Then

$$\|u - \widehat{v}\| \leqslant \|u - v\| + \|\widehat{v} - v\| \leqslant \mathrm{M}(v, \tau, q) + \|\widehat{v} - v\|, \qquad (2.19)$$

where v is an arbitrary function from $\overset{\circ}{J}{}^1_2(\Omega) + u_0$.

Henceforth, $\overline{\alpha}$, $\overline{\nu}$, and $\overline{\mu}$ (cf. 2.8) denote maximal values in Ω of $\alpha(x)$, $\nu(x)$, and $\mu(x)$, respectively ($\underline{\alpha}$, $\underline{\nu}$, and $\underline{\mu}$ denote minimal values), and ζ_i denote various constants, which depend only on the problem data.

Since

$$\begin{aligned}\|\nu^{-1/2}(\tau - \nu\nabla v + q\mathbb{1})\| &\leqslant \|\nu^{-1/2}(\tau - \nu\nabla\widehat{v} + q\mathbb{1})\| + \|\nu^{1/2}\nabla(\widehat{v} - v)\| \\ &\leqslant \|\nu^{-1/2}(\tau - \nu\nabla\widehat{v} + q\mathbb{1})\| + \overline{\nu}^{1/2}\|\nabla(\widehat{v} - v)\|\end{aligned}$$

and

$$\begin{aligned}\|\mu^{1/2}\,\mathcal{R}(v, \tau)\| &\leqslant \|\mu^{1/2}\,\mathcal{R}(\widehat{v}, \tau)\| + \|\mu^{1/2}(\mathrm{a}\cdot\nabla(\widehat{v} - v) + \alpha^2(\widehat{v} - v))\| \\ &\leqslant \|\mu^{1/2}\,\mathcal{R}(\widehat{v}, \tau)\| + \zeta_1\|\nabla(\widehat{v} - v)\|,\end{aligned}$$

where

$$\zeta_1 = \overline{\mu}^{1/2}\big(\|\mathrm{a}\|_\infty + \overline{\alpha}^2\, C_{F\Omega}\big),$$

we find that

$$\begin{aligned}\mathrm{M}(v, \tau, q) &\leqslant \mathrm{M}(\widehat{v}, \tau, q) \\ &\quad + \Big(\overline{\nu}^{1/2} + \zeta_1 C_{F\Omega}\Big)\,\|\nabla(\widehat{v} - v)\| \leqslant \mathrm{M}(\widehat{v}, \tau, q) + \zeta_2\rho(\widehat{v}),\end{aligned} \qquad (2.20)$$

where

$$\zeta_2 = \overline{\nu}^{1/2} + \zeta_1 C_{F\Omega}, \quad \text{and} \quad \rho(\widehat{v}) = \inf_{v \in \overset{\circ}{J}{}^1_2(\Omega) + u_0} \|\nabla(\widehat{v} - v)\|.$$

Analogously,

$$\|\widehat{v} - v\|^2 \leqslant \overline{\nu}\|\nabla(\widehat{v} - v)\|^2 + \overline{\alpha}^2\|\widehat{v} - v\|^2 \leqslant \zeta_3^2\rho^2(\widehat{v}), \qquad (2.21)$$

where

$$\zeta_3^2 = \overline{\nu} + C^2_{F\Omega}\overline{\alpha}^2.$$

Now, the relations (2.19)–(2.21) yield the estimate

$$|\!|\!|u - \widehat{v}|\!|\!| \leqslant \mathrm{M}(\widehat{v}, \tau, q) + (\zeta_2 + \zeta_3)\rho(\widehat{v}). \tag{2.22}$$

If the term $\rho(\widehat{v})$ is estimated by (2.18), then (2.22) has the form

$$|\!|\!|u - \widehat{v}|\!|\!| \leqslant \mathrm{M}(\widehat{v}, \tau, q) + (\zeta_2 + \zeta_3)\kappa_\Omega \|\mathrm{div}\widehat{v}\|. \tag{2.23}$$

If $\widehat{v} \in V^N_{\mathrm{div}}(\Omega)$, then we use Lemma 2.3 and arrive at the estimate

$$|\!|\!|u - \widehat{v}|\!|\!| \leqslant \mathrm{M}(\widehat{v}, \tau, q) + (\zeta_2 + \zeta_3)\left(\sum_{i=1}^{N} \kappa^2_{\Omega_i}\|\mathrm{div}\widehat{v}\|^2_{\Omega_i}\right)^{1/2}. \tag{2.24}$$

§3. Estimates of deviations from p and σ

3.1. Estimates for the pressure. Let $q \in \widetilde{L}^2(\Omega)$ be an approximation of the pressure field p. Estimates of $\|p - q\|$ can be also derived with the help of Lemma 2.2.

Theorem 3.1. *The following estimate holds*

$$\frac{1}{\kappa_\Omega}\|p - q\| \leqslant C_1 \mathrm{M}(\widehat{v}, \tau, q) + C_2\|\mathrm{div}\widehat{v}\|, \tag{3.1}$$

where

$$C_1 = 2\zeta_3 + \zeta_4, \quad C_2 = (\zeta_3 + \zeta_4)(\zeta_2 + \zeta_3),$$
$$\zeta_4 = \min\left\{\frac{\|\mathrm{a}\|_\infty}{\underline{\alpha}^2}, \frac{C_{F\Omega}\|\mathrm{a}\|_\infty}{\underline{\nu}}\right\},$$

and $\widehat{v}$ and τ are arbitrary functions in $V_0(\Omega) + u_0$ and $H(\Omega, \mathrm{Div})$, respectively.

Proof. Since $(p - q) \in \widetilde{L}^2(\Omega)$, by Lemma 1 we know that there exists a vector valued function $\widetilde{w} \in V_0(\Omega)$ such that

$$\mathrm{div}\widetilde{w} = p - q, \quad \text{and} \quad \|\nabla\widetilde{w}\| \leqslant \kappa_\Omega\|p - q\|.$$

We use the first relation and represent the norm of $p - q$ as follows:

$$\|p - q\|^2 = \int_\Omega \mathrm{div}\widetilde{w}\,(p - q)\,dx.$$

Next, we use (1.11) and obtain

$$\|p-q\|^2 = \int\limits_{\Omega} \left(\nu\nabla u : \nabla\widetilde{w} + \alpha^2 u\cdot\widetilde{w} - (\mathrm{a}\otimes u) : \nabla\widetilde{w} - f\cdot\widetilde{w} - q\mathrm{div}\widetilde{w}\right)dx = I_1 + I_2, \tag{3.2}$$

where

$$I_1 = \int\limits_{\Omega} (\nu\nabla(u-\widehat{v}) : \nabla\widetilde{w} + \alpha^2(u-\widehat{v})\cdot\widetilde{w} - (\mathrm{a}\otimes(u-\widehat{v})) : \nabla\widetilde{w})\,dx$$

and

$$I_2 = \int\limits_{\Omega} \left(\nu\nabla\widehat{v} : \nabla\widetilde{w} + \alpha^2\widehat{v}\cdot\widetilde{w} - (\mathrm{a}\otimes\widehat{v}) : \nabla\widetilde{w} - f\cdot\widetilde{w} - q\mathrm{div}\widetilde{w}\right)dx.$$

It is not difficult to see that

$$\int\limits_{\Omega} \Big(\nu\nabla(u-\widehat{v}) : \nabla\widetilde{w} + \alpha^2(u-\widehat{v})\cdot\widetilde{w}\Big)\,dx \leqslant \|\!|u-\widehat{v}|\!\|\,\|\!|\widetilde{w}|\!\| \leqslant \kappa_\Omega\zeta_3\|\!|u-\widehat{v}|\!\|\,\|p-q\|$$

and

$$\int\limits_{\Omega} (\mathrm{a}\otimes(u-\widehat{v})) : \nabla\widetilde{w}\,dx \leqslant \|\mathrm{a}\|_\infty\kappa_\Omega\,\|u-\widehat{v}\|\,\|p-q\| \leqslant \kappa_\Omega\zeta_4\|\!|u-\widehat{v}|\!\|\,\|p-q\|.$$

Thus,

$$I_1 \leqslant \kappa_\Omega(\zeta_3+\zeta_4)\|\!|u-\widehat{v}|\!\|\,\|p-q\|. \tag{3.3}$$

In order to estimate I_2, we introduce $\tau \in H(\Omega, \mathrm{Div})$ and use the same decomposition as in Theorem 2.1

$$\int\limits_{\Omega} \left(\nu\nabla\widehat{v} : \nabla\widetilde{w} + \alpha^2\widehat{v}\cdot\widetilde{w} + (\mathrm{a}\cdot\nabla\widehat{v})\cdot\widetilde{w} - f\cdot\widetilde{w} - q\mathrm{div}\widetilde{w}\right)dx$$
$$= \int\limits_{\Omega} (\nu\nabla\widehat{v} - \tau - q\mathbb{1}) : \nabla\widetilde{w}\,dx - \int\limits_{\Omega} \left(\mathrm{Div}\tau + \mathrm{a}\cdot\nabla\widehat{v} + \alpha^2\widehat{v} + f\right)\cdot\widetilde{w}dx$$
$$\leqslant \|\nu\nabla\widehat{v} - \tau - q\mathbb{1}\|\|\nabla\widehat{w}\| + \mathcal{L}_{\tau,\widehat{v}}(\widetilde{w}). \tag{3.4}$$

Since

$$|\mathcal{L}_{\tau,\widehat{v}}(\widetilde{w})| \leqslant |\mathcal{L}_{\tau,\widehat{v}}| \, |\!|\!|\widetilde{w}|\!|\!|,$$

we use (2.10) and find that

$$I_2 \leqslant \|\nu\nabla\widehat{v} - \tau - q\mathbb{1}\| \|\nabla\widehat{w}\| + \left(\int\limits_{\Omega} \frac{C^2_{F\Omega}}{\nu + C^2_{F\Omega}\alpha^2} |\mathcal{R}(\widehat{v},\tau)|^2 \, dx\right)^{1/2} |\!|\!|\widetilde{w}|\!|\!| \leqslant \mathrm{M}(\widehat{v},\tau,q)\zeta_3\kappa_\Omega\|p-q\|. \quad (3.5)$$

In view of (3.2), (3.4), and (3.5), we obtain

$$\frac{1}{\kappa_\Omega}\|p-q\| \leqslant (\zeta_3+\zeta_4)|\!|\!|u-\widehat{v}|\!|\!| + \zeta_3\mathrm{M}(\widehat{v},\tau,q) \leqslant (2\zeta_3+\zeta_4)\mathrm{M}(\widehat{v},\tau,q) + (\zeta_3+\zeta_4)(\zeta_2+\zeta_3)\|\mathrm{div}\widehat{v}\|.$$

and arrive at the estimate (3.1). □

Theorem 3.1 shows that the majorant is controlled by the same quantity M and a term, which serves as a penalty for violations of the divergence–free condition. It is easy to see that the right–hand side of (3.1) vanishes if and only if,

$$\widehat{v} = u, \quad \tau = \sigma, \quad \text{and} \quad p = q.$$

However, in this case, the dependence of the penalty multipliers on the constant κ_Ω is stronger.

Remark 3.1. If $\alpha = 0$, $\mathrm{a} = 0$, and $\nu = const$, then we arrive at the classical Stokes problem. In this case, $\zeta_1 = 0$, $\zeta_2 = \zeta_3 = \nu^{1/2}$, $(\zeta_2+\zeta_3) = 2\nu^{1/2}$, and $\zeta_4 = 0$. Then, the estimate (2.5) has the form

$$\|\nu\nabla(u-v)\| \leqslant C_{F\Omega}\|\mathrm{Div}\tau + f\| + \|\tau - \nu\nabla v + q\mathbb{1}\|,$$

and (3.1) has the form

$$\frac{1}{\kappa_\Omega}\|p-q\| \leqslant 2\nu^{1/2}\mathrm{M}(\widehat{v},\tau) + 2\nu\|\mathrm{div}\widehat{v}\| = 2C_{F\Omega}\|\mathcal{R}(\widehat{v},\tau)\| + 2\|\tau - \nu\nabla v + q\mathbb{1}\| + 2\nu\|\mathrm{div}\widehat{v}\|$$

what coincides with the known estimate for the Stokes problem (see, e.g., [22], (6.2.13)).

3.2. Estimates for stresses. Estimates (2.5) and (3.1) imply majorants of deviations in terms of stresses. For this purpose we use the norm of $H(\Omega, \mathrm{Div})$.

$$|||\tau - \sigma|||_{\mathrm{Div}} := \|\tau - \sigma\| + C_{F\Omega}\|\mathrm{Div}(\tau - \sigma)\|,$$

which is equivalent to the standard norm of $H(\Omega, \mathrm{Div})$, but contains Let $\widehat{v} \in V_0$, $\tau \in H(\Omega, \mathrm{Div})$, and $q \in \widetilde{L}^2(\Omega)$ approximate u, σ, and p, respectively. We have

$$\begin{aligned}\|\mathrm{Div}(\tau - \sigma)\| &= \|\mathrm{Div}\tau - \mathrm{a}\cdot\nabla u - \alpha^2 u + f\| = \|\mathcal{R}(u,\tau)\| \\ &\leqslant \|\mathcal{R}(\widehat{v},\tau)\| + \|\mathrm{a}\|_\infty\|\nabla(\widehat{v} - u)\| + \|\alpha^2(\widehat{v} - u)\|\end{aligned} \tag{3.6}$$

Analogously,

$$\begin{aligned}\|\tau - \sigma\| = \|\tau - \nu\nabla u + p\mathbb{1}\| &\leqslant \|\tau - \nu\nabla\widehat{v} + q\mathbb{1}\| + \|\nu\nabla(u - \widehat{v})\| + \sqrt{d}\|p - q\| \\ &\leqslant \|\tau - \nu\nabla\widehat{v} + q\mathbb{1}\| + \overline{\nu}^{1/2}\|\nu^{1/2}\nabla(\widehat{v} - u)\| + \sqrt{d}\|p - q\|.\end{aligned} \tag{3.7}$$

In view of (3.6) and (3.7),

$$\begin{aligned}&|||\tau - \sigma|||_{\mathrm{Div}} \leqslant C_{F\Omega}\|\mathcal{R}(\widehat{v},\tau)\| + \|\tau - \nu\nabla\widehat{v} + q\mathbb{1}\| \\ &+ \left(\overline{\nu}^{1/2} + \frac{\|\mathrm{a}\|_\infty}{\underline{\nu}^{1/2}}\right)\|\nu^{1/2}\nabla(\widehat{v} - u)\| + C_{F\Omega}\overline{\alpha}\|\alpha(\widehat{v} - u)\| + \sqrt{d}\|p - q\| \\ &\qquad \leqslant C_{F\Omega}\|\mathcal{R}(\widehat{v},\tau)\| + \|\tau - \nu\nabla\widehat{v} + q\mathbb{1}\| \\ &\qquad\qquad + \zeta_5 |||\widehat{v} - u||| + \sqrt{d}\|p - q\|,\end{aligned}$$

where

$$\zeta_5 = \left(\left(\overline{\nu}^{1/2} + \frac{\|\mathrm{a}\|_\infty}{\underline{\nu}^{1/2}}\right)^2 + C_{F\Omega}^2\overline{\alpha}^2\right)^{1/2}.$$

Hence, we arrive at the estimate

$$\begin{aligned}|||\tau - \sigma|||_{\mathrm{Div}} \leqslant C_{F\Omega}\|\mathcal{R}(\widehat{v},\tau)\| &+ \|\tau - \nu\nabla\widehat{v} + q\mathbb{1}\| \\ &+ \zeta_6\mathrm{M}(\widehat{v},\tau,q) + \zeta_7\kappa_\Omega\|\mathrm{div}\widehat{v}\|,\end{aligned} \tag{3.8}$$

where

$$\zeta_6 = \zeta_5 + \kappa_\Omega\sqrt{d}C_1, \quad \zeta_7 = \zeta_5(\zeta_2 + \zeta_3) + \sqrt{d}C_2,$$

and $\widehat{v}$ and q are arbitrary functions in $V_0 + u_0$ and $\widetilde{L}^2(\Omega)$, respectively.

§4. Equivalence of the majorant and combined error norm

Now, we deduce two–sided estimates of the error in terms of the combined norm

$$\big|[u-\widehat{v}, p-q, \sigma-\tau]\big| := \|u-\widehat{v}\| + \|p-q\| + \|\sigma-\tau\|_{\mathrm{Div}},$$

which characterizes the accuracy of $(\widehat{v},\tau,q)$ in $W(\Omega)\times \widetilde{L}^2(\Omega)\times H(\Omega,\mathrm{Div})$. Note that

$$\underline{\mu} = \frac{1}{\overline{\nu} + C^2_{F\Omega}\overline{\alpha}^2} \leqslant \frac{1}{\overline{\nu}}$$

and, therefore,

$$\mathrm{M}(\widehat{v},\tau,q) \geqslant \underline{\mu}^{1/2} C_{F\Omega}\|\,\mathcal{R}(\widehat{v},\tau)\| + \frac{1}{\overline{\nu}^{1/2}}\|\tau - \nu\nabla\widehat{v} + q\mathbb{1}\|.$$

Thus,

$$\mathrm{M}(\widehat{v},\tau,q) \geqslant \underline{\mu}^{1/2}\big(\|\,\mathcal{R}(\widehat{v},\tau)\| + \|\tau-\nu\nabla\widehat{v}+q\mathbb{1}\|\big). \tag{4.1}$$

In view of (3.8), we conclude that

$$\|\tau-\sigma\|_{\mathrm{Div}} \leqslant \left(\zeta_6 + \underline{\mu}^{-1/2}\right)\mathrm{M}(\widehat{v},\tau,q) + \zeta_7\kappa_\Omega\|\mathrm{div}\widehat{v}\|. \tag{4.2}$$

We recall that

$$\|p-q\| \leqslant \kappa_\Omega\left(C_1\mathrm{M}(\widehat{v},\tau,q) + C_2\|\mathrm{div}\widehat{v}\|\right).$$

and obtain

$$\big|[u-\widehat{v}, p-q, \sigma-\tau]\big| \leqslant \mathbb{C}_1\mathrm{M}(\widehat{v},\tau,q) + \mathbb{C}_2\kappa_\Omega\|\mathrm{div}\widehat{v}\|, \tag{4.3}$$

where

$$\mathbb{C}_1 = 1 + C_1\kappa_\Omega + \zeta_6 + \underline{\mu}^{-1/2} \text{ and } \mathbb{C}_2 = (\zeta_2+\zeta_3) + C_2 + \zeta_7.$$

Now, our goal is to establish the estimate, which in a sense is opposite to (4.4). First, we note that

$$\mathrm{M}(\widehat{v},\tau,q) \leqslant \overline{\mu}^{1/2} C_{F\Omega}\|\,\mathcal{R}(\widehat{v},\tau)\| + \|\nu^{-1/2}(\tau-\nu\nabla\widehat{v}+q\mathbb{1})\|.$$

Since

$$\begin{aligned}\mathcal{R}(\widehat{v},\tau) &= \|\mathrm{Div}(\tau-\sigma) - \mathrm{a}\cdot\nabla(\widehat{v}-u) - \alpha^2(\widehat{v}-u)\| \\ &\leqslant \|\mathrm{Div}(\tau-\sigma)\| + \frac{\|a\|_\infty}{\underline{\nu}^{1/2}}\|\nu^{1/2}\nabla(\widehat{v}-u)\| + \overline{\alpha}\|\alpha(\widehat{v}-u)\|,\end{aligned}$$

and

$$\|\nu^{-1/2}(\tau - \nu\nabla\widehat{v} + q\mathbb{1})\| = \|\nu^{-1/2}((\tau-\sigma) - \nu\nabla(\widehat{v}-u) + (q-p)\mathbb{1})\| \leqslant \frac{1}{\underline{\nu}^{1/2}}\|\tau-\sigma\| + \|\nu^{1/2}\nabla(\widehat{v}-u)\| + \sqrt{d}\|p-q\|$$

we find that

$$\begin{aligned}
&\mathrm{M}(\widehat{v},\tau,q)\\
&= C_{F\Omega}\left\|\mu^{1/2}(\mathrm{Div}\tau - \mathrm{a}\cdot\nabla v - \alpha^2 v + f)\right\| + \left\|\nu^{-1/2}(\tau - \nu\nabla v + q\mathbb{1})\right\|\\
&= C_{F\Omega}\left\|\mu^{1/2}(\mathrm{Div}(\tau-\sigma) - \mathrm{a}\cdot\nabla(v-u) - \alpha^2(v-u))\right\|\\
&\quad + \left\|\nu^{-1/2}((\tau-\sigma) - \nu\nabla(v-u) + (q-p)\mathbb{1})\right\|\\
&\leqslant \frac{1}{\underline{\nu}^{1/2}}\|\tau-\sigma\| + \overline{\mu}^{1/2}C_{F\Omega}\|\mathrm{Div}(\tau-\sigma)\|\\
&+\left(1+\overline{\mu}^{1/2}C_{F\Omega}\frac{\|a\|_\infty}{\underline{\nu}^{1/2}}\right)\|\nu^{1/2}\nabla(\widehat{v}-u)\| + \overline{\mu}^{1/2}C_{F\Omega}\overline{\alpha}\|\alpha(\widehat{v}-u)\| + \sqrt{d}\|p-q\|\\
&\leqslant \max\left\{\frac{1}{\underline{\nu}^{1/2}},\overline{\mu}^{1/2}\right\}\|\sigma-\tau\|_{\mathrm{Div}} + \sqrt{d}\|p-q\| + \zeta_8|\!|\!|\widehat{v}-u)|\!|\!|, \qquad (4.4)
\end{aligned}$$

where $\zeta_8 = \left(\left(1+\overline{\mu}^{1/2}C_{F\Omega}\frac{\|a\|_\infty}{\underline{\nu}^{1/2}}\right)^2 + \overline{\mu}C_{F\Omega}^2\overline{\alpha}^2\right)^{1/2}$.

Note that $\overline{\mu} = \frac{1}{\underline{\nu}+C_{F\Omega}^2\underline{\alpha}^2} \leqslant \frac{1}{\underline{\nu}}$. Therefore, (4.4) infers

$$\mathbb{C}_3\mathrm{M}(\widehat{v},\tau,q) \leqslant \left|[u-\widehat{v}, p-q, \sigma-\tau]\right|, \qquad (4.5)$$

where $\mathbb{C}_3 = \min\left\{\underline{\nu}^{1/2}, \zeta_8^{-1}, \frac{1}{\sqrt{d}}\right\}$.

Estimates (4.3) and (4.5) show that the functional $\mathrm{M}(\widehat{v},\tau,q)$ controls deviations from (u,σ,p). Assume that $\{\widehat{v}_k\}$, $\{\tau_h\}$, and $\{q_k\}$ are sequences of functions such that $\mathrm{M}(\widehat{v}_k,\tau_k,q_k)\to 0$ and $\mathrm{div}\widehat{v}_k \to 0$. Then (4.3) shows that such sequences tend to the corresponding exact solutions. This fact suggests the idea to minimize the right hand side of (4.3) over certain sets of finite dimensional approximations in order to construct such sequences.

If approximations of the velocity field are solenoidal ($v \in \overset{\circ}{J}{}^1_2(\Omega) + u_0$), then the following simplified estimate holds:

$$\mathbb{C}_3 \mathrm{M}(v,\tau,q) \leqslant \big|[u-v, p-q, \sigma-\tau]\big| \leqslant \mathbb{C}_1 \mathrm{M}(v,\tau,q). \tag{4.6}$$

In other words, the functional $\mathrm{M}(v,\tau,q)$ is *equivalent* to the deviation from the exact solution. It vanishes if and only if its arguments coincide with the exact velocity, stress, and pressure. Hence, we conclude that *minimization of* $\mathrm{M}(v,\tau,q)$ *over* $(\overset{\circ}{J}{}^1_2(\Omega)+u_0)\times \widetilde{L}^2(\Omega)\times H(\Omega,\mathrm{Div})$ *is an adequate method of solving the generalized Oseen problem.* Indeed, the corresponding minimizing sequence automatically converges to the exact solution and the values of M show the accuracy that has been achieved.

Finally, we comment on the values of constants in the inequalities establishing equivalence of the error and the majorant. Let ν be a constant lesser than $\frac{1}{d}$.

Consider the Stokes problem. In this case,

$$\mathbb{C}_1 = 1 + 2\overline{\nu}^{1/2}((1+\sqrt{d})\kappa_\Omega + 1), \quad \mathbb{C}_2 = 2\overline{\nu}^{1/2} + 2\nu(2+\sqrt{d}), \quad \mathbb{C}_3 = \nu^{1/2},$$

and for solenoidal approximations we have

$$\frac{\mathbb{C}_1}{\mathbb{C}_3} = \frac{1}{\nu^{1/2}} + 2((1+\sqrt{d})\kappa_\Omega + 1).$$

If $d = 3$, then $\frac{\mathbb{C}_1}{\mathbb{C}_3} \leqslant \frac{1}{\nu^{1/2}} + 6\kappa_\Omega + 2$, so that the ratio can be large if ν is very small. For nonsolenoidal approximations the ratio of the right hand side of (4.3) and left hand side of (4.5) is larger, but has the same order with respect to ν.

Also, it is interesting to analyze the behavior of the constants for the Oseen problem with respect to large α. In this case,

$$\frac{\mathbb{C}_1}{\mathbb{C}_3} = \frac{1}{\nu^{1/2}} + 2C_{F\Omega}\frac{\overline{\alpha}}{\nu^{1/2}}((1+\sqrt{d})\kappa_\Omega + 1).$$

If α is generated by the semidiscrete scheme discussed in the Introduction (so that $\alpha^2 = \frac{1}{\Delta t}$), then we see that the ratio of the constants deteriorates proportionally to $\sqrt{\frac{1}{\nu\Delta t}}$. It is not difficult to find that the ratio $\frac{\mathbb{C}_1}{\mathbb{C}_3}$ is always larger than $\nu^{-1/2}$. Certainly, if ν diminishes and tends to zero, then all aspects of quantitative analysis of the problem become more complicated, so that this effect is natural to await. However, this relatively slow rate of deterioration shows that the estimates can be used even for small values of viscosity.

References

1. I. Babuška, *The finite element method with Lagrangian multipliers.* — Numer. Math., **20** (1973), 179–192.
2. I. Babuška and A. K. Aziz, *Surway lectures on the mathematical foundations of the finite element method.* The mathematical formulations of the finite element method with applications to partial differential equations, Academic Press, New York, 1972.
3. L. Badea, M. Discacciati, A. Quarteroni, *Numerical analysis of the Navier-Stokes/Darcy coupling*, Numerische Mathematik, **115** (2010), 195–227.
4. J. Bramble, *A proof of the inf-sup condition for the Stokes equations on Lipschitz domains.* — Math. Models Methods Appl. Sci. **13**, No. 3 (2003), 361–371.
5. J. Bonvin, M. Picasso, R. Stenberg, *GLS and EVSS methods for a three–field Stokes problem arising from viscoelastic flows.* — Comput. Methods Appl. Mech. Engrg. **190** (2001), 3893–3914.
6. F. Brezzi, *On the existence, uniqueness and approximation of saddle-point problems arising from Lagrange multipliers.* — R.A.I.R.O., Annal. Numer. **R2** (1974), 129–151.
7. M. Dobrowolski, *On the LBB constant on stretched domains.* — Math. Nachr. **254-255** (2003), 64-67.
8. R. Finn, D. R. Smith, *On the linearized hydrodynamical equations in two dimensions.* — Arch. Rational Mech. Anal. **25** (1967), 1–25.
9. M. Fuchs, S. Repin, *Estimates for the deviation from the exact solutions of variational problems modeling certain classes of generalized Newtonian fluids.* — Math. Methods Applied Sciences (M2AS) **29** (2006), 2225–2244.
10. V. Girault, P. A. Raviart, *Finite element approximation of the Navier–Stokes equations*, Springer, Berlin, 1986.
11. J. G. Heywood, W. Nagata, W. Xie, *A numerically based existence theorem for the Navier–Stokes equations.* — J. Math. Fluid Mech. **1** (1999), 5–23.
12. O. A. Ladyzhenskaya, *Mathematical Problems in the Dynamics of a Viscous Incompressible Fluid*, Nauka, M., 1970.
13. O. A. Ladyzenskaja, V. A. Solonnikov, *Some problems of vector analysis, and generalized formulations of boundary value problems for the Navier–Stokes equation.* — Zap. Nauchn, Semin. LOMI **59** (1976), 81–116.
14. A. Mikhailov, S. Repin. *Estimates of deviations from exact solution of the Stokes problem in the velocity-vorticity-pressure formulation.* — Zap. Nauchn. Semin. POMI **397** (2011), 73–87.
15. M.A. Olshanskii, E.V. Chizhonkov, *On the best constant in the $\inf\sup$ condition for prolonged rectangular domains.* — Mat. Zametki **67**, No. 3 (2000), 387–396.
16. F. K. G. Oseen, *Neuere Methoden und Ergebnisse in der Hydrodynamik.* Akademische Velagsgesellschaft, Leipzig, 1927.
17. L. E. Payne, *A bound for the optimal constant in an inequality of Ladyzhenskaya and Solonnikov.* — IMA J. Appl. Math. **72** (2007), 563–569.
18. R. Rannacher, *Finite element methods for the incompressible Navier-Stokes equations.* — In: Fundamental directions in mathematical fluid mechanics (G. P. Galdi ed.) Birkhauser, Basel, 2000, 191–293.

19. S. Repin, *A posteriori error estimation for variational problems with uniformly convex functionals.* — Math. Comput. **69**, No. 230 (2000), 481–500.
20. S. Repin, *Estimates of deviations from exact solutions for some boundary-value problems with incompressibility condition.* — Algebra Analiz **16**, No. 5 (2004), 124–161.
21. S. I. Repin, *Aposteriori estimates for the Stokes problem.* — J. Math. Sci. **109**, No. 5 (2002), 1950–1964.
22. S. Repin, *A posteriori estimates for partial differential equations.* Walter de Gruyter, Berlin, 2008.
23. V. A. Solonnikov, *Estimates for solutions of a non-stationary linearized system of Navier–Stokes equations* — Trudy Mat. Inst. Steklov. **70** (1964), 213–317.

St.Petersburg Department
of Steklov Mathematical Institute RAS
Fontanka 27, St.Petersburg 191023, Russia
E-mail: repin@pdmi.ras.ru

Поступило 24 декабря 2012 г.