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THE LINEARIZATION PRINCIPLE FOR A FREE BOUNDARY PROBLEM FOR VISCOUS, CAPILLARY INCOMPRESSIBLE FLUIDS

Abstract. We consider the free boundary problem associated to a viscous incompressible surface wave subjected to capillary force on the free upper surface and Dirchlet boundary condition on the fixed bottom surface. In the spatially periodic case, we prove a general linearization principle which gives, for sufficiently small perturbations from a linearly stable stationary solution, existence of a global solution of the associated system and exponential convergence of the latter to the stationary one. Convergence of the velocity, the pressure and the free boundary is proved in anisotropic Sobolev–Slobodetskii spaces, after a suitable change of variables is performed to formulate the problem in a fixed domain. We apply this linearization principle to the study of the rest state's stability in the case of general potential forces.

Dedicated to the memory of Professor M. Padula

§1. Introduction and auxiliary propositions

A viscous, incompressible fluid, with velocity field $\boldsymbol{v}$ and pressure p, fills at any time $t \geqslant 0$ a domain Ω_t, where it satisfies the incompressible Navier–Stokes equations with external force $\boldsymbol{f}$ and viscosity ν. The density of the fluid is supposed to be 1. We suppose that this domain can be described as $\Omega_t := \{(y_1, y_2, y_3) : 0 \leqslant y_3 \leqslant \phi(y', t)\}$, where $y' = (y_1, y_2)$ and ϕ is a sufficiently regular function whose graph in $\mathbb{R}^3$ is the free boundary of the fluid, Γ_t, with exterior normal $\boldsymbol{n}$. We suppose that the velocity field, the pressure and the free boundary function ϕ are periodic for every $t \geqslant 0$, with periodic cell Σ being a fixed rectangle in $\mathbb{R}^2$.

On the bottom part of the boundary we impose nonhomogeneous Dirichlet boundary conditions $\boldsymbol{v}((y', 0), t) = \boldsymbol{\alpha}(y', t)$, for some sufficiently

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smooth, Σ-periodic $\boldsymbol{\alpha} = (\alpha^1, \alpha^2, \alpha^3)$ such that

$$\int_\Sigma \alpha^3 \, dy' = 0.$$

We suppose that the free boundary is subjected to capillary forces, with surface tension coefficient $\sigma > 0$ and to an external pressure p_e. Both p_e and $\boldsymbol{f}$ are supposed Σ-periodic and defined in all $\{x_3 \geqslant 0\}$. Given a suitable Σ-periodic initial velocity field $\boldsymbol{v}_0$ at time $t = 0$, defined in a Σ-periodic domain Ω_0, whose upper boundary Γ_0 is the graph of $\phi_0 = \phi(\cdot, 0)$, one is thus lead to the following evolution problem:

$$\begin{cases} \boldsymbol{v}_{,t} + (\boldsymbol{v} \cdot \nabla)\boldsymbol{v} - \nabla \cdot \mathbb{T}(\boldsymbol{v}, p) = \boldsymbol{f} & \text{in} \quad \Omega_t, \\ \nabla \cdot \boldsymbol{v} = 0 & \text{in} \quad \Omega_t, \\ \mathbb{T}(\boldsymbol{v}, p)\boldsymbol{n} = -\sigma H_t \boldsymbol{n} - p_e \boldsymbol{n} & \text{on} \quad \Gamma_t, \\ V_{\boldsymbol{n}} = \boldsymbol{v} \cdot \boldsymbol{n} & \text{on} \quad \Gamma_t, \\ \boldsymbol{v}(y, 0) = \boldsymbol{v}_0(y) & \text{in} \quad \Omega_0, \\ \boldsymbol{v}((y', 0), t) = \boldsymbol{\alpha}(y', t) & \text{on} \quad \Sigma \quad \text{for } t \geqslant 0, \end{cases} \tag{1}$$

where the underscript comma in $\boldsymbol{v}_{,t}$ denotes the partial derivative w.r.t. t (we will always assume such a notation). Here $\mathbb{T}$ is the stress tensor of the fluid, $V_{\boldsymbol{n}}$ is the normal velocity of the free surface Γ_t, H_t is the doubled mean curvature of Γ_t, positive for boundaries of convex bodies. We suppose the fluid is Newtonian, and thus

$$\mathbb{T}(\boldsymbol{v}, p) = -pI + \nu \mathbb{D}(\boldsymbol{v}),$$

where $\mathbb{D}(\boldsymbol{v})$ is the doubled symmetric rate-of-strain tensor $\mathbb{D}(\boldsymbol{v}) := \nabla \boldsymbol{v} + (\nabla \boldsymbol{v})^T$. If Π_0 is the orthogonal projection on the tangent space to Γ_0 and $\boldsymbol{n}_0$ its exterior normal, this system is coupled with the compatibility conditions

$$\begin{cases} \nabla \cdot \boldsymbol{v}_0 = 0 & \text{in} \quad \Omega_0, \\ \Pi_0 \mathbb{D}(\boldsymbol{v}_0)\boldsymbol{n}_0 = 0 & \text{on} \quad \Gamma_0, \\ \boldsymbol{v}_0(y', 0) = \boldsymbol{\alpha}(y', 0) & \text{on} \quad \Sigma. \end{cases} \tag{2}$$

We are concerned with a linearization principle for this problem. When $\boldsymbol{f}$, p_e and $\boldsymbol{\alpha}$ are independent of time, we consider a smooth stationary solution $(\boldsymbol{v}_b, p_b)$, in some domain $\Omega_b := \{(x', x_3) : 0 \leqslant x_3 \leqslant \phi_b(x')\}$ of

$$\begin{cases} (\boldsymbol{v}_b \cdot \nabla)\boldsymbol{v}_b - \nabla \cdot T(\boldsymbol{v}_b, p_b) = \boldsymbol{f} & \text{in} \quad \Omega_b, \\ \nabla \cdot \boldsymbol{v}_b = 0 & \text{in} \quad \Omega_b, \\ T(\boldsymbol{v}_b, p_b)\boldsymbol{N} = -\sigma H_b \boldsymbol{N} - p_e \boldsymbol{N} & \text{on} \quad \mathcal{G}, \\ \boldsymbol{v}_b \cdot \boldsymbol{N} = 0 & \text{on} \quad \mathcal{G}, \\ \boldsymbol{v}_b(x', 0, t) = \boldsymbol{\alpha}(x') & \text{on} \quad \Sigma \quad \text{for } t \geqslant 0, \end{cases} \tag{3}$$

where $\mathcal{G}$ is the surface defined by $x_3 = \phi_b(x')$, $\boldsymbol{N}$ its exterior normal and H_b its doubled mean curvature. Notice that $\nabla \cdot \boldsymbol{v}_b = 0$ actually forces the condition $\int\limits_\Sigma \alpha^3 \, dx' = 0$. We then consider solutions of (1) with initial data $(\boldsymbol{v}_0, \Omega_0)$ which are small perturbations of $(\boldsymbol{v}_b, \Omega_b)$, thus imposing $|\Omega_0| = |\Omega_b|$. We linearize system (1) near this stationary solution, performing the transformation of coordinates

$$\Omega_b \ni x \to e_\rho(x) := x + \theta(x)\rho(x_1, x_2, t)\boldsymbol{e}_3 = y \in \Omega_t, \tag{4}$$

for a smooth cutoff function $\theta = \theta(x_3)$ equal to 1 near $\mathcal{G}$ and zero near Σ, to obtain a problem in the fixed domain Ω_b (this transformation is well defined as long as $\sup_\Sigma |\rho|$ is sufficiently small). We let, in the new coordinates $x \in \Omega_b$, $\boldsymbol{u} = \boldsymbol{v} - \boldsymbol{v}_b$, $\rho = \phi - \phi_b$, $q = p - p_b$, with $\rho(x', \phi_b(x')) = \phi(x') - \phi_b(x')$, $x' = (x_1, x_2)$. Neglecting the nonlinear terms and setting $\nabla'_x = (\partial_{x_1}, \partial_{x_2})$, we obtain, for suitable linear differential operators $\boldsymbol{\Phi}_1$, Φ_2 and $\boldsymbol{\Phi}_3$, the system

$$\begin{cases} \boldsymbol{u}_{,t} - \nu \Delta_x \boldsymbol{u} + \nabla_x q - \boldsymbol{\Phi}_1(\boldsymbol{u}, \rho) = 0 & \text{in} \quad \Omega_b, \\ \nabla_x \cdot \boldsymbol{u} - \Phi_2(\rho) = 0 & \text{in} \quad \Omega_b, \\ \mathbb{T}_x(\boldsymbol{u}, q)\boldsymbol{N} + \sigma L \rho \boldsymbol{N} - \boldsymbol{\Phi}_3(\rho) = 0 & \text{on} \quad \mathcal{G}, \\ \rho_{,t} + (\nabla'_x \phi_b, -1) \cdot \boldsymbol{u} + (\nabla'_x \rho, 0) \cdot \boldsymbol{v}_b = 0 & \text{on} \quad \mathcal{G}, \\ \boldsymbol{u}(\cdot, t) \equiv 0 & \text{on} \quad \Sigma, \\ \boldsymbol{u}(\cdot, 0) = \boldsymbol{u}_0 := \boldsymbol{v}_0 - \boldsymbol{v}_b & \text{in} \quad \Omega_b, \\ \rho(\cdot, 0) = \rho_0 := \phi_0 - \phi_b & \text{on} \quad \mathcal{G}, \end{cases} \tag{5}$$

for all $t \geqslant 0$, subjected to the compatibility conditions

$$\begin{cases} \nabla_x \cdot \boldsymbol{u}_0 - \Phi_2(\rho_0) = 0, \\ \Pi_b\big(\nu \mathbb{D}_x(\boldsymbol{u}_0)\boldsymbol{N} - \boldsymbol{\Phi}_3(\rho_0)\big) = 0, \\ \int\limits_\Sigma \rho_0 \, dx' = 0, \end{cases} \tag{6}$$

where Π_b is the orthogonal projection on the tangent space to $\mathcal{G}$. Here the first two conditions correspond to the first two in (2) in the new variables, while the third to the volume preservation of the perturbation, i.e., $|\Omega_0| = |\Omega_b|$, where $|A|$ stands here for the volume of $A \subseteq \mathbb{R}^3$.

For $l \geqslant 0$, let K^l denote the parabolic anisotropic Sobolev–Slobodetskii space, W_2^l the isotropic one (both precisely defined below), $Q_T = \Omega_b \times [0,T)$, $G_T = \mathcal{G} \times [0,T)$ for $0 < T \leqslant +\infty$. We define

$$\|(\boldsymbol{u}, p, \rho)\|_{l,T} = \|\boldsymbol{u}\|_{K^{l+2}(Q_T)} + \|\nabla p\|_{K^l(Q_T)} + \|p\|_{K^{l+\frac{1}{2}}(G_T)} + \|\rho\|_{K^{l+\frac{5}{2}}(G_T)} + \|\rho_{,t}\|_{K^{l+\frac{3}{2}}(G_T)}, \quad (7)$$

and say that the stationary solution $(\boldsymbol{v}_b, p_b)$ in Ω_b of (3) is linearly exponentially stable if there exists $\gamma > 0$ such that every solution of (5) with compatibility conditions (6) decays exponentially in time, i.e.,

$$\|e^{\gamma t}(\boldsymbol{u}, p, \rho)\|_{l,\infty} \leqslant c\Big(\|\boldsymbol{u}_0\|_{W_2^{l+1}(\Omega_b)} + \|\rho_0\|_{W_2^{l+2}(\mathcal{G})}\Big). \quad (8)$$

We will prove the following theorem.

Theorem 1.1. *Let $l \in (\frac{1}{2}, 1)$ and $(\boldsymbol{v}_b, p_b, \phi_b)$ be a linearly exponentially stable solution of* (3)*. There exists $\delta > 0$ such that for any $\boldsymbol{v}_0$, $\phi_0 = \phi_b + \rho_0$ satisfying*

$$\|\boldsymbol{v}_0(e_{\rho_0}(x)) - \boldsymbol{v}_b(x)\|_{W_2^{l+1}(\Omega_b)} + \|\phi_0 - \phi_b\|_{W_2^{l+2}(\Sigma)} < \delta,$$

the compatibility conditions (2) *and $\int_\Sigma \rho_0 \, dx' = 0$, there exists a unique global periodic solution $(\boldsymbol{v}, p, \phi)$ of* (1) *such that for some $\gamma > \gamma' > 0$,*

$$\|e^{\gamma' t}(\boldsymbol{v} - \boldsymbol{v}_b, p - p_b, \phi - \phi_b)\|_{l,\infty} \leqslant c\Big(\|\boldsymbol{v}_0 - \boldsymbol{v}_b\|_{W_2^{l+1}(\Omega_b)} + \|\phi_0 - \phi_b\|_{W_2^{l+2}(\Sigma)}\Big),$$

where the norms of $\boldsymbol{v}$ and p are calculated in Ω_b according to $\boldsymbol{v}(x) = \boldsymbol{v}(e_\rho(x))$, $p(x) = p(e_\rho(x))$, with $\rho = \phi - \phi_b$.

This theorem is applied in the last section to the rest state $\boldsymbol{v}_b \equiv 0$ in a horizontal layer of fluid $\{0 < x_3 < h\}$ subjected to a potential hydrodinamic force $\boldsymbol{f} = \nabla V$ and an external pressure p_e. Under the natural condition that the free boundary $\{x_3 = h\}$ is an equipotential surface for the total potential $V - p_e$, we will prove the following result.

Theorem 1.2. *Let $l \in (\frac{1}{2}, 1)$ and $\boldsymbol{f} = \nabla V$ and p_e be Σ-periodic, sufficiently smooth (say, $W_{2,\mathrm{loc}}^{l+\frac{5}{2}}(\mathbb{R}^+)$) and satisfy $\nabla'(V - p_e) = 0$ on $\{x_3 = h\}$,* *so that we may suppose $p_e = V$ there. If the quadratic form*

$$\mathcal{B}(\rho) = \int_{\Sigma} \sigma |\nabla' \rho(x')|^2 + (p_e(x', h) - V(x', h))_{,x_3} \rho^2(x')\, dx',$$

is positive definite, the rest state $\boldsymbol{v}_b \equiv 0$, $p_b = V$, $\phi_b \equiv h$ is linearly exponentially stable.

Thus for any sufficiently small perturbation from the rest state, there exists unique global in time solution of the free boundary problem (1), which exponentially converges to the rest state. We will provide some concrete examples, such as surface waves on a small scale (constant gravity force and external pressure) and large scale (decaying gravity force and pressure) and sufficient conditions for the stability of an upside-down capillary layer of fluid.

Without periodicity assumption the problem has been treated in [2, 12, 14, 16] for a heavy fluid without capillarity. When surface tension is present, it has been treated in [3, 4, 13, 14, 15] without periodicity assumption, and in [7] for periodic motions. Theorem 1.1 seems optimal in its regularity assumptions. It is worth noting that in [3, 4, 13, 14, 15], a nonoptimal regularity on the initial perturbation is required, mostly asking $\boldsymbol{v} - \boldsymbol{v}_b$ to be small in $W_2^{l+\frac{3}{2}}$. Exponential stability results for the rest state (without periodicity assumptions) are addressed in [3] for $1 < l < 3/2$ with nonoptimal regularity on the initial velocity and in [13] for $1/2 < l < 1$ but $\phi_0 - h \in W_2^{l+\frac{5}{2}}(\Sigma) < \delta$. In [8], exponential stability is proved for $l = -1$ regardless of the size of the initial data, provided a global in time and smooth solution exists. In [7], the periodic case is studied, a weak form of the exponential stability of the rest state is proved for $l = 1$. Both these latter two works employ energy methods. In all of the aforementioned literature, a small scale model for the gravity and pressure is assumed: thus, the general sufficient condition on the positivity of the quadratic form $\mathcal{B}$ in order to obtain stability is, to the best of our knowledge, new.

Although in most of the literature some kind of linearization around the rest state is used, a general linearization principle was still unproved. Consequently, exponential stability has been obtained only in special cases ($l = -1, 1$, near the rest state with small scale models). Due to its generality, the linearization principle may allow to obtain exponential stability not only of the rest state, but also of other known stationary solutions of the free boundary problems, e.g. suitable flows down an inclined plane.

Auxiliary propositions

If l is a nonnegative integer, the isotropic Sobolev–Slobodetskii space on a bounded domain $\Omega \subset \mathbb{R}^N$ coincides with the usual Sobolev space, i.e., the set of functions $u : \Omega \to \mathbb{R}$ with finite norm

$$\|u\|^2_{W^l_2(\Omega)} = \sum_{|j|\leqslant l} \int_\Omega |D^j u(x)|^2\, dx,$$

where $D^j u$ is the jth distributional derivative. Here j is a multiindex $j = (j_1, \dots, j_N)$ and $|j| = j_1 + \dots + j_N$. When $l = [l] + \{l\}$, where $\{l\} \in (0,1)$ is the fractional part of l, the norm is

$$\|u\|^2_{W^l_2(\Omega)} := \|u\|^2_{W^{[l]}_2(\Omega)} + \sum_{|j|=[l]} \int_\Omega \int_\Omega \frac{|D^j u(x) - D^j u(y)|^2}{|x-y|^{N+2\{l\}}}\, dx\, dy.$$

Given a continuation operator $C : W^l_2(\Omega) \to W^l_2(\mathbb{R}^N)$, an equivalent norm, with constant depending on Ω, is

$$\|u\|^2_{W^{[l]}_2(\Omega)} + \sum_{|j|=[l]} \int_{\{|z|\leqslant 1\}} \|\Delta_z D^j C(u)(x)\|^2_{L^2(\Omega)} \frac{dz}{|z|^{N+2\{l\}}},$$

for which we will use the same symbol $\| \ \|_{W^l_2(\Omega)}$, while for the last addend of the previous definition we will use the symbol $\| \ \|_{\mathring{W}^l_2(\Omega)}$, which will denote the principal part of the norm. It will be useful to recall the algebra properties of $W^l_2(\Omega)$. To this end, we recall the following theorem (see [10] for a a refined statement using Besov spaces).

Proposition 1.3. *For arbitrary functions u, v given in a smooth bounded domain $\Omega \subset \mathbb{R}^N$ it holds*

(1) *If $0 \leqslant l \leqslant N/2$,*

$$\|uv\|_{L^2(\Omega)} \leqslant c\|u\|_{W^l_2(\Omega)} \|v\|_{W_2^{\frac{N}{2}-l}(\Omega)}.$$

(2) *If $0 \leqslant l \leqslant N/2 < s$,*

$$\begin{aligned} \|uv\|_{W^l_2(\Omega)} &\leqslant c\|u\|_{W^l_2(\Omega)} (\|v\|_{W_2^{\frac{N}{2}}(\Omega)} + \|v\|_{L^\infty(\Omega)}) \\ &\leqslant c\|u\|_{W^l_2(\Omega)} \|v\|_{W^s_2(\Omega)}. \end{aligned} \tag{9}$$

(3) *If $l > s > N/2$,*

$$\begin{aligned} \|uv\|_{W^l_2(\Omega)} &\leqslant c\big(\|u\|_{W^l_2(\Omega)} \|v\|_{W^s_2(\Omega)} + \|v\|_{W^l_2(\Omega)} \|u\|_{W^s_2(\Omega)}\big) \\ &\leqslant c\|u\|_{W^l_2(\Omega)} \|v\|_{W^l_2(\Omega)}. \end{aligned} \tag{10}$$

When $\Omega = (0,T)$, the constants in the previous inequalities do not depend on T as long as $T \geqslant 1$.

For $\lambda \in \mathbb{C}$ the weighted Sobolev–Slobodetskii norm on $W_2^l(\Omega)$ is

$$\|u\|^2_{H^l_\lambda(\Omega)} = \|u\|^2_{W_2^l(\Omega)} + |\lambda|^l \|u\|^2_{L^2(\Omega)}.$$

The classical interpolation inequality reads

$$|\lambda|^\eta \|u\|^2_{W_2^l(\Omega)} \leqslant c\|u\|^2_{H^{l+\eta}_\lambda(\Omega)}, \tag{11}$$

which clearly implies

$$|\lambda|^\eta \|u\|^2_{H^l_\lambda(\Omega)} \leqslant c\|u\|^2_{H^{l+\eta}_\lambda(\Omega)}.$$

Another type of interpolation inequality is the following:

$$|\lambda|^\eta \|u\|^2_{L^2(\partial\Omega)} \leqslant c\|u\|^2_{H^{\eta+\frac{1}{2}}_\lambda(\Omega)}, \tag{12}$$

which toghether with standard restriction estimates for Sobolev–Slobodetsii spaces, implies for any $\eta \geqslant 0$, $\lambda \neq 0$,

$$\|u\|^2_{H^\eta_\lambda(\partial\Omega)} \leqslant c\|u\|^2_{H^{\eta+\frac{1}{2}}_\lambda(\Omega)}.$$

(Notice that for $\eta = 0$ this inequality fails in non weighted Sobolev–Slobodetskii spaces.) All the constants in the previous inequalities do not depend on u or λ, as long as $|\lambda|$ is bounded away from 0, say $|\lambda| \geqslant 1$.

The anisotropic Sobolev–Slobodetskii space is defined as the set of functions $u = u(x,t)$, defined in $Q_T := \Omega \times [0,T)$, $0 < T \leqslant +\infty$ such that

$$u \in K^l(Q_T) := L^2(0,T; W_2^l(\Omega)) \cap W_2^{\frac{l}{2}}(0,T; L^2(\Omega))$$

with norm

$$\|u\|^2_{K^l(Q_T)} := \int_0^T \|u(\cdot,t)\|^2_{W_2^l(\Omega)}\,dt + \int_0^T \sum_{0\leqslant j\leqslant\left[\frac{l}{2}\right]} \|D_t^j u(\cdot,t)\|^2_{L^2(\Omega)}\,dt$$
$$+ \int_0^T \frac{dh}{h^{1+2\left\{\frac{l}{2}\right\}}} \int_h^T \|\Delta_{-h} D_t^{\left[\frac{l}{2}\right]} u(\cdot,t)\|^2_{L^2(\Omega)}\,dx.$$

An equivalent norm, which will still be denoted with the same symbol, is

$$\|u\|^2_{K^l(Q_T)} := \int_0^T \|u(\cdot,t)\|^2_{W_2^l(\Omega)}\,dt + \int_\Omega \|u(x,\cdot)\|^2_{W_2^{\frac{l}{2}}(0,T)}\,dx,$$

and the first and second term will be denoted respectively by $\|u\|^2_{W_2^{l,0}(Q_T)}$ and $\|u\|^2_{W_2^{0,l/2}(Q_T)}$. The seminorms $\|u\|_{\mathring{W}_2^{l,0}(Q_T)}$ and $\|u\|_{\mathring{W}_2^{0,l/2}(Q_T)}$ are similarly defined in the natural way. Applying proposition 1.3 gives, for any smooth function v,

$$\|uv\|_{K^l(Q_T)} \leqslant c_v \|u\|_{K^l(Q_T)}, \quad \forall u \in K^l(Q_T). \tag{13}$$

We will sometimes use the following mixed norm on functions $u : Q_T \to \mathbb{R}$

$$|u|^2_{l/2,r} := \sum_{0 \leqslant j \leqslant \left[\frac{l}{2}\right]} \|D_t^j u\|^2_{L^2(0,T;W_2^r(\Omega))} + \int_0^T \frac{dh}{h^{1+2\left\{\frac{l}{2}\right\}}} \int_h^T \|\Delta_{-h} D_t^{\left[\frac{l}{2}\right]} u(\cdot,t)\|^2_{W_2^r(\Omega)}\, dt,$$

which well defines the Banach space $W_2^{l/2}(0,T;W_2^r(\Omega))$. Interpolation inequalities ensure that

$$K^{l+r}(Q_T) \hookrightarrow W_2^{l/2}(0,T;W_2^r(\Omega)). \tag{14}$$

Finally we recall the following result, which follows from standard interpolation arguments.

Proposition 1.4. *Let $\mathcal{G}$ be a smooth bounded submanifold of $\mathbb{R}^N$, and $T > 0$. For any $l > \frac{1}{2}$ it holds the estimate*

$$\sup_{0 \leqslant t < T} \|\rho(\cdot,t)\|_{W_2^l(\mathcal{G})} \leqslant c\Big(\|\rho\|_{W_2^{l+\frac{1}{2},0}(\mathcal{G}\times[0,T))} + \|\rho_{,t}\|_{W_2^{l-\frac{1}{2},0}(\mathcal{G}\times[0,T))}\Big),$$

for a constant c independent of $\rho : \mathcal{G}\times[0,T) \to \mathbb{R}$ and T, as long as $T \geqslant 1$.

§2. Reduction to a fixed domain

If $\mathcal{G}$ is the graph of ϕ_b over Σ,

$$\boldsymbol{N} = \frac{(-\nabla'\phi_b, 1)}{\sqrt{1+|\nabla'\phi_b|^2}}, \quad \Pi_b(\boldsymbol{V}) = \boldsymbol{V} - (\boldsymbol{N}\cdot\boldsymbol{V})\boldsymbol{N},$$

are its normal and projection operator on the tangent space of $\mathcal{G}$, respectively. Letting $\rho := \phi - \phi_b$, we rewrite problem (1) in terms of the new variable $x \in \Omega_b$ defined as

$$\Omega_t \ni y = e_\rho(x) = x + \theta(x)\rho(x',t)\boldsymbol{e}_3,$$

where θ is a C^∞ cutoff function with suitable regularity. We will assume, to simplify some calculations, that $\theta = \theta(y_3)$ and $\theta(s) = 0$ for $s < h$ and $\theta = 1$ for $s > 2h$, with $\inf_\Sigma \phi_b > 3h > 0$. We suppose that, for some $l \geqslant \frac{1}{2}$,

$$\sup_{t<T} \|\rho(\cdot,t)\|_{W_2^{l+\frac{3}{2}}(\Sigma)} \ll 1,$$

so that the transformation (4) is at least of class $C^{1,\alpha}$ and invertible. Moreover, we will henceforth write $\rho^*(x,t) = \theta(x_3)\rho(x',t)$. This change of variable transforms Ω_b to Ω_t, and we will denote by $\mathcal{L} = \mathcal{L}(x,\rho)$ the Jacobi matrix of this transformation:

$$\mathcal{L}(x,\rho) = \left(\frac{\partial y_i}{\partial x_j}\right)_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \theta\rho_{,x_1} & \theta\rho_{,x_2} & 1+\theta'\rho \end{pmatrix}. \tag{15}$$

Furthermore, we will set $L = \det\mathcal{L}$, $\widehat{\mathcal{L}} = L\mathcal{L}^{-1}$ so that $\widehat{\mathcal{L}} = \mathrm{cof}(\mathcal{L})^T$. One has

$$L = 1+\theta'\rho, \quad \mathcal{L}^{-T} = \begin{pmatrix} 1 & 0 & \frac{-\theta\rho_{,x_1}}{1+\theta'\rho} \\ 0 & 1 & \frac{-\theta\rho_{,x_2}}{1+\theta'\rho} \\ 0 & 0 & \frac{1}{1+\theta'\rho} \end{pmatrix}, \quad I - \mathcal{L}^{-T} = \frac{1}{L}\nabla\rho^* \otimes \boldsymbol{e}_3. \tag{16}$$

The transformation e_ρ converts the operator ∇_y to $\widetilde{\nabla} = \mathcal{L}^{-T}\nabla_x$, and we will henceforth write ∇ for ∇_x. We now rewrite system (1) in the new variables (x,t). For the term $\boldsymbol{v}_{,t}$ we have

$$\frac{d}{dt}\boldsymbol{v}(e_\rho(x),t) = \nabla_y \boldsymbol{v}\frac{\partial y}{\partial t} + \boldsymbol{v}_{,t} = \rho^*_{,t}(\mathcal{L}^{-1}\boldsymbol{e}_3 \cdot \nabla)\boldsymbol{v} + \boldsymbol{v}_{,t}.$$

The term $(\boldsymbol{v}\cdot\nabla_y)\boldsymbol{v}$ corresponds to $(\mathcal{L}^{-1}\boldsymbol{v}\cdot\nabla)\boldsymbol{v}$ and all the other differential operators are substituted with the rule $\nabla_y \to \widetilde{\nabla}$ and $\Delta_y \to \widetilde{\Delta} = \widetilde{\nabla}^2$. Letting then $\boldsymbol{v} = \boldsymbol{v}(e_\rho(x))$, $\boldsymbol{f} = \boldsymbol{f}(e_\rho(x))$ and $p_e = p_e(e_\rho(x))$, the system (1) becomes, in the new variables:

$$\begin{cases} \boldsymbol{v}_{,t} - \rho^*_{,t}(\mathcal{L}^{-1}\boldsymbol{e}_3\cdot\nabla)\boldsymbol{v} - \nu\widetilde{\Delta}\boldsymbol{v} + \widetilde{\nabla}p + (\boldsymbol{v}\cdot\widetilde{\nabla})\boldsymbol{v} = \boldsymbol{f} & \text{in } \Omega_b, \\ \widetilde{\nabla}\cdot\boldsymbol{v} = 0 & \text{in } \Omega_b, \\ \widetilde{\mathbb{T}}(\boldsymbol{v},p)\boldsymbol{n} = -\sigma H\boldsymbol{n} - p_e\boldsymbol{n} & \text{on } \mathcal{G}, \\ \rho_{,t} + \nabla'\phi_b\cdot\boldsymbol{v}(x',\phi_b) - v^3(x',\phi_b) + \nabla'\rho\cdot\boldsymbol{v}(x',\phi_b) = 0 & \text{on } \Sigma, \\ \boldsymbol{v}(x,0) = \widetilde{\boldsymbol{v}}_0(x), \quad \text{in } \Omega_b, \quad \rho(x',0) = \rho_0(x'), & \text{on } \Sigma, \\ \boldsymbol{v}(x',t) = \boldsymbol{\alpha}(x') \quad \text{for} \quad t \geqslant 0,\, x'\in\Sigma, \end{cases} \tag{17}$$

where $\widetilde{\boldsymbol{v}}_0 = \boldsymbol{v}_0(e_{\rho_0})$, H is the doubled mean curvature of the graph of $\phi_b+\rho$, $\boldsymbol{n}$ the normal of the latter and $\widetilde{\mathbb{T}} = -p\boldsymbol{n}+\widetilde{\mathbb{D}}(\boldsymbol{v})\boldsymbol{n}$ with $\mathbb{D}(\boldsymbol{v}) = \widetilde{\nabla}\boldsymbol{v}+(\widetilde{\nabla}\boldsymbol{v})^T$. The equation for $\rho_{,t}$ can and will be equivalently written with variables in $\mathcal{G}$ instead of Σ, simply letting $\rho(x', \phi_b(x')) = \rho(x')$.

Recall now that $(\boldsymbol{v}_b, p_b, \phi_b)$ is a smooth solution of (3). In order to linearize problem (17) near $(\boldsymbol{v}_b, p_b, \phi_b)$, we set $\boldsymbol{u} = \boldsymbol{v} - \boldsymbol{v}_b$, $q = p - p_b$ and, subtracting (3) to (17), obtain a system of the form

$$\begin{cases} \boldsymbol{u}_{,t} - \nu\Delta\boldsymbol{u} + \nabla q - \boldsymbol{\Phi}_1(\boldsymbol{u},\rho) = \boldsymbol{l}_0(\boldsymbol{u},\rho) + \boldsymbol{l}_1(\boldsymbol{u},\rho,q) & \text{in} \quad \Omega_b, \\ \nabla\cdot\boldsymbol{u} - \Phi_2(\rho) = l_2(\boldsymbol{u},\rho) = \nabla\cdot\boldsymbol{G}(\boldsymbol{u},\rho) & \text{in} \quad \Omega_b, \\ \mathbb{T}(\boldsymbol{u},q)\boldsymbol{N} + \sigma L\rho\boldsymbol{N} - \boldsymbol{\Phi}_3(\rho) = \boldsymbol{l}_3(\boldsymbol{u},\rho) & \text{on} \quad \mathcal{G}, \\ \rho_{,t} + \nabla'\phi_b\cdot\boldsymbol{u} - u^3 + \nabla'\rho\cdot\boldsymbol{v}_b = l_4(\boldsymbol{u},\rho) & \text{on} \quad \mathcal{G}, \\ \boldsymbol{u}(x,0) = \boldsymbol{u}_0(x) \quad \text{in } \Omega_b, \quad \rho(x',0) = \rho_0(x') & \text{on} \quad \Sigma, \\ \boldsymbol{u}(x',t) = 0 \quad \forall x'\in\Sigma, \quad t\geqslant 0. \end{cases} \tag{18}$$

Problem (18) is subjected to the following compatibility conditions:

$$\begin{cases} \nabla\cdot\boldsymbol{u}_0 - \Phi_2(\rho_0) = l_2(\boldsymbol{u}_0,\rho_0), \\ \Pi_b\big(\nu\mathbb{D}(\boldsymbol{u}_0)\boldsymbol{N} - \boldsymbol{\Phi}_3(\rho_0)\big) = \Pi_b\boldsymbol{l}_3(\boldsymbol{u}_0,\rho_0), \\ \int\limits_\Sigma \rho_0\,dx' = 0. \end{cases} \tag{19}$$

The first two conditions are the simplest compatibility conditions at the initial time, while the third one is the preservation of mass for the perturbation, and a straightforward calculation shows that this implies that for any solution of (18) the mean value of ρ vanishes identically for $t\geqslant 0$.

We will now calculate the expressions Φ_i and $\boldsymbol{l}_i$ which we will construct as the first order and higher order term respectively in $\boldsymbol{u}$, q and ρ. First note that the exact equation for $\rho_{,t}$ is

$$\rho_{,t} + \nabla'\rho\cdot\boldsymbol{v}_b + \nabla'\phi_b\cdot\boldsymbol{u} - u^3 = -\nabla'\rho\cdot\boldsymbol{u},$$

and therefore

$$l_4(\boldsymbol{u},\rho) = -\nabla'\rho\cdot\boldsymbol{u}. \tag{20}$$

From the explicit matrix given in (16), we have

$$\delta\mathcal{L}^{-T} = \begin{pmatrix} 0 & 0 & -\theta\rho_{,x_1} \\ 0 & 0 & -\theta\rho_{,x_2} \\ 0 & 0 & -\theta'\rho \end{pmatrix} := -\nabla\rho^*\otimes\boldsymbol{e}_3, \quad \delta\mathcal{L}^{-1} = -\boldsymbol{e}_3\otimes\nabla\rho^*,$$

and we will let

$$\delta^2\mathcal{L}^{-T} := \mathcal{L}^{-T} - I - \delta\mathcal{L}^{-T} = \nabla\rho^*\frac{\theta'\rho}{1+\theta'\rho}\otimes \boldsymbol{e}_3.$$

We now define

$$\boldsymbol{v}_{,t} - \nu\Delta\boldsymbol{v} + \nabla p = \widetilde{\boldsymbol{l}}_0(\boldsymbol{v},\rho) + \widetilde{\boldsymbol{l}}_1(\boldsymbol{v},p,\rho)$$

where, if $\boldsymbol{f}(\rho) = \boldsymbol{f}(e_\rho(x))$ and simply $\boldsymbol{f}(0) = \boldsymbol{f}$,

$$\widetilde{\boldsymbol{l}}_0(\boldsymbol{v},\rho) = -(\mathcal{L}^{-1}\boldsymbol{v}\cdot\nabla)\boldsymbol{v} + \boldsymbol{f}(\rho),$$
$$\widetilde{\boldsymbol{l}}_1(\boldsymbol{v},p,\rho) = \nu(\widetilde{\Delta}-\Delta)\boldsymbol{v} + (\nabla-\widetilde{\nabla})p + \rho^*_{,t}(\mathcal{L}^{-1}\boldsymbol{e}_3\cdot\nabla)\boldsymbol{v},$$

noting that

$$\boldsymbol{u}_{,t} - \nu\Delta\boldsymbol{u} + \nabla p = \widetilde{\boldsymbol{l}}_0(\boldsymbol{v},\rho) - \widetilde{\boldsymbol{l}}_0(\boldsymbol{v}_b,0) + \widetilde{\boldsymbol{l}}_1(\boldsymbol{v},p,\rho) - \widetilde{\boldsymbol{l}}_1(\boldsymbol{v}_b,p_b,0).$$

Since $\widetilde{\boldsymbol{l}}_1$ is linear in the arguments $\boldsymbol{v}$ and p, it suffices to compute the linearization $\delta\boldsymbol{l}_1(\boldsymbol{v}_b,p_b,\rho)$ of $\widetilde{\boldsymbol{l}}_1(\boldsymbol{v}_b,p_b,\rho)$ with respect to ρ.

$$\delta\boldsymbol{l}_1(\boldsymbol{v}_b,p_b,\rho) = \nu\big(\delta\mathcal{L}^{-T}\nabla\cdot\nabla + \nabla\cdot\delta\mathcal{L}^{-T}\nabla\big)\boldsymbol{v}_b + \nabla\rho^*\frac{\partial p_b}{\partial y_3} + \delta\rho^*_{,t}(\mathcal{L}^{-1}\boldsymbol{e}_3\cdot\nabla)\boldsymbol{v}_b.$$

For the last term, we have that

$$\rho^*_{,t}(\mathcal{L}^{-1}\boldsymbol{e}_3\cdot\nabla)\boldsymbol{v}_b = \frac{\rho^*_{,t}}{1+\theta'\rho}\boldsymbol{v}_{b,x_3} = \rho^*_{,t}\boldsymbol{v}_{b,x_3} + \frac{\theta'\rho\rho^*_{,t}}{1+\theta'\rho}\boldsymbol{v}_{b,x_3}.$$

Therefore the linear part is

$$\nu\big(\delta\mathcal{L}^{-T}\nabla\cdot\nabla + \nabla\cdot\delta\mathcal{L}^{-T}\nabla\big)\boldsymbol{v}_b + \nabla\rho^* p_{b,x_3} + \rho^*_{,t}\boldsymbol{v}_{b,x_3}, \tag{21}$$

and the nonlinear one is

$$\begin{aligned}\boldsymbol{l}_1(\boldsymbol{u},q,\rho) &= \widetilde{\boldsymbol{l}}_1(\boldsymbol{u},q,\rho) - \delta^2\mathcal{L}^{-T}\nabla p_b + \frac{\theta'\rho\rho^*_{,t}}{1+\theta'\rho}\boldsymbol{v}_{b,x_3}\\ &+ \nu\big((I-\mathcal{L}^{-T})\nabla\cdot(I-\mathcal{L}^{-T})\nabla + \delta^2\mathcal{L}^{-T}\nabla\cdot\nabla + \nabla\cdot\delta^2\mathcal{L}^{-T}\nabla\big)\boldsymbol{v}_b.\end{aligned} \tag{22}$$

For $\boldsymbol{l}_0$ we have

$$\widetilde{\boldsymbol{l}}_0(\boldsymbol{v},\rho) - \widetilde{\boldsymbol{l}}_0(\boldsymbol{v}_b,0) = (\delta\mathcal{L}^{-1}\boldsymbol{v}_b\cdot\nabla + \boldsymbol{u}\cdot\nabla)\boldsymbol{v}_b + (\boldsymbol{v}_b\cdot\nabla)\boldsymbol{u} + \widetilde{\boldsymbol{l}}_0(\boldsymbol{u},\rho) + \boldsymbol{f}(\rho) - \boldsymbol{f},$$

which adds a further linear term to (21), giving

$$\begin{aligned}\boldsymbol{\Phi}_1(\boldsymbol{u},\rho) &= \nabla\rho^* p_{b,x_3} + \theta\rho_{,t}\boldsymbol{v}_{b,x_3} + (\boldsymbol{v}_b\cdot\nabla)\boldsymbol{u} + \boldsymbol{f}_{,x_3}\rho^*\\ &+ \nu\big(\delta\mathcal{L}^{-T}\nabla\cdot\nabla + \nabla\cdot\delta\mathcal{L}^{-T}\nabla + \frac{1}{\nu}\delta\mathcal{L}^{-1}\boldsymbol{v}_b\cdot\nabla + \frac{1}{\nu}\boldsymbol{u}\cdot\nabla\big)\boldsymbol{v}_b.\end{aligned} \tag{23}$$

This is a linear combination of terms of the form $\boldsymbol{h}$ times

$$u^k, \quad \rho^*_{,x_i}, \quad \rho^*_{,t}, \quad u^k_{,x_i}, \quad \rho^*_{,x_ix_j}, \tag{24}$$

for suitable smooth $\boldsymbol{h}$'s depending on the indexes and on $\boldsymbol{v}_b$, p_b, ϕ_b, $\boldsymbol{f}$ and θ. The nonlinear term is

$$\boldsymbol{l}_0(\boldsymbol{u},\rho) = (\delta^2\mathcal{L}^{-1}\boldsymbol{v}_b\cdot\nabla)\boldsymbol{u} + (\delta^2\mathcal{L}^{-1}\boldsymbol{u}\cdot\nabla)\boldsymbol{v}_b + (\delta^2\mathcal{L}^{-1}\boldsymbol{v}_b\cdot\nabla)\boldsymbol{v}_b \\ + (\delta^2\mathcal{L}^{-1}\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + \delta^2\boldsymbol{f}, \tag{25}$$

where

$$\delta^2\boldsymbol{f} := \boldsymbol{f}(\rho) - \boldsymbol{f} - \boldsymbol{f}_{,x_3}\rho^* = \int_0^1 (1-s)\frac{d^2}{ds^2}\boldsymbol{f}(e_{s\rho}(x))\,ds.$$

Inspecting the terms in (22) and (25), one sees that the sum $\boldsymbol{l}_1 + \boldsymbol{l}_0$ is a linear combination of terms of the form $\boldsymbol{h}(x,\rho^*(x),\nabla\rho^*(x))$ times

$$\begin{aligned} &\rho^{*2}, \quad \rho^*\rho^*_{,x_i}, \quad \rho^*_{,x_i}\rho^*_{,x_j}, \quad \rho^*\rho^*_{,t}, \quad \rho^*\rho^*_{,x_ix_j}, \quad \rho^*_{,t}u^k_{,x_i}, \\ &\rho^*_{,x_i}u^k_{,x_j}, \quad \rho^*_{,x_ix_j}u^k_{,x_l}, \quad \rho^*_{,x_i}u^k_{,x_jx_l}, \quad \rho^*_{,x_i}q_{,x_j}, \quad u^ju^k_{,x_i} \end{aligned} \tag{26}$$

for suitable smooth $\boldsymbol{h}$'s depending on the indexes and on $\boldsymbol{v}_b$, p_b, ϕ_b, $\boldsymbol{f}$ and θ. We will henceforth set $\boldsymbol{h}_\rho = \boldsymbol{h}(x,\rho^*(x),\nabla\rho^*(x))$ (and similar notation in the scalar case) always assuming such a smooth dependence on the indexes, the datas and the stationary solution. Regarding the divergence, notice that

$$\widetilde{\nabla}\cdot\boldsymbol{v} = \mathcal{L}^{-T}\nabla\cdot\boldsymbol{v} = \frac{1}{L}(\widehat{\mathcal{L}}^T\nabla)\cdot\boldsymbol{v} = 0 \quad \Leftrightarrow \quad (\widehat{\mathcal{L}}^T\nabla)\cdot\boldsymbol{v} = 0,$$

and

$$(\widehat{\mathcal{L}}^T\nabla)\cdot\boldsymbol{v} = \nabla\cdot(\widehat{\mathcal{L}}\boldsymbol{v}),$$

since the cofactor matrix has divergence free rows. Therefore

$$\nabla\cdot\boldsymbol{v} = \big((I-\widehat{\mathcal{L}}^T)\nabla\big)\cdot\boldsymbol{v} = \nabla\cdot(I-\widehat{\mathcal{L}})\boldsymbol{v}. \tag{27}$$

Since

$$\widehat{\mathcal{L}} = \begin{pmatrix} 1+\theta'\rho & 0 & 0 \\ 0 & 1+\theta'\rho & 0 \\ -\theta\rho_{,x_1} & -\theta\rho_{,x_2} & 1 \end{pmatrix} = I(1+\theta'\rho) - \boldsymbol{e}_3\otimes\nabla\rho^*, \tag{28}$$

the matrix $I-\widehat{\mathcal{L}}$ is linear in ρ; therefore, from (27) we get

$$\nabla\cdot\boldsymbol{u} = \nabla\rho^*\cdot\boldsymbol{v}_{b,x_3} - \theta'\rho\nabla\cdot\boldsymbol{v}_b + \nabla\rho^*\cdot\frac{\partial\boldsymbol{u}}{\partial y_3} - \theta'\rho\nabla\cdot\boldsymbol{u},$$

giving

$$\Phi_2(\rho) = \nabla\rho^* \cdot \boldsymbol{v}_{b,x_3} - \theta'\rho\nabla\cdot\boldsymbol{v}_b = \nabla\cdot(I-\widehat{\mathcal{L}})\boldsymbol{v}_b, \tag{29}$$

$$l_2(\boldsymbol{u},\rho) = \nabla\rho^*\cdot\boldsymbol{u}_{,x_3} - \theta'\rho\nabla\cdot\boldsymbol{u}, \quad \boldsymbol{G}(\boldsymbol{u},\rho) = (\nabla\rho^*\cdot\boldsymbol{u})\boldsymbol{e}_3 - \theta'\rho\boldsymbol{u}. \tag{30}$$

Notice that

$$(I-\widehat{\mathcal{L}})\boldsymbol{v}_b = \boldsymbol{G}(\boldsymbol{u},\rho) = 0, \tag{31}$$

in a neighbourhood of Σ, since θ identically vanishes for sufficiently small x_3. We now look at the equation for $\mathbb{T}(\boldsymbol{u},p)$. It holds

$$\begin{aligned}\widetilde{\mathbb{T}}(\boldsymbol{v},p)\boldsymbol{n} - \mathbb{T}(\boldsymbol{v}_b,p_b)\boldsymbol{N} &= \mathbb{T}(\boldsymbol{u},q)\boldsymbol{N} - p_b(\boldsymbol{n}-\boldsymbol{N}) - q(\boldsymbol{n}-\boldsymbol{N}) \\ &+ (\widetilde{\mathbb{D}}-\mathbb{D})(\boldsymbol{v}_b)\boldsymbol{N} + \mathbb{D}(\boldsymbol{v}_b)(\boldsymbol{n}-\boldsymbol{N}) + \mathbb{D}(\boldsymbol{u})(\boldsymbol{n}-\boldsymbol{N}) \\ &+ (\widetilde{\mathbb{D}}-\mathbb{D})(\boldsymbol{u})\boldsymbol{N} + (\widetilde{\mathbb{D}}-\mathbb{D})(\boldsymbol{v}_b)(\boldsymbol{n}-\boldsymbol{N}) + (\widetilde{\mathbb{D}}-\mathbb{D})(\boldsymbol{u})(\boldsymbol{n}-\boldsymbol{N}).\end{aligned}$$

If $\boldsymbol{n}_s$ is the upward normal to the cartesian surface with equation $y_3 = \phi_b(x') + s\rho(x')$, we define $g_s = 1 + |\nabla(\phi_b + s\rho)|^2$, and thus $\boldsymbol{n}_s = (-\nabla'(\phi_b + s\rho), 1)/\sqrt{g_s}$. The first variation w.r.t. $s\rho$ of $\boldsymbol{N}$ and $\mathbb{D}$ is readily computed as

$$\delta\boldsymbol{N} = \Pi_b\frac{(-\nabla'\rho,0)}{\sqrt{1+|\nabla'\phi_b|^2}}, \quad \delta\mathbb{D}(\boldsymbol{v}) = -\nabla\rho^*\otimes\nabla v^3 - \nabla v^3\otimes\nabla\rho^*,$$

and it holds

$$\delta^2\boldsymbol{N} := \boldsymbol{n} - \boldsymbol{N} - \delta\boldsymbol{N} = \int_0^1 (1-s)\frac{d^2}{ds^2}\boldsymbol{n}_s\,ds = \rho_{,x_i}\rho_{,x_j}\boldsymbol{A}^{ij}(y',\nabla\rho)\,ds,$$

$$\boldsymbol{n} - \boldsymbol{N} = \rho_{,x_i}\boldsymbol{B}^i(y',\nabla\rho),$$

for smooth $\boldsymbol{A}^{ij}$ and $\boldsymbol{B}^i$. Since $\mathbb{D}(\boldsymbol{v}) = 2\mathrm{Sym}(\nabla\boldsymbol{v})$ where

$$\mathrm{Sym}(M) = (M + M^T)/2,$$

$$(\widetilde{\mathbb{D}} - \mathbb{D} - \delta\mathbb{D})(\boldsymbol{v}) = 2\mathrm{Sym}\big(\delta^2\mathcal{L}^{-T}\nabla\boldsymbol{v}\big) = 2\mathrm{Sym}\left(\nabla\rho^*\frac{\theta'\rho}{1+\theta'\rho}\otimes\nabla v^3\right),$$

which vanishes on $\mathcal{G}$ supposing $\theta \equiv 1$ near $\mathcal{G}$. Therefore

$$\widetilde{\mathbb{T}}(\boldsymbol{v},p)\boldsymbol{n} - \mathbb{T}(\boldsymbol{v}_b,p_b)\boldsymbol{N} = \mathbb{T}(q,\boldsymbol{u})\boldsymbol{N} - p_b\delta\boldsymbol{N} + \delta\mathbb{D}(\boldsymbol{v}_b)\boldsymbol{N} + \mathbb{D}(\boldsymbol{v}_b)\delta\boldsymbol{N} + \boldsymbol{l}(\boldsymbol{u},q,\rho),$$

where $\boldsymbol{l}(\boldsymbol{u},q,\rho)$ collects the higher order terms:

$$\begin{aligned}\boldsymbol{l}(\boldsymbol{u},q,\rho) := &- p_b\delta^2\boldsymbol{N} - q(\boldsymbol{n}-\boldsymbol{N}) + \mathbb{D}(\boldsymbol{v}_b)\delta^2\boldsymbol{N} + \mathbb{D}(\boldsymbol{u})(\boldsymbol{n}-\boldsymbol{N}) \\ &+ \delta\mathbb{D}(\boldsymbol{u})\boldsymbol{N} + \delta\mathbb{D}(\boldsymbol{v}_b)(\boldsymbol{n}-\boldsymbol{N}) + \delta\mathbb{D}(\boldsymbol{u})(\boldsymbol{n}-\boldsymbol{N}).\end{aligned}$$

It is convenient to eliminate the pressure term by looking at the normal component of $\mathbb{T}(p, \boldsymbol{v})$ and $\mathbb{T}(p_b, \boldsymbol{v}_b)$: setting $p_e(\rho) = p_e(x', \phi_b + \rho)$ and $p_e = p_e(0) = p_e(x', \phi_b)$, it holds

$$p = \sigma H + p_e(\rho) + \boldsymbol{n}\widetilde{\mathbb{D}}(\boldsymbol{v})\boldsymbol{n}$$
$$p_b = \sigma H_b + p_e + \boldsymbol{N}\mathbb{D}(\boldsymbol{v}_b)\boldsymbol{N},$$

from which

$$q = \sigma(H - H_b) + (p_e(\rho) - p_e) + \boldsymbol{n}\widetilde{\mathbb{D}}(\boldsymbol{v})\boldsymbol{n} - \boldsymbol{N}\mathbb{D}(\boldsymbol{v})\boldsymbol{N}. \tag{32}$$

We now consider the curvature term. Recall that given a surface with normal $\boldsymbol{n}$, smoothly extended in a neighbourhood of the surface, the doubled mean curvature H is defined as $\nabla \cdot \boldsymbol{n}$. Definining $H_s = \nabla_x \cdot \boldsymbol{n}_s$, it holds

$$H_1 = H_0 + \frac{d}{ds} H_s \lfloor_{s=0} + \int_0^1 (1-s) \frac{d^2}{ds^2} H_s \, ds.$$

Moreover, for α, $\beta = 1, 2$

$$\begin{aligned} \frac{d}{ds} H_s \lfloor_{s=0} &= -\frac{1}{g_b} \partial_\alpha \left(\delta_{\alpha\beta} \sqrt{g_b} - \frac{\phi_{b,x_\alpha} \phi_{b,x_\beta}}{\sqrt{g_b}} \right) \rho_{,x_\beta} \\ &\quad - \frac{1}{g_b^{\frac{5}{2}}} \nabla' \phi_b \cdot \nabla' |\nabla' \phi_b|^2 \nabla \phi_b \cdot \nabla' \rho + \frac{\nabla' |\nabla' \phi_b|^2 \cdot \nabla' \rho}{g_b^{\frac{3}{2}}} \\ &= -\frac{1}{\sqrt{g_b}} \Delta_{\mathcal{G}} \rho + \boldsymbol{b} \cdot \nabla' \rho =: L\rho. \end{aligned} \tag{33}$$

Here $\Delta_{\mathcal{G}}$ is the Laplace–Beltrami operator on the surface $\mathcal{G}$, and $\boldsymbol{b}$ is a smooth field depending on ϕ_b. Finally, a lengthy but straightforward calculation shows that

$$\frac{d^2}{ds^2} H_s = \rho_{,x_\alpha} \rho_{,x_\beta} \sum_{m=1}^{3} \frac{p_{\alpha\beta m}}{g_s^{m+\frac{1}{2}}} + \rho_{,x_\alpha} \rho_{,x_\beta x_\gamma} \sum_{m=1}^{3} \frac{q_{\alpha\beta\gamma m}}{g_s^{m+\frac{1}{2}}},$$

where $p_{\alpha\beta m}$ and $q_{\alpha\beta\gamma m}$ are polynomials in the variables s, $\nabla'\rho$, $\nabla'\phi_b$, $\nabla^2\phi_b$. Since $H_0 = H_b$, we have

$$H_1 \boldsymbol{n} - H_b \boldsymbol{N} = H_b \delta \boldsymbol{N} + L\rho \boldsymbol{N} + \boldsymbol{l}'(\rho),$$

where

$$\boldsymbol{l}'(\rho) := (\mathcal{H} - \mathcal{H}_b)(\boldsymbol{n} - \boldsymbol{N}) + \boldsymbol{N} \int_0^1 (1-s) \frac{d^2}{ds^2} H_s \, ds + \mathcal{H}_b \delta^2 \boldsymbol{N},$$

collects the higher order terms and is a linear combination of terms of the form $\boldsymbol{h}_\rho \rho_{,x_i}\rho_{,x_j}$ or $\boldsymbol{h}_\rho \rho_{,x_i}\rho_{,x_j x_k}$. Finally we consider the external pressure. Setting $p_e(\rho) = p_e(x', \phi_b + \rho)$ and $p_e = p_e(0) = p_e(x', \phi_b)$

$$p_e(\rho)\boldsymbol{n} - p_e\boldsymbol{N} = \delta p_e \boldsymbol{N} + (p_e(\rho) - p_e - \delta p_e)\boldsymbol{N} + p_e \delta\boldsymbol{N} + p_e(\boldsymbol{n} - \boldsymbol{N} - \delta\boldsymbol{N}) + (p_e(\rho) - p_e)(\boldsymbol{n} - \boldsymbol{N}),$$

where

$$\delta p_e(x) = p_{e,x_3}(x)\rho(x').$$

Thus the linear part is $\delta p_e \boldsymbol{N} + p_e \delta\boldsymbol{N}$, which is a linear combination of terms of the form $\boldsymbol{h}_\rho \rho$ and $\boldsymbol{h}_\rho \rho_{,x_i}$, while the nonlinear one is

$$\boldsymbol{l}''(\rho) := (p_e(\rho) - p_e - \delta p_e)\boldsymbol{N} + p_e(\boldsymbol{n} - \boldsymbol{N} - \delta\boldsymbol{N}) + (p_e(\rho) - p_e)(\boldsymbol{n} - \boldsymbol{N}),$$

which is a linear combination of terms of the form $\boldsymbol{h}\rho^2$ and $\boldsymbol{h}\rho_{,x_i}\rho_{,x_j}$ for smooth $\boldsymbol{h}$'s depending on $\boldsymbol{v}_b$, p_b, ϕ_b and p_e. All in all, we have

$$\boldsymbol{\Phi}_3(\rho) = \big(p_b - p_e - \sigma\mathcal{H}_b - \mathbb{D}(\boldsymbol{v}_b)\big)\delta\boldsymbol{N} - \delta p_e \boldsymbol{N} - \delta\mathbb{D}(\boldsymbol{v}_b)\boldsymbol{N}, \tag{34}$$

which is a linear combination of terms of the form $\boldsymbol{h}_\rho \rho_{,x_i}$ and $\boldsymbol{h}_\rho \rho$. For the nonlinear part, using (32) to get rid of q, we get

$$\boldsymbol{l}_3(\boldsymbol{u}, \rho) = -\boldsymbol{l}(\boldsymbol{u}, q, \rho) - \sigma\boldsymbol{l}'(\rho) - \boldsymbol{l}''(\rho),$$

which is a linear combination of terms of the form $\boldsymbol{h}_\rho$ times

$$\rho^2, \quad \rho_{,x_i}\rho_{,x_j}, \quad u^k_{,x_i}\rho_{,x_j}, \quad \rho_{,x_i}\rho_{,x_j x_k}. \tag{35}$$

Neglecting all the nonlinear terms, one is thus lead to the study of the the optimal regularity properties of the linearized problem

$$\begin{cases} \boldsymbol{u}_{,t} - \nu\Delta\boldsymbol{u} + \nabla q - \boldsymbol{\Phi}_1(\boldsymbol{u}, \rho) = \boldsymbol{f} & \text{in} \quad \Omega_b, \\ \nabla\cdot\boldsymbol{u} - \Phi_2(\rho) = h = \nabla\cdot\boldsymbol{F} & \text{in} \quad \Omega_b, \\ \mathbb{T}(\boldsymbol{u}, q)\boldsymbol{N} + \sigma L\rho\boldsymbol{N} - \boldsymbol{\Phi}_3(\rho) = \boldsymbol{d} & \text{on} \quad \mathcal{G}, \\ \rho_{,t} + \nabla'\phi_b\cdot\boldsymbol{u} - u^3 + \nabla'\rho\cdot\boldsymbol{v}_b = g & \text{on} \quad \mathcal{G}, \\ \boldsymbol{u} = 0 & \text{on} \quad \Sigma \quad \text{for all} \quad t \geqslant 0, \\ \boldsymbol{u}(x,0) = \boldsymbol{u}_0(x), \quad \rho(x',0) = \rho_0(x') & \text{for} \quad x\in\Omega_b, \quad x'\in\Sigma, \end{cases} \tag{36}$$

with suitable regularity conditions on the right-hand terms and compatibility conditions on $\boldsymbol{u}_0$, ρ_0. Here Φ_i are given in (23), (29), and (34), and L in (33).

Using (29), an explicit calculation shows that for any solution of (36),

$$\frac{d}{dt}\int_\Sigma \rho\, dx' = \int_{\Omega_b} h\, dx + \int_\Sigma g\, dx'$$

holds, therefore we have preservation of mass for the linear problem when the right-hand side is identically zero. As previously noted, the right-hand side above vanishes for $h = l_2(\boldsymbol{u}, \rho)$ and $g = l_4(\boldsymbol{u}, \rho)$ whenever $\int_\Sigma u^3\,dx' \equiv 0$, which is clearly the case.

Using Laplace transform, we consider the associated complex parameter dependent problem

$$\begin{cases} \lambda \boldsymbol{u} - \nu\Delta\boldsymbol{u} + \nabla q - \widehat{\boldsymbol{\Phi}}_1(\boldsymbol{u},\rho) = \boldsymbol{f} & \text{in} \quad \Omega_b, \\ \nabla\cdot\boldsymbol{u} - \Phi_2(\rho) = h = \nabla\cdot\boldsymbol{F} & \text{in} \quad \Omega_b, \\ \mathbb{T}(\boldsymbol{u},q)\boldsymbol{N} + \sigma L\rho\boldsymbol{N} - \boldsymbol{\Phi}_3(\rho) = \boldsymbol{d} & \text{on} \quad \mathcal{G}, \\ \lambda\rho + \nabla'\phi_b\cdot\boldsymbol{u} - u^3 + \nabla'\rho\cdot\boldsymbol{v}_b = g & \text{on} \quad \mathcal{G}, \\ \boldsymbol{u} = 0 & \text{on} \quad \Sigma, \end{cases} \tag{37}$$

where $\widehat{\boldsymbol{\Phi}}_1$ is given as in (23) substituting the term $\rho^*_{,t}$ with $\lambda\rho^*$.

The reason we keep separated the linear operator L given in (33) from $\boldsymbol{\Phi}_3$ is apparent with the following lemma.

Lemma 2.1. *The bilinear form*

$$B_s(\rho) = \int_\Sigma L\rho(s\rho + \nabla'\rho\cdot\boldsymbol{v}_b)\,dx',$$

is positive definite for sufficiently large real s, (depending on ϕ_b and $\boldsymbol{v}_b$).

Proof. A straightforward calculation shows that, summing for $\alpha,\beta = 1,2$,

$$L\rho = -\frac{1}{g_b}\partial_\alpha\left[\left(\delta_{\alpha\beta}\sqrt{g_b} - \frac{\phi_{b,\alpha}\phi_{b,\beta}}{\sqrt{g_b}}\right)\rho_{,\beta}\right] - \left(\delta_{\alpha\beta}\sqrt{g_b} - \frac{\phi_{b,\alpha}\phi_{b,\beta}}{\sqrt{g_b}}\right)\rho_{,\beta}\partial_\alpha\frac{1}{g_b}.$$

We integrate by parts one derivative in the Laplace–Beltrami operator: by periodicity there is no boundary term and by the previous formula the terms in $\partial_\alpha(1/g_b)$ cancel out, giving

$$\int_\Sigma L\rho\cdot s\rho\,dx' = s\int_\Sigma \frac{1}{g_b}\left(\delta_{\alpha\beta}\sqrt{g_b} - \frac{\phi_{b,\alpha}\phi_{b,\beta}}{\sqrt{g_b}}\right)\rho_{,\beta}\rho_{,\alpha}\,dx'.$$

From Schwartz inequality, one immediately obtains

$$\frac{1}{g_b}\left(\delta_{\alpha\beta}\sqrt{g_b} - \frac{\phi_{b,\alpha}\phi_{b,\beta}}{\sqrt{g_b}}\right)\rho_{,\beta}\rho_{,\alpha} = \frac{|\nabla'\rho|^2(1+|\nabla'\phi_b|^2) - (\nabla'\rho\cdot\nabla'\phi_b)^2}{g_b^{\frac{3}{2}}} \geqslant \frac{|\nabla'\rho|^2}{g_b^{\frac{3}{2}}},$$

and thus

$$\int_\Sigma L\rho \cdot s\rho \, dx' \geqslant s \int_\Sigma \frac{|\nabla'\rho|^2}{g_b^{\frac{3}{2}}} \, dx' \geqslant cs \int_\Sigma |\nabla'\rho|^2 \, dx'.$$

Similarly, for the other term we have

$$\int_\Sigma L\rho \nabla'\rho \cdot \boldsymbol{v}_b \, dx' = \int_\Sigma \frac{1}{g_b} \left(\delta_{\alpha\beta}\sqrt{g} - \frac{\phi_{b,\alpha}\phi_{b,\beta}}{\sqrt{g_b}} \right) \rho_{,\beta} \left(\rho_{,\alpha\gamma} v_b^\gamma + \rho_{,\gamma} v_{b,\alpha}^\gamma \right) dx'.$$

Clearly

$$\frac{1}{g_b} \left(\delta_{\alpha\beta}\sqrt{g_b} - \frac{\phi_{b,\alpha}\phi_{b,\beta}}{\sqrt{g_b}} \right) \rho_{,\beta}\rho_{,\gamma} v_{b,\alpha}^\gamma \geqslant -c'|\nabla'\rho|^2,$$

with a constant depending on ϕ_b and $\boldsymbol{v}_b$. It remains to estimate

$$\int_\Sigma \frac{1}{g_b} \left(\delta_{\alpha\beta}\sqrt{g_b} - \frac{\phi_{b,\alpha}\phi_{b,\beta}}{\sqrt{g_b}} \right) \rho_{,\beta}\rho_{,\alpha\gamma} v_b^\gamma \, dx',$$

but since this expression is symmetric in α and β, integrating by parts on the term $\rho_{,\alpha\gamma}$ with respect to x_γ gives

$$2\int_\Sigma \left(\delta_{\alpha\beta}\sqrt{g_b} - \frac{\phi_{b,\alpha}\phi_{b,\beta}}{\sqrt{g_b}} \right) \frac{\rho_{,\beta}\rho_{,\alpha\gamma} v_b^\gamma}{g_b} \, dx' =$$

$$= -\int_\Sigma \rho_{,\alpha}\rho_{,\beta}\partial_\gamma \left[\left(\delta_{\alpha\beta}\sqrt{g_b} - \frac{\phi_{b,\alpha}\phi_{b,\beta}}{\sqrt{g_b}} \right) \frac{v_b^\gamma}{g_b} \right] dx' \geqslant -c' \int_\Sigma |\nabla'\rho|^2.$$

The claim follows, since gathering together the previous estimates gives

$$B_s(\rho) \geqslant (cs - c') \int_\Sigma |\nabla'\rho|^2 \, dx'. \tag{38}$$

□

§3. Model problems in the half-space

In this section, we study the model problems in the half space arising from the system

$$\begin{cases} \lambda \boldsymbol{u} - \nu \Delta \boldsymbol{u} + \nabla q = \boldsymbol{f} & \text{in} \quad \Omega, \\ \nabla \cdot \boldsymbol{u} = h = \nabla \cdot \boldsymbol{F} & \text{in} \quad \Omega, \\ \mathbb{T}(\boldsymbol{u}, p)\boldsymbol{N} + \sigma L\rho \boldsymbol{N} = \boldsymbol{d} & \text{on} \quad \mathcal{G}, \\ \lambda\rho + \nabla'\phi \cdot \boldsymbol{u} - u^3 + \boldsymbol{v}_b \cdot \nabla'\rho = g & \text{on} \quad \mathcal{G}, \\ \boldsymbol{u} = \boldsymbol{a} & \text{on} \quad \Sigma \end{cases} \tag{39}$$

obtained neglecting the lower order terms in (37).

The first one has been treated in [11] and is defined as

$$\begin{cases} \lambda \boldsymbol{u} - \nu \Delta \boldsymbol{u} + \nabla q = \boldsymbol{f} & \text{in} \quad \mathbb{R}^3_+, \\ \nabla \cdot \boldsymbol{u} = h := \nabla \cdot \boldsymbol{F} + h' & \text{in} \quad \mathbb{R}^3_+, \\ \nu\big(u^3_{,x_j} + u^j_{,x_3}\big) = d^j, \quad j = 1,2 & \text{in} \quad \mathbb{R}^2, \\ -q + 2\nu u^3_{,x_3} - \sigma \Delta' \rho = d^3 & \text{in} \quad \mathbb{R}^2, \\ \lambda \rho + \boldsymbol{V} \cdot \nabla' \rho + u^3 = g & \text{in} \quad \mathbb{R}^2, \end{cases} \tag{40}$$

where $\mathbb{R}^2 \subset \mathbb{R}^3_+$ as $\{x_3 = 0\}$ and primed variables and differential operators are to be meant in $\mathbb{R}^2$.

We set $\Sigma^\infty = \Sigma \times [0, +\infty)$, and consider first an auxiliary problem.

Theorem 3.1. *Let $l \geqslant 0$, and $\boldsymbol{V}' = (V_1, V_2)$ a constant vector. For sufficiently large $\operatorname{Re} \lambda$ there exists a unique Σ-periodic solution of*

$$\begin{cases} \lambda \boldsymbol{u} + (\boldsymbol{V}' \cdot \nabla')\boldsymbol{u} - \nu \Delta \boldsymbol{u} + \nabla q = 0 & \textit{in} \quad \mathbb{R}^3_+, \\ \nabla \cdot \boldsymbol{u} = 0 & \textit{in} \quad \mathbb{R}^3_+, \\ \nu\big(u^3_{,x_j} + u^i_{,x_3}\big) = d^j, \quad j = 1,2 & \textit{in} \quad \mathbb{R}^2, \\ -q + 2\nu u^3_{,x_3} - \sigma \Delta' \rho = d^3, & \textit{in} \quad \mathbb{R}^2, \\ \lambda \rho + \boldsymbol{V}' \cdot \nabla' \rho + u^3 = g, & \textit{in} \quad \mathbb{R}^2, \end{cases} \tag{41}$$

such that $\boldsymbol{u} \to 0$ and $q \to c$ for $x_3 \to +\infty$. It satisfies the estimates

$$\begin{aligned} &\|\boldsymbol{u}\|^2_{H^{l+2}_\lambda(\Sigma^\infty)} + \|\nabla q\|^2_{H^l_\lambda(\Sigma^\infty)} + \|q(0)\|^2_{H^{l+\frac12}_\lambda(\Sigma)} \\ &\quad + \|\rho\|^2_{H^{l+\frac52}_\lambda(\Sigma)} + \|\lambda \rho\|^2_{H^{l+\frac32}_\lambda(\Sigma)} \leqslant c\Big(\|\boldsymbol{d}\|^2_{H^{l+\frac12}_\lambda(\Sigma)} + \|g\|^2_{H^{l+\frac32}_\lambda(\Sigma)}\Big), \end{aligned} \tag{42}$$

$$\int\limits_{\{x_3 = s\}} q \, dx' = -\int\limits_{\Sigma} d^3 \, dx' =: \overline{d}, \quad \forall s \geqslant 0, \tag{43}$$

$$\|q - \overline{d}\|^2_{L^2(\Sigma^\infty)} \leqslant c\Big(\|\boldsymbol{d}\|^2_{W_2^{-\frac12}(\Sigma)} + \|g\|^2_{W_2^{\frac12}(\Sigma)}\Big). \tag{44}$$

Proof. The proof of the first estimate is a straightforward modification of the one in [11], using Fourier series instead of Fourier transforms, and a slightly different norm. For any $\xi \in \mathbb{Z}^2$, let $\boldsymbol{u}_\xi$, p_ξ, and ρ_ξ be the ξth Fourier coefficient with respect to (x_1, x_2) of $\boldsymbol{u}$, p, and ρ, respectively. System (41) is then reduced to

$$\begin{cases} \nu\left(r_1^2-\dfrac{d^2}{dx_3^2}\right)u_\xi^j+i\xi_j q_\xi=0 & \text{for}\quad j=1,2,\quad x_3>0,\\ \nu\left(r_1^2-\dfrac{d^2}{dx_3^2}\right)u_\xi^3+\dfrac{dq_\xi}{dx_3}=0 & \text{for}\quad x_3>0,\\ i\xi_1u_\xi^1+i\xi_2u_\xi+\dfrac{du_\xi^3}{dx_3}=0 & \text{for}\quad x_3>0,\\ \nu\left(\dfrac{du^j}{dx_3}+i\xi_ju_\xi^3\right)=d_\xi^j & \text{for}\quad j=1,2,\quad x_3=0,\\ -q_\xi+2\nu\dfrac{du_\xi^3}{dx_3}+\sigma|\xi|^2\rho_\xi=d_\xi^3 & \text{for}\quad x_3=0,\\ \lambda_1\rho_\xi+u_\xi^3=g_\xi & \text{for}\quad x_3=0,\\ \boldsymbol{u}_\xi\to0,\quad q_\xi\to c & \text{for}\quad x_3\to+\infty, \end{cases}$$

where $r_1=r_1(\lambda,\xi)=\sqrt{\lambda_1\nu^{-1}+|\xi|^2}$, $-\pi\leqslant\operatorname{Arg}(r_1)<\pi$, $\lambda_1=\lambda+i\boldsymbol{V}'\cdot\xi$. This system of ODE can be explicitly solved for $\operatorname{Re}\lambda>0$ as

$$u_\xi^i=-\frac{1-\delta_{i3}}{\nu r_1}e_0(x_3)d_\xi^i+\frac{e_0(x_3)}{\nu^2r_1(r_1+|\xi|)P}\sum_{j=1}^3U_{ij}d_\xi^j$$
$$+\frac{e_1(x_3)}{\nu^2(r_1+|\xi|)P}\sum_{j=1}^3V_{ij}d_\xi^j-\frac{\sigma|\xi|^2(e_0(x_3)U_{i3}+r_1e_1(x_3)V_{i3})}{\nu^2\lambda_1r_1(r_1+|\xi|)P}g_\xi,\qquad i=1,2,3,$$

$$q_\xi=\frac{r_1\lambda_1}{\nu^2P}\left[\left(2\nu+\frac{\sigma|\xi|^2}{r_1\lambda_1}\right)(i\xi_1d_\xi^1+i\xi_2d_\xi^2)\right.$$
$$\left.-\nu\left(r_1+\frac{|\xi|^2}{r_1}\right)\left(d_\xi^3-\frac{\sigma}{\lambda_1}|\xi|^2g_\xi\right)\right]e^{-|\xi|x_3},$$

$$\rho_\xi=(g_\xi-u_\xi^3)/\lambda_1,$$

where

$$e_0(x_3)=e^{-r_1x_3},\quad e_1(x_3)=\frac{e^{-r_1x_3}-e^{-|\xi|x_3}}{r_1-|\xi|},$$

$$P = (r_1^2 + |\xi|^2)^2 - 4r_1|\xi|^2 + \frac{\sigma}{\nu^2}|\xi|^3 = \frac{\lambda_1}{\nu}\left(\frac{\lambda_1}{\nu} + 4|\xi|^2\left(1 - \frac{|\xi|}{r_1 + |\xi|}\right) + \frac{\sigma|\xi|^3}{\nu\lambda_1}\right),$$

and U_{ij}, V_{ij} are the elements of the matrices

$$\begin{pmatrix} \xi_1^2((3r_1-|\xi|)\lambda_1+\frac{\sigma}{\nu}|\xi|^2) & \xi_1\xi_2((3r_1-|\xi|)\lambda_1+\frac{\sigma}{\nu}|\xi|^2) & i\xi_1 r_1\lambda_1(r_1-|\xi|) \\ \xi_1\xi_2((3r_1-|\xi|)\lambda_1+\frac{\sigma}{\nu}|\xi|^2) & \xi_2^2((3r_1-|\xi|)\lambda_1+\frac{\sigma}{\nu}|\xi|^2) & i\xi_2 r_1\lambda_1(r_1-|\xi|) \\ -i\xi_1 r_1\lambda_1(r_1-|\xi|) & -i\xi_2 r_1\lambda_1(r_1-|\xi|) & -|\xi| r_1\lambda_1(r_1-|\xi|) \end{pmatrix},$$

and

$$\begin{pmatrix} -\xi_1^2(2r_1\lambda_1+\frac{\sigma}{\nu}|\xi|^2) & -\xi_1\xi_2(2r_1\lambda_1+\frac{\sigma}{\nu}|\xi|^2) & -i\xi_1\lambda_1(r_1^2+|\xi|^2) \\ -\xi_1\xi_2(2r_1\lambda_1+\frac{\sigma}{\nu}|\xi|^2) & -\xi_2^2(2r_1\lambda_1+\frac{\sigma}{\nu}|\xi|^2) & -i\xi_2\lambda_1(r_1^2+|\xi|^2) \\ -i\xi_1|\xi|(2r_1\lambda_1+\frac{\sigma}{\nu}|\xi|^2) & -i\xi_2|\xi|(2r_1\lambda_1+\frac{\sigma}{\nu}|\xi|^2) & |\xi|\lambda_1(r_1^2+|\xi|^2) \end{pmatrix},$$

respectively. For the constant mode $\xi = (0,0)$ this reduces to

$$u_0^i(x_3) = -\frac{d_0^i}{\nu\sqrt{\lambda}}e^{-\sqrt{\lambda}x_3}, \quad i = 1,2, \quad u_0^3 \equiv 0, \quad q_0 \equiv -d_0^3, \quad \rho_0 = \frac{g_0}{\lambda},$$

and thus $q \to -d_0^3$ for $x_3 \to +\infty$. If $\gamma > |\boldsymbol{V}'|^2/\nu$ and $\operatorname{Re}\lambda \geqslant \gamma$, it holds

$$\frac{1}{c}|r_1| \leqslant \frac{1}{2}\Big(\sqrt{|\lambda|} + |\xi|\Big) \leqslant \sqrt{|\lambda| + |\xi|^2} \leqslant \sqrt{|\lambda|} + |\xi| \leqslant c|r_1|, \tag{45}$$

and the same estimate for λ_1. Moreover $|r_1 + |\xi|| \geqslant \max\{|r_1|, |\xi|\}$, and

$$|P|^2 \geqslant c(\gamma)\Big(|\xi|^6 + |\xi|^4|\lambda_1|^2 + |\xi|^2|\lambda_1|^3 + |\lambda_1|^4\Big). \tag{46}$$

The principal parts of the norms of e_i on $[0,+\infty)$ are estimated as

$$\|e_0\|^2_{\mathring{W}_2^\eta([0,+\infty))} \leqslant c|r_1|^{2\eta-1},$$
$$\|e_1\|^2_{\mathring{W}_2^\eta([0,+\infty))} \leqslant c\frac{|r_1|^{2\eta-1} + |\xi|^{2\eta-1}}{|r_1|^2}$$

for any $\eta \geqslant 0$. Finally for $\xi \in \mathbb{Z}^2$ and $\operatorname{Re}\lambda \geqslant \gamma$, it holds

$$\begin{gathered} |U_{ij}|^2 + |V_{ij}|^2 \leqslant c(|\xi|^2|\lambda_1|^4 + |\xi|^4|\lambda_1|^3 + |\xi|^6|\lambda_1|^2 + |\xi|^8), \\ |U_{i3}|^2 + |U_{3i}|^2 + |V_{i3}|^2 \leqslant c(|\xi|^2|\lambda_1|^4 + |\xi|^6|\lambda_1|^2). \end{gathered} \tag{47}$$

From these inequalities one gets the following estimates for $\boldsymbol{u}_\xi$:

$$\begin{gathered} |r_1|^{2(l+2)}\|\boldsymbol{u}_\xi\|^2_{L^2(\mathbb{R}_+)} \leqslant c|r_1|^{2l+1}(|\boldsymbol{d}_\xi|^2 + |\xi|^2|g_\xi|^2), \\ \|\boldsymbol{u}_\xi\|^2_{\mathring{W}_2^{l+2}(\mathbb{R}_+)} \leqslant c|r_1|^{2l+1}(|\boldsymbol{d}_\xi|^2 + |\xi|^2|g_\xi|^2). \end{gathered} \tag{48}$$

One can similarly estimate the pressure, considering the cases $|\lambda_1| \leqslant |\xi|^2$, which implies $|r_1| \leqslant c|\xi|$, and $|\lambda_1| > |\xi|^2$, which implies $|r_1| \leqslant c\sqrt{|\lambda_1|}$. Supposing $\xi \neq (0,0)$ and noting that $\|e^{-|\xi|x_3}\|_{\mathring{W}_2^l(\mathbb{R}_+)} \leqslant c|\xi|^{2l-1}$ one obtains

$$\begin{aligned} &|r_1|^{2l}|\xi|^2\|q_\xi\|^2_{L^2(\mathbb{R}_+)} \leqslant c|r_1|^{2l+1}(|\boldsymbol{d}_\xi|^2 + |\xi|^2|g_\xi|^2),\\ &\|\frac{dq_\xi}{dx_3}\|^2_{\mathring{W}_2^l(\mathbb{R}_+)} \leqslant c|r_1|^{2l}|\xi|^2\|q_\xi\|^2_{L^2(\mathbb{R}_+)} \leqslant c|r_1|^{2l+1}(|\boldsymbol{d}_\xi|^2 + |\xi|^2|g_\xi|^2). \end{aligned} \tag{49}$$

Summing in $\xi \in \mathbb{Z}^2$ inequalities (48), (49) and using (45), we get through Parceval identity

$$\|\boldsymbol{u}\|^2_{H_\lambda^{l+2}(\Sigma^\infty)} + \|\nabla q\|^2_{H_\lambda^l(\Sigma^\infty)} \leqslant c\Big(\gamma)(\|\boldsymbol{d}\|^2_{H_\lambda^{l+\frac12}(\Sigma)} + \|g\|^2_{W_2^{l+\frac32}(\Sigma)} + |\lambda|^{l+\frac12}\|g\|^2_{W_2^1(\Sigma)}\Big).$$

To estimate q at $x_3 = 0$, one has, with the same method

$$|r_1|^{2l+1}|q_\xi(0)|^2 \leqslant c|r_1|^{2l+1}(|\boldsymbol{d}_\xi|^2 + (1+|\xi|^2)|g_\xi|^2),$$

which gives

$$\|q(0)\|^2_{H_\lambda^{l+\frac12}(\Sigma)} \leqslant c(\gamma)\Big(\|\boldsymbol{d}\|^2_{H_\lambda^{l+\frac12}(\Sigma)} + \|g\|^2_{W_2^{l+\frac32}(\Sigma)} + |\lambda|^{l+\frac12}\|g\|^2_{W_2^1(\Sigma)}\Big).$$

So far we have obtained the estimate

$$\begin{aligned} &\|\boldsymbol{u}\|^2_{H_\lambda^{l+2}(\Sigma^\infty)} + \|\nabla q\|^2_{H_\lambda^l(\Sigma^\infty)} + \|q(0)\|^2_{H_\lambda^{l+\frac12}(\Sigma)}\\ &\qquad \leqslant c(\gamma)\Big(\|\boldsymbol{d}\|^2_{H_\lambda^{l+\frac12}(\Sigma)} + \|g\|^2_{W_2^{l+\frac32}(\Sigma)} + |\lambda|^{l+\frac12}\|g\|^2_{W_2^1(\Sigma)}\Big), \end{aligned}$$

and thus, by interpolation

$$\begin{aligned} &\|\boldsymbol{u}\|^2_{H_\lambda^{l+2}(\Sigma^\infty)} + \|\nabla q\|^2_{H_\lambda^l(\Sigma^\infty)} + \|q(0)\|^2_{H_\lambda^{l+\frac12}(\Sigma)}\\ &\qquad \leqslant c(\gamma)\Big(\|\boldsymbol{d}\|^2_{H_\lambda^{l+\frac12}(\Sigma)} + \|g\|^2_{H_\lambda^{l+\frac32}(\Sigma)}\Big). \end{aligned}$$

We now estimate ρ_ξ, using the relations

$$\begin{aligned} &\lambda_1\rho_\xi = g_\xi - u_\xi^3(0),\\ &\sigma|\xi|^2\rho_\xi = d_\xi^3 + q_\xi(0) - 2\nu\frac{du_\xi^3(0)}{dx_3} = d_\xi^3 + q_\xi(0) + 2\nu\big(i\xi_1 u_\xi^1 + i\xi_2 u_\xi^2\big). \end{aligned}$$

Since $\operatorname{Re}\lambda$ is supposed to be large, it suffices to obtain the estimates

$$\|\rho\|^2_{W_2^{l+\frac{5}{2}}(\Sigma)} \leqslant c\Big(\|\boldsymbol{d}\|^2_{W_2^{l+\frac{1}{2}}(\Sigma)} + \|g\|^2_{W_2^{l+\frac{3}{2}}(\Sigma)}\Big), \tag{50}$$

$$\|\lambda\rho\|^2_{W_2^{l+\frac{3}{2}}(\Sigma)} \leqslant c\Big(\|\boldsymbol{d}\|^2_{W_2^{l+\frac{1}{2}}(\Sigma)} + \|g\|^2_{W_2^{l+\frac{3}{2}}(\Sigma)}\Big), \tag{51}$$

$$|\lambda|^{l+\frac{3}{2}}\|\lambda\rho\|^2_{L^2(\Sigma)} \leqslant c\Big(|\lambda|^{l+\frac{1}{2}}\|\boldsymbol{d}\|^2_{L^2(\Sigma)} + |\lambda|^{l+\frac{3}{2}}\|g\|^2_{L^2(\Sigma)}\Big). \tag{52}$$

From the explicit formula for $\boldsymbol{u}_\xi$ and the bounds (46), (47), we obtain

$$\begin{aligned} |\lambda_1|^2|\rho_\xi|^2 &\leqslant c\bigg(\frac{|\boldsymbol{d}_\xi|^2}{|r_1|^2} + |g_\xi|^2\bigg), \\ |\xi|^4|\rho_\xi|^2 &\leqslant c\Big(|\boldsymbol{d}_\xi|^2 + (1+|\xi|^2)|g_\xi|^2\Big). \end{aligned} \tag{53}$$

Since $|\lambda_1| \geqslant c|\lambda| \geqslant c\gamma$, and recalling (45), we get from the first one

$$|\lambda|^{l+\frac{3}{2}}|\lambda\rho_\xi|^2 \leqslant c\Big(|\lambda|^{l+\frac{1}{2}}|\boldsymbol{d}_\xi|^2 + |\lambda|^{l+\frac{3}{2}}|g_\xi|^2\Big),$$

which, summed on $\xi \in \mathbb{Z}^2$, gives (52). By the first inequality (53) and $|r_1| \geqslant c|\xi|$,

$$|\xi|^{2l+3}|\lambda\rho_\xi|^2 \leqslant c\Big(|\xi|^{2l+1}|\boldsymbol{d}_\xi|^2 + |\xi|^{2l+3}|g_\xi|^2\Big),$$

which gives, together with $|\rho_\xi|^2 \leqslant c(|\boldsymbol{d}_\xi|^2 + |g_\xi|^2)$, inequality (51). Finally (50) is given by the second inequality in (53), which implies

$$|\xi|^{2l+5}|\rho_\xi|^2 \leqslant c\Big(|\xi|^{2l+1}|\boldsymbol{d}_\xi|^2 + |\xi|^{2l+3}|g_\xi|^2\Big).$$

We conclude estimating $q - d_0^3$ on Σ^∞: for any $\xi \neq 0$ we have

$$\|q_\xi\|^2_{L^2(\mathbb{R}_+)} = \frac{1}{2|\xi|}|q_\xi|^2 \leqslant \frac{c}{|\xi|}\Big(|\boldsymbol{d}_\xi|^2 + (1+|\xi|^2)|g_\xi|^2\Big),$$

which gives (44), summing over $\xi \in \mathbb{Z}\backslash\{0\}$ and recalling that $q_0 = -d_0^3$. $\square$

We now consider the full model problem (40).

Theorem 3.2. *Let $l \geqslant 0$ and $\boldsymbol{V}' = (V_1, V_2)$ a constant vector. Suppose h decays for $x_3 \to +\infty$ sufficiently rapidly and h' is compactly supported in*

x_3. *For sufficiently large* $\operatorname{Re}\lambda$, *there is a unique periodic solution to* (40), *with* $\boldsymbol{u} \to 0$ *and* $q \to c$ *for* $x_3 \to +\infty$ *and*

$$\|\boldsymbol{u}\|^2_{H^{l+2}_\lambda(\Sigma^\infty)} + \|\nabla q\|^2_{H^l_\lambda(\Sigma^\infty)} + \|q\|^2_{H^{l+\frac{1}{2}}_\lambda(\Sigma)} + \|\rho\|^2_{H^{l+\frac{5}{2}}_\lambda(\Sigma)} + \|\lambda\rho\|^2_{H^{l+\frac{3}{2}}_\lambda(\Sigma)}$$
$$\leqslant c\Big(\|\boldsymbol{f}\|^2_{H^l_\lambda(\Sigma^\infty)} + |\lambda|^{l+2}\|\boldsymbol{F}\|^2_{L^2(\Sigma^\infty)} + |\lambda|^{l+2}\|h'\|^2_{L^2(\Sigma^\infty)}$$
$$+ \|h\|^2_{W_2^{l+1}(\Sigma^\infty)} + \|\boldsymbol{d}\|^2_{H^{l+\frac{1}{2}}_\lambda(\Sigma)} + \|g\|^2_{W_2^{l+\frac{3}{2}}(\Sigma)} + |\lambda|^{l+\frac{1}{2}}\|g\|^2_{W_2^1(\Sigma)}\Big). \quad (54)$$

Proof. Let us call $X(\boldsymbol{u}, q, \rho)$ the left-hand side of (54) and $Y(\boldsymbol{f}, \boldsymbol{d}, g, h, \boldsymbol{F}, h')$ the right-hand side. First of all we solve the corresponding problem with solenoidal velocity, i.e., we consider the case $h = 0$. To this end, consider the problem

$$\begin{cases} \lambda\boldsymbol{u} + (\boldsymbol{V}'\cdot\nabla')\boldsymbol{u} - \nu\Delta\boldsymbol{u} + \nabla q = \boldsymbol{f} & \text{in } \mathbb{R}^3_+, \\ \nabla\cdot\boldsymbol{u} = 0 & \text{in } \mathbb{R}^3_+, \\ \nu\big(u^3_{,x_j} + u^j_{,x_3}\big) = d^j, \quad j = 1,2 & \text{in } \mathbb{R}^2, \\ -q + 2\nu u^3_{,x_3} - \sigma\Delta'\rho = d^3 & \text{in } \mathbb{R}^2, \\ \lambda\rho + \boldsymbol{V}'\cdot\nabla'\rho + u^3 = g & \text{in } \mathbb{R}^2. \end{cases} \quad (55)$$

Notice that the solution of

$$\begin{cases} \Delta p = \nabla\cdot\boldsymbol{f} & \text{in } \Sigma^\infty, \\ p = 0 & \text{if } x_3 = 0, \end{cases}$$

satisfies $\|\nabla p\|_{H^l_\lambda(\Sigma^\infty)} \leqslant c\|\boldsymbol{f}\|_{H^l_\lambda(\Sigma^\infty)}$. Set then $\widehat{\boldsymbol{f}} = \boldsymbol{f} - \nabla p$, for which it holds $\nabla\cdot\widehat{\boldsymbol{f}} = 0$: we extend $\widehat{\boldsymbol{f}}$ to the whole space with preservation of class, periodicity and solenoidality, and define for $\xi \in \mathbb{Z}^2$ and $x_3 \in \mathbb{R}$ the Fourier coefficients

$$\boldsymbol{v}_\xi(x_3) = \frac{\widehat{\boldsymbol{f}}_\xi(x_3)}{\lambda_1 + \nu|\xi|^2}.$$

For $\operatorname{Re}\lambda > |\boldsymbol{V}'|/\nu$, it holds $(1+|\xi|^2)^2 + |\lambda|^2 \leqslant c|\lambda_1 + |\xi|^2|^2$ and thus these coefficients define a Σ-periodic solution in $\mathbb{R}^3$ to

$$\lambda\boldsymbol{v} - \nu\Delta\boldsymbol{v} + (\boldsymbol{V}'\cdot\nabla')\boldsymbol{v} = \widehat{\boldsymbol{f}}, \quad \nabla\cdot\boldsymbol{v} = 0,$$

satisfying $\|\boldsymbol{v}\|_{H^{l+2}_\lambda(\Sigma\times\mathbb{R})} \leqslant c\|\boldsymbol{f}\|_{H^l_\lambda(\Sigma^\infty)}$. Thus, solving (55) in $(\boldsymbol{u}, q, \rho)$ is equivalent to solving (41) in $(\boldsymbol{u}-\boldsymbol{v}, q-p, \rho)$, since, by standard restriction theorems, the right-hand sides in the latter case are modified by terms whose corresponding norms are bounded by $\|\boldsymbol{f}\|_{H^l_\lambda(\Sigma^\infty)}$.

We now can get rid of the term $(\boldsymbol{V}'\cdot\nabla')\boldsymbol{u}$ in the equation for the velocity by a standard iteration argument, defining $(\boldsymbol{u}_1, q_1, \rho_1)$ as the solution to (55), and $(\boldsymbol{u}_{n+1}, q_{n+1}, \rho_{n+1})$ as the solution to (55) with the right-hand side $\boldsymbol{f} + (\boldsymbol{V}'\cdot\nabla')\boldsymbol{u}_n$ of the velocity equation. If

$$(\boldsymbol{w}_n, p_n, \mu_n) := (\boldsymbol{u}_n - \boldsymbol{u}_{n-1}, q_n - q_{n-1}, \rho_n - \rho_{n-1}),$$

then we note that $(\boldsymbol{w}_{n+1}, p_{n+1}, \mu_{n+1})$ satisfies (55) with the right-hand side $(\boldsymbol{V}'\cdot\nabla')\boldsymbol{w}_n$ on the velocity equation and zero elsewhere. From the interpolation inequality

$$\|(\boldsymbol{V}'\cdot\nabla')\boldsymbol{w}_n\|^2_{H^l_\lambda(\Sigma^\infty)} \leqslant \frac{c}{|\lambda|}\|\boldsymbol{w}_n\|^2_{H^{l+2}_\lambda(\Sigma^\infty)},$$

and the estimate (54) for problem (55), we get that

$$X(\boldsymbol{w}_{n+1}, p_{n+1}, \mu_{n+1}) \leqslant \frac{c}{|\lambda|}X(\boldsymbol{w}_n, p_n, \mu_n),$$

which in turn gives, for $c/|\lambda| < 1$, strong convergence of the sequence $(\boldsymbol{u}_n, q_n, \rho_n)$ to a solution of

$$\begin{cases} \lambda\boldsymbol{u} - \nu\Delta\boldsymbol{u} + \nabla q = \boldsymbol{f} & \text{in} \quad \mathbb{R}^3_+, \\ \nabla\cdot\boldsymbol{u} = 0 & \text{in} \quad \mathbb{R}^3_+, \\ \nu\big(u^3_{,x_j} + u^j_{,x_3}\big) = d^j, \quad j = 1,2 & \text{in} \quad \mathbb{R}^2, \\ -q + 2\nu u^3_{,x_3} - \sigma\Delta'\rho = d^3 & \text{in} \quad \mathbb{R}^2, \\ \lambda\rho + \boldsymbol{V}'\cdot\nabla'\rho + u^3 = g & \text{in} \quad \mathbb{R}^2, \end{cases} \tag{56}$$

and the estimate $X(\boldsymbol{u}, q, \rho) \leqslant cY(\boldsymbol{f}, \boldsymbol{d}, g, 0, 0, 0)$.

We finally take care of the divergence term, defining $\boldsymbol{w} = \nabla\psi$, where ψ is the stable periodic solution of

$$\begin{cases} \Delta\psi = h = \nabla\cdot\boldsymbol{F} + h' & \text{in} \quad \mathbb{R}^3_+, \\ \psi = 0 & \text{on} \quad \mathbb{R}^2. \end{cases} \tag{57}$$

It satisfies

$$\|\boldsymbol{w}\|^2_{H^{l+2}_\lambda(\Sigma^\infty)} \leqslant c\Big(\|h\|^2_{W^{l+1}_2(\Sigma^\infty)} + |\lambda|^{l+2}\Big(\|\boldsymbol{F}\|^2_{L^2(\Sigma^\infty)} + \|h'\|^2_{L^2(\Sigma^\infty)}\Big)\Big). \tag{58}$$

The solution defined as $(\boldsymbol{w} + \boldsymbol{u}, q, \rho)$, where $(\boldsymbol{u}, q, \rho)$ solves (56) with modified the right-hand sides. Inequality (58) and standard restriction theorems ensure the validity of (54). □

The second model problem arises from the need of a correction in solenoidality, together with Dirichlet boundary conditions, on the bottom surface.

Theorem 3.3. *Let $l \geqslant 0$. Assume that $\boldsymbol{f} \in W_2^l(\Sigma^\infty)$, $\boldsymbol{a} = (a^1, a^2, 0) \in W_2^{l+\frac{3}{2}}(\Sigma)$, $h \in W^{l+1}(\Sigma^\infty)$, and $\boldsymbol{F}, h' \in L^2(\Sigma^\infty)$ are Σ-periodic, $h'(x) = 0$, whenever $x_3 \geqslant L$, $F^3 = 0$ for $x_3 = 0$ and $\int\limits_{\Sigma^\infty} h'\,dx = 0$. For any $\operatorname{Re}\lambda \geqslant \gamma > 0$, there is a unique Σ-periodic solution to*

$$\begin{cases} \lambda \boldsymbol{u} - \nu \Delta \boldsymbol{u} + \nabla q = \boldsymbol{f} & \text{in} \quad \mathbb{R}^3_+, \\ \nabla \cdot \boldsymbol{u} = h = \nabla \cdot \boldsymbol{F} + h' & \text{in} \quad \mathbb{R}^3_+, \\ \boldsymbol{u} = \boldsymbol{a} & \text{in} \quad \mathbb{R}^2; \end{cases} \tag{59}$$

it satisfies the estimate

$$\begin{aligned} &\|\boldsymbol{u}\|^2_{H_\lambda^{l+2}(\Sigma^\infty)} + \|\nabla p\|^2_{H_\lambda^l(\Sigma^\infty)} \leqslant c\Big(\|\boldsymbol{f}\|^2_{H_\lambda^l(\Sigma^\infty)} + \|\boldsymbol{a}\|^2_{H_\lambda^{l+\frac{3}{2}}(\Sigma)} \\ &+ \|h\|^2_{W_2^{l+1}(\Sigma^\infty)} + |\lambda|^{l+2}(\|\boldsymbol{F}\|^2_{L^2(\Sigma^\infty)} + \|h'\|^2_{L^2(\Sigma^\infty)})\Big). \end{aligned} \tag{60}$$

We omit the proof of this proposition, since in the case $h \equiv 0$, optimal regularity estimates for periodic solutions are proved, e.g., in [7], and one can always reduce (59) to a similar one with $h \equiv 0$ subtracting the gradient of the Neumann problem corresponding to (57). Estimate (58) still holds true (see the proof of Lemma 4.1 below), and thus gives (60).

§4. Parameter dependent linear problem

In this section, we prove the solvability and the coercive estimates for sufficiently large $\operatorname{Re}\lambda$, of problem (39), where $a^3 = F^3 = 0$ on Σ. The method of proof follows [1]. We start with a lemma which allows to extend the equation $h = \nabla \cdot \boldsymbol{F}$ from Ω to $\mathbb{R}^3_+$ controlling the norms.

Lemma 4.1. *Let $h, h' \in W_2^{l+1}(\Omega)$, $\boldsymbol{F} \in W_2^{l+2}(\Omega)$ be Σ-periodic and such that*

$$h = \nabla \cdot \boldsymbol{F} + h'$$

holds in Ω. There exist a Σ-periodic extensions $\overline{h}$ of h to $\mathbb{R}^3_+$ and an $\overline{\boldsymbol{F}} \in W_2^{l+2}(\Sigma^\infty)$ such that

$$\overline{h} = \nabla \cdot \overline{\boldsymbol{F}},$$

in $\mathbb{R}^3_+$, $\overline{h} = \overline{\boldsymbol{F}} = 0$ *for sufficiently large* x_3, $\overline{\boldsymbol{F}} \cdot \boldsymbol{n} = \boldsymbol{F} \cdot \boldsymbol{n}$ *on* Σ *and*

$$\begin{aligned} \|\overline{h}\|_{W_2^{l+1}(\Sigma^\infty)} &\leqslant c\|h\|_{W_2^{l+1}(\Omega)}, \\ \|\overline{\boldsymbol{F}}\|_{L^2(\Sigma^\infty)} &\leqslant c\Big(\|\boldsymbol{F}\|_{L^2(\Omega)} + \|h'\|_{L^2(\Omega)}\Big). \end{aligned} \tag{61}$$

Proof. Let ψ be the periodic solution of

$$\begin{cases} \Delta\psi = h = \nabla \cdot \boldsymbol{F} + h' & \text{in} \quad \Omega, \\ \psi = 0 & \text{on} \quad \mathcal{G}, \\ \frac{\partial \psi}{\partial \boldsymbol{n}} = \boldsymbol{F} \cdot \boldsymbol{n} & \text{on} \quad \Sigma. \end{cases} \tag{62}$$

Standard elliptic estimates guarantee that

$$\|\psi\|_{\mathring{W}_2^{l+3}(\Omega)} \leqslant c\|h\|_{W_2^{l+1}(\Omega)},$$

and the weak formulation of (62) reads

$$\int_\Omega \nabla\psi \cdot \nabla\eta \, dx = \int_\Omega \boldsymbol{F} \cdot \nabla\eta - h'\eta \, dx,$$

for all $\eta \in C^\infty(\overline{\Omega})$ such that $\eta\lfloor_{\mathcal{G}} = 0$, which gives

$$\|\nabla\psi\|^2_{L^2(\Omega)} \leqslant \|\boldsymbol{F}\|_{L^2(\Omega)} \|\nabla\psi\|_{L^2(\Omega)} + \|h'\|_{L^2(\Omega)} \|\psi\|_{L^2(\Omega)}. \tag{63}$$

Since $\psi = 0$ on $\mathcal{G}$, a form of Poincaré inequality gives

$$\|\psi\|_{L^2(\Omega)} \leqslant c\|\nabla\psi\|_{L^2(\Omega)},$$

and thus (63) becomes

$$\|\nabla\psi\|_{L^2(\Omega)} \leqslant c(\|\boldsymbol{F}\|_{L^2(\Omega)} + \|h'\|_{L^2(\Omega)}). \tag{64}$$

We now consider a vector field $\overline{\boldsymbol{F}}$ defined in the whole $\mathbb{R}^3_+$ extending $\nabla\psi$ with controlled norms and vanishing for sufficiently large x_3. Setting then $\overline{h} := \nabla \cdot \overline{\boldsymbol{F}}$ gives the claim, since

$$\|\overline{h}\|_{W_2^{l+1}(\Sigma^\infty)} \leqslant \|\overline{\boldsymbol{F}}\|_{W_2^{l+2}(\Sigma^\infty)} \leqslant c\|\nabla\psi\|_{W_2^{l+2}(\Omega)} \leqslant c\|h\|_{W_2^{l+1}(\Omega)},$$

while the inequality for $\overline{\boldsymbol{F}}$ follows from (64). □

We will use the following proposition.

Proposition 4.2. *Let $l \geqslant 0$. For any sufficiently large $\operatorname{Re}\lambda$, there is a unique periodic solution to the problem*

$$\begin{cases} \lambda \boldsymbol{u} - \nu \Delta \boldsymbol{u} + \nabla q = \boldsymbol{f} & \text{in} \quad \Omega, \\ \nabla \cdot \boldsymbol{u} = 0 & \text{in} \quad \Omega, \\ \mathbb{T}(\boldsymbol{u}, q)\boldsymbol{N} = 0 & \text{on} \quad \mathcal{G}, \\ \boldsymbol{u} = 0 & \text{on} \quad \Sigma \end{cases} \tag{65}$$

for any $\boldsymbol{f} \in W_2^l(\Omega)$ and it satisfies the inequality

$$\|\boldsymbol{u}\|^2_{H^{l+2}_\lambda(\Omega)} + \|\nabla q\|^2_{H^l_\lambda(\Omega)} + \|q\|^2_{H^{l+\frac{1}{2}}_\lambda(\mathcal{G})} \leqslant c\|\boldsymbol{f}\|^2_{H^l_\lambda(\Omega)}. \tag{66}$$

Proof. The existence of a weak solution

$$\boldsymbol{u} \in \mathcal{J} := \Big\{ \boldsymbol{v} \in W_2^1(\Omega) : \nabla \cdot \boldsymbol{v} = 0, \ \boldsymbol{v}\lfloor_\Sigma = 0 \Big\}$$

can be proved through Lax–Milgram theorem, since the weak formulation of (65) is

$$\lambda \int\limits_\Omega \boldsymbol{u} \cdot \boldsymbol{\phi}\, dx + \frac{\nu}{2} \int\limits_\Omega \mathbb{D}(\boldsymbol{u}) : \mathbb{D}(\boldsymbol{\phi})\, dx = \int\limits_\Omega \boldsymbol{f} \cdot \boldsymbol{\phi}\, dx, \quad \forall \boldsymbol{\phi} \in W_2^1(\Omega), \quad \boldsymbol{\phi}\lfloor_\Sigma = 0.$$

Indeed, the Korn inequality gives coerciveness of the bilinear form defined from the left-hand side, and Sobolev inequality the continuity of the right-hand side for $\boldsymbol{f} \in L^2(\Omega)$. The pressure can be recovered through standard methods, see, e.g., [9]. The estimate (66) follows, for example, from Shauder localisation method and the analogous estimates for the related problems in the half-space. □

Lemma 4.3. *Let $\delta \leqslant 1$ and η, ψ be smooth functions such that $\operatorname{supp} \eta \in B(0, \delta) \subset \Omega$ for some smooth bounded $\Omega \subset \mathbb{R}^N$, $N = 2, 3$. If*

$$\sup_{B(0,\delta)} |\eta| + \delta|\nabla \eta| + \delta^2 |\nabla^2 \eta| + \frac{|\psi|}{\delta} + |\nabla \psi| + |\nabla^2 \psi| \leqslant k \tag{67}$$

for k independent of δ, then for any $f \in W_2^l(\Omega)$,

$$\|\eta\psi f\|^2_{W_2^l(\Omega)} \leqslant \delta \|f\|^2_{W_2^l(\Omega)} + c_2(\delta)\|f\|^2_{L^2(\Omega)},$$

where $c_2(\delta)$ depends on δ, η, ψ, lm and k.

Proof. We consider an extension f^* of f to the whole $\mathbb{R}^N$, with controlled norm. Let us consider the case $N = 3$ first. From (67) we get

$$\|\eta\psi\|^2_{W_2^2(\mathbb{R}^3)} \leqslant ck^4\delta.$$

If $l \leqslant \frac{3}{2}$ we use (9) to obtain

$$\|\eta\psi f\|^2_{W^l_2(\Omega)} \leqslant c\|\eta\psi f^*\|^2_{W^l_2(\mathbb{R}^3)} \leqslant \|\eta\psi\|^2_{W^2_2(\mathbb{R}^3)}\|f^*\|^2_{W^l_2(\mathbb{R}^3)} \leqslant ck^4\delta\|f\|^2_{W^l_2(\Omega)},$$

and conclude by interpolation inequality. Otherwise we use (10) with $\min\{2,l\} > s > \frac{3}{2}$ to obtain

$$\begin{aligned}\|\eta\psi f\|^2_{W^l_2(\Omega)} &\leqslant \|\eta\psi\|^2_{W^s_2(\mathbb{R}^3)}\|f^*\|^2_{W^l_2(\mathbb{R}^3)} + \|\eta\psi\|^2_{W^l_2(\mathbb{R}^3)}\|f^*\|^2_{W^s_2(\mathbb{R}^3)}\\ &\leqslant \delta\|f^*\|^2_{W^l_2(\mathbb{R}^3)} + c(\delta)\|f^*\|^2_{L^2(\mathbb{R}^3)}\end{aligned}$$

by the interpolation inequality. For $N = 2$ we proceed in a similar way: since $\|\eta\psi\|^2_{W^2_2(\mathbb{R}^2)} \leqslant ck^4$ and $\|\eta\psi\|^2_{L^2(\mathbb{R}^2)} \leqslant ck^4\delta^4$, the interpolation inequality gives $\|\eta\psi\|^2_{W^{3/2}_2(\mathbb{R}^2)} \leqslant ck\delta$, and the rest of the proof is entirely analogous. □

Theorem 4.4. *Let $l \geqslant 0$. For any sufficiently large $\operatorname{Re}\lambda$, there exists a unique periodic solution of (39), for any choice of periodic $\boldsymbol{f} \in W^l_2(\Omega)$, $\boldsymbol{d} \in W^{l+\frac{1}{2}}_2(\mathcal{G})$, $g \in W^{l+\frac{3}{2}}_2(\Omega)$, $h \in W^{l+1}_2(\Omega)$, $\boldsymbol{F} \in W^1_2(\Omega)$, and $\boldsymbol{a} \in W^{l+\frac{3}{2}}_2(\Sigma)$ with $a^3 = F^3\lfloor_\Sigma = 0$. The solution satisfies the estimate:*

$$\begin{aligned}\|\boldsymbol{u}\|^2_{H^{l+2}_\lambda(\Omega)} + \|\nabla q\|^2_{H^l_\lambda(\Omega)} + \|q\|^2_{H^{l+\frac{1}{2}}_\lambda(\mathcal{G})} + \|\rho\|^2_{H^{l+\frac{5}{2}}_\lambda(\mathcal{G})} + \|\lambda\rho\|^2_{H^{l+\frac{3}{2}}_\lambda(\mathcal{G})}\\ \leqslant c\Big(\|\boldsymbol{f}\|^2_{H^l_\lambda(\Omega)} + \|h\|^2_{W^{l+1}_2(\Omega)} + |\lambda|^{l+2}\|\boldsymbol{F}\|^2_{L^2(\Omega)} + \\ + \|\boldsymbol{d}\|^2_{H^{l+\frac{1}{2}}_\lambda(\mathcal{G})} + \|g\|^2_{H^{l+\frac{3}{2}}_\lambda(\mathcal{G})} + \|\boldsymbol{a}\|^2_{H^{l+\frac{3}{2}}_\lambda(\Sigma)}\Big).\end{aligned} \tag{68}$$

Proof. We first show that it suffices to prove the existence of a solution of

$$\begin{cases}\lambda\boldsymbol{u} - \nu\Delta\boldsymbol{u} + \nabla q = 0 & \text{in } \Omega,\\ \nabla\cdot\boldsymbol{u} = 0 & \text{in } \Omega,\\ \mathbb{T}(\boldsymbol{u},q)\boldsymbol{N} + \sigma L\rho\boldsymbol{N} = \boldsymbol{d} & \text{on } \mathcal{G},\\ \lambda\rho + \nabla'\phi_b\cdot\boldsymbol{u} - u^3 + \boldsymbol{v}_b\cdot\nabla'\rho = g & \text{on } \mathcal{G},\\ \boldsymbol{u} = 0 & \text{on } \Sigma.\end{cases} \tag{69}$$

Indeed one can extend $\boldsymbol{f}$ with preservation of class and controlled norms, as well as apply Lemma 4.1 to $\boldsymbol{F}$ and h. We consider a solution $\boldsymbol{v}_1$, p_1 of (59) with the right-hand sides, then a solution $\boldsymbol{v}_2$, p_2 of (65) with the right-hand side ∇p_1. Given a solution $\boldsymbol{v}_3$, q_3, ρ of (69) with the right-hand sides

$$\widetilde{\boldsymbol{d}} := \boldsymbol{d} - \nu\mathbb{D}(\boldsymbol{v}_1),\quad \widetilde{g} := g - \nabla'\phi_b\cdot(\boldsymbol{v}_1 + \boldsymbol{v}_2) + v^3_1 + v^3_2,$$

the triple

$$\boldsymbol{v}_1 + \boldsymbol{v}_2 + \boldsymbol{v}_3, \quad p_2 + p_3, \quad \rho,$$

satisfies (39). From estimates (60) and (66) for problems (59) and (65), respectively, we readily get

$$\begin{aligned}\|\widetilde{\boldsymbol{d}}\|^2_{H^{l+\frac12}_\lambda(\mathcal{G})} + \|\widetilde{g}\|^2_{H^{l+\frac32}_\lambda(\mathcal{G})} \leqslant c\Big(\|\boldsymbol{f}\|^2_{H^l_\lambda(\Omega)} + \|h\|^2_{W^{l+1}_2(\Omega)} \\ + |\lambda|^{l+2}\|\boldsymbol{F}\|^2_{L^2(\Omega)} + \|\boldsymbol{d}\|^2_{H^{l+\frac12}_\lambda(\mathcal{G})} + \|g\|^2_{H^{l+\frac32}_\lambda(\mathcal{G})} + \|\boldsymbol{a}\|^2_{H^{l+\frac32}_\lambda(\Sigma)}\Big).\end{aligned}$$

Also by the same estimates, it is clear that a bound of the form

$$\begin{aligned}\|\boldsymbol{u}\|^2_{H^{l+2}_\lambda(\Omega)} + \|\nabla q\|^2_{H^l_\lambda(\Omega)} + \|q\|^2_{H^{l+\frac12}_\lambda(\mathcal{G})} + \|\rho\|^2_{H^{l+\frac52}_\lambda(\mathcal{G})} + \|\lambda\rho\|^2_{H^{l+\frac32}_\lambda(\mathcal{G})} \\ \leqslant c\Big(\|\boldsymbol{d}\|^2_{H^{l+\frac12}_\lambda(\mathcal{G})} + \|g\|^2_{H^{l+\frac32}_\lambda(\mathcal{G})}\Big)\end{aligned}$$

for the solution of (69), implies (68) for the solution just constructed.

We now prove the existence of a solution to (69). For any fixed $\delta > 0$, we consider a finite covering of $\mathcal{G}$ with balls $B(x_i, \delta)$,$x_i \in \mathcal{G}$, and this can be done in such a way that the number of balls containing any point of Ω_b is bounded independently of δ. We now choose a periodic partition of unity φ_i with each φ_i having support in $B(x_i, 2\delta)$, $\sum_i \varphi_i = 1$ on $V := \Omega_b \cap \cup_i B(x_i, \delta) \subset \Omega_b \cap \cup_i B(x_i, 2\delta) =: U$. For each φ_i we choose η_i, with support in $B(x_i, 3\delta)$ such that $\varphi_i = \eta_i\varphi_i$. Any norm of the φ_i, η_i is bounded by a suitable constant depending only on δ, and in particular we can suppose

$$|\nabla\varphi_i| + |\nabla\eta_i| \leqslant \frac{c}{\delta}, \quad |\nabla^2\varphi_i| + |\nabla^2\eta_i| \leqslant \frac{c}{\delta^2}. \tag{70}$$

Moreover, $\boldsymbol{N}_i$ will be the normal to $\mathcal{G}$ at x_i, $\boldsymbol{V}_i = \boldsymbol{v}_b(x_i)$, Π_i the projection on the tangent space to $\mathcal{G}$ at x_i, C_i an isometry bringing $\boldsymbol{N}_i$ to $-\boldsymbol{e}_3$ and we write $\boldsymbol{N}_i' = C_i\boldsymbol{N}$, $\boldsymbol{V}_i' = C_i\boldsymbol{V}_i$. For each i we will set, as in (4)

$$y_i = C_i x, \quad y_i = e_{\phi_i}(z_i),$$

where ϕ_i is defined through $C_i(x', \phi_b(x')) = (z_i', -\phi_i(z_i'))$ and e_{ϕ_i} is the transformation defined in (4). Here we suppose that $\phi_i^* = \theta_i(z_{i3})\phi_i(z_i')$ with $\theta_i = 1$ on the support of $\varphi_i(x(z_i))$. Recall that for any isometry C, it holds

$$\nabla_x = C^T\nabla_y = C^{-1}\nabla_y, \quad \Delta_x = C_{ij}C_{kj}\frac{\partial^2}{\partial y_i \partial y_k} = \Delta_y, \tag{71}$$

and thus these formulas hold for each of the C_i with respect to the coordinates y_i. Moreover $\nabla_{y_i} = \mathcal{L}_i^{-T}\nabla_{z_i}$, where $\mathcal{L}_i$ is the Jacoby matrix of the transformation e_{ϕ_i}. Since ϕ_b is smooth for $z' \in \Sigma \cap B(0, 2\delta)$, we have

$$|\phi_i(z')| \leqslant c|z'|^2, \quad |\nabla'\phi_i(z')| \leqslant c|z'|, \tag{72}$$

which, toghether with (70), implies (67) for $\eta = \eta_i$ and $\psi = \frac{\partial\phi_i}{\partial z_j}$. Therefore, by the previous lemma and Proposition 1.3, it holds

$$\left\|\eta_i \frac{\partial\phi_i}{\partial z_j} h\right\|^2_{W_2^\mu(\Sigma)} \leqslant \delta\|h\|^2_{W_2^\mu(\Sigma)} + c(\delta)\|h\|^2_{L^2(\Sigma)}, \tag{73}$$

and the same inequality for the norms on Σ^∞. In the following we will shorten somewhat the notation, letting, for example, $z(x) = e_{\phi_i}^{-1}(C_i(x))$ (and similar expressions) whenever the dependance on i will be clear.

We define a linear operator $R(\boldsymbol{d}, g) = (\widehat{\boldsymbol{u}}, \widehat{q}, \widehat{\rho})$, where we construct $\widehat{\boldsymbol{u}}$, $\widehat{q}$ and $\widehat{\rho}$ linearly in $\boldsymbol{d}$ and g in the following. We let

$$\boldsymbol{v} = \sum_i \eta_i C_i^{-1}\boldsymbol{v}_i(z_i(x)), \quad p = \sum_i \eta_i p_i(z_i(x)), \quad \rho = \sum_i \eta_i\rho_i(z_i(x)),$$

where $\boldsymbol{v}_i = \boldsymbol{v}_i(z)$, $p_i = p_i(z)$, and ρ_i are periodic and solve a problem of the type (41), namely.

$$\begin{cases} \lambda\boldsymbol{v}_i - \nu\Delta_z\boldsymbol{v}_i + (\boldsymbol{V}_i'\cdot\nabla_z)\boldsymbol{v}_i + \nabla_z p_i = 0 & \text{in} \quad \mathbb{R}^3_+, \\ \nabla_z\cdot\boldsymbol{v}_i = 0 & \text{in} \quad \mathbb{R}^3_+, \\ \nu\big(\frac{\partial v_i^3}{\partial z_j} + \frac{\partial v_i^j}{\partial z_3}\big) = \eta_i(C_i\boldsymbol{d})^j(x(z)) \quad j = 1,2 & \text{in} \quad \mathbb{R}^2, \\ -p_i + 2\nu\frac{\partial v_i^3}{\partial z_3} - \sigma\Delta'\rho_i = \varphi_i(C_i\boldsymbol{d})^3(x(z)) & \text{in} \quad \mathbb{R}^2, \\ \lambda\rho_i + \boldsymbol{V}_i'\cdot\nabla'\rho_i + v_i^3 = \varphi_i g(x(z)) & \text{in} \quad \mathbb{R}^2. \end{cases} \tag{74}$$

We then set

$$\begin{cases} \lambda\boldsymbol{v} - \nu\Delta_x\boldsymbol{v} + \nabla_x p =: \widehat{\boldsymbol{f}} \\ \nabla_x\cdot\boldsymbol{v} =: \widehat{h} \\ \mathbb{T}(\boldsymbol{v}, p)\boldsymbol{N} + \sigma L\rho\boldsymbol{N} =: \boldsymbol{d} + \boldsymbol{A}(\boldsymbol{d}, g) \\ \lambda\rho + \nabla'\phi_b\cdot\boldsymbol{v} - v^3 + \boldsymbol{v}_b\cdot\nabla'\rho =: g + A(\boldsymbol{d}, g) \end{cases}$$

noting that both $\boldsymbol{v}$ and p vanish in a neighbourhood of Σ. We have

$$\widehat{h} = \nabla\cdot\widehat{\boldsymbol{F}} + \widehat{h}'$$

for sufficiently regular $\widehat{\boldsymbol{F}}$ and $\widehat{h}'$ that will be specified later. We can apply Lemma 4.1 on $\widehat{h}$, $\widehat{\boldsymbol{F}}$ and $\widehat{h}'$ and suppose that $\widehat{h}' = 0$ and $\widehat{h}$, $\widehat{\boldsymbol{F}}$ are defined in the whole Σ^∞ with controlled norms. We also extend $\widehat{\boldsymbol{f}}$ to Σ^∞ with preservation of class and controlled norm. We keep the notation unchanged for the extensions of $\widehat{h}$ and $\widehat{\boldsymbol{f}}$, and call $\overline{\boldsymbol{F}}$ the vector arising from Lemma 4.1. Let then $(\overline{\boldsymbol{v}}_1, \overline{p}_1)$ be the periodic solution of

$$\begin{cases} \lambda\overline{\boldsymbol{v}}_1 - \nu\Delta\overline{\boldsymbol{v}}_1 + \nabla\overline{p}_1 = \widehat{\boldsymbol{f}} & \text{in} \quad \mathbb{R}^3_+, \\ \nabla\cdot\overline{\boldsymbol{v}}_1 = -\widehat{h} = -\nabla\cdot\overline{\boldsymbol{F}} & \text{in} \quad \mathbb{R}^3_+, \\ \overline{\boldsymbol{v}}_1 = 0 & \text{on} \quad \Sigma, \end{cases}$$

(notice that $\overline{F}^3\lfloor_\Sigma = 0$, since all the η_i vanish in a neighbourhood of Σ), for which it holds estimate (60), which, together with (61), implies

$$\begin{aligned} &\|\overline{\boldsymbol{v}}_1\|^2_{H^{l+2}_\lambda(\Omega)} + \|\nabla\overline{p}_1\|^2_{H^l_\lambda(\Omega)} \\ &\quad \leqslant c\Big(\|\widehat{\boldsymbol{f}}\|^2_{H^l_\lambda(\Omega)} + \|\widehat{h}\|^2_{W^{l+1}_2(\Omega)} + |\lambda|^{l+2}(\|\widehat{\boldsymbol{F}}\|^2_{L^2(\Omega)} + \|\widehat{h}'\|^2_{L^2(\Omega)})\Big). \end{aligned} \tag{75}$$

Finally, we let $\overline{\boldsymbol{v}}_2$, $\overline{p}_2$ be solution of (65) with the right-hand side $\nabla\overline{p}_1$. We let

$$R(\boldsymbol{d}, g) = (\widehat{\boldsymbol{u}}, \widehat{q}, \widehat{\rho}) := (\boldsymbol{v} + \overline{\boldsymbol{v}}_1 + \overline{\boldsymbol{v}}_2, p + \overline{p}_2, \rho). \tag{76}$$

This triple solves

$$\begin{cases} \lambda\widehat{\boldsymbol{u}} - \nu\Delta\widehat{\boldsymbol{u}} + \nabla\widehat{q} = 0 & \text{in} \quad \Omega, \\ \nabla\cdot\widehat{\boldsymbol{u}} = 0 & \text{in} \quad \Omega, \\ \mathbb{T}(\widehat{\boldsymbol{u}}, \widehat{q})\boldsymbol{N} + \sigma L\widehat{\rho}\boldsymbol{N} = \boldsymbol{d} + \widehat{\boldsymbol{A}}(\boldsymbol{d}, g) & \text{on} \quad \mathcal{G}, \\ \lambda\widehat{\rho} + \nabla'\phi_b\cdot\widehat{\boldsymbol{u}}(e_\rho) - \widehat{u}^3(e_\rho) + \boldsymbol{v}_b\cdot\nabla'\widehat{\rho} = g + \widehat{A}(\boldsymbol{d}, g) & \text{on} \quad \mathcal{G}, \\ \boldsymbol{u} = 0 & \text{on} \quad \Sigma, \end{cases}$$

where

$$\begin{aligned} &\widehat{\boldsymbol{A}}(\boldsymbol{d}, g) = \boldsymbol{A}(\boldsymbol{d}, g) + \nu\mathbb{D}(\overline{\boldsymbol{v}}_1)\boldsymbol{N}, \\ &\widehat{A}(\boldsymbol{d}, g) = A(\boldsymbol{d}, g) + \nabla'\phi_b\cdot(\overline{\boldsymbol{v}}_1 + \overline{\boldsymbol{v}}_2) - \overline{v}^3_1 - \overline{v}^3_2. \end{aligned}$$

We will prove that $(\widehat{\boldsymbol{A}}, \widehat{A})$ is a contraction operator from $W_2^{l+\frac{1}{2}}(\mathcal{G}) \times W_2^{l+\frac{3}{2}}(\mathcal{G})$ to itself, therefore, establishing the invertibility of $I + (\widehat{\boldsymbol{A}}, \widehat{A})$, and obtaining the solution $R(I + (\widehat{\boldsymbol{A}}, \widehat{A}))^{-1}(\boldsymbol{d}, g)$. Instead of using the usual norm, we will perform the estimates w.r.t. the norm

$$\|(\boldsymbol{d}, g)\|^2_\lambda := \sum_i \|\varphi_i\boldsymbol{d}\|^2_{H_\lambda^{l+\frac{1}{2}}(\mathcal{G})} + \|\varphi_i g\|^2_{H_\lambda^{l+\frac{3}{2}}(\mathcal{G})},$$

which is equivalent to the usual norm with constant independent of δ and λ, whenever $\operatorname{Re}\lambda \geqslant c(\delta)$, where $c(\delta) \to +\infty$ for $\delta \to 0$.

1. *Estimate of* $\widehat{\boldsymbol{f}}$.

Transforming coordinates near each point x_i and using (71) and(74), we find that

$$\widehat{\boldsymbol{f}} = \sum_i \eta_i C_i^{-1} \left[-\nu(\Delta_y - \Delta_z)\boldsymbol{v}_i + (\nabla_y - \nabla_x)p_i \right] + \sum_i p_i \nabla_x \eta_i$$
$$- \sum_i C_i^{-1} \nu (2\nabla_x \eta_i \nabla_x \boldsymbol{v}_i + \boldsymbol{v}_i \Delta_x \eta_i + \frac{\eta_i}{\nu} (\boldsymbol{V}_i' \cdot \nabla_z)\boldsymbol{v}_i).$$

The lower order terms of the second line are estimated in the z coordinates, through interpolation inequality:

$$\Big\| \sum_i C_i^{-1} \nu (2\nabla_x \eta_i \nabla_x \boldsymbol{v}_i + \boldsymbol{v}_i \Delta_x \eta_i + \frac{\eta_i}{\nu} (\boldsymbol{V}_i' \cdot \nabla_z)\boldsymbol{v}_i) \Big\|^2_{H^l_\lambda(\Omega)}$$

$$\leqslant c(\delta) \sum_i \|\boldsymbol{v}_i\|^2_{W_2^{l+1}(\Sigma^\infty)} + |\lambda|^l \|\boldsymbol{v}_i\|^2_{W_2^1(\Sigma^\infty)}$$

$$\leqslant \frac{c(\delta)}{|\lambda|} \sum_i \|\boldsymbol{v}_i\|^2_{H_\lambda^{l+2}(\Sigma^\infty)} \leqslant \frac{c(\delta)}{|\lambda|} \|(\boldsymbol{d}, g)\|^2_\lambda,$$

where we used estimate (42) for solution of (74). For the higher order terms, we recall that

$$\nabla_y - \nabla_z = (\mathcal{L}_i^{-T} - I)\nabla_z,$$

$$\Delta_y - \Delta_z = (\mathcal{L}_i^{-T} - I)(\mathcal{L}_i^{-T} + I) : D_z^2 + \mathcal{L}_i^{-T} D_z \mathcal{L}_i^{-T} \nabla_z.$$

The last term in the previous formula is still a lower order term which can be estimated as before, while by (16), the terms involving $\mathcal{L}_i^{-T} - I$ have coefficients of the form $\frac{\partial \phi_i^*}{\partial z_k} m_k$ with smooth m_k's depending on z, ϕ_i^*, and $\nabla \phi_i^*$. Therefore, we have to estimate terms of the form

$$\eta_i \frac{\partial \phi_i^*}{\partial z_k} \boldsymbol{m}_{kj}^i \frac{\partial p_i}{\partial z_j} \quad \text{and} \quad \eta_i \frac{\partial \phi_i^*}{\partial z_k} m_{kjl}^i \frac{\partial^2 \boldsymbol{v}_i}{\partial z_j \partial z_l},$$

in the W_2^l and L^2 norm. The L^2 norm is estimated using (72) and (42) giving

$$
\begin{aligned}
|\lambda|^l \sum_{i,k,j,l} \|\eta_i \frac{\partial \phi_i^*}{\partial z_k} \boldsymbol{m}_{kj}^i \frac{\partial p_i}{\partial z_j}\|_{L^2(\Sigma^\infty)}^2 + \|\eta_i \frac{\partial \phi_i^*}{\partial z_k} m_{kjl}^i \frac{\partial^2 \boldsymbol{v}_i}{\partial z_j \partial z_l}\|_{L^2(\Sigma^\infty)}^2 \\
\leqslant c\delta^2 \sum_i |\lambda|^l (\|\nabla p_i\|_{L^2(\Sigma^\infty)}^2 + \|\boldsymbol{v}_i\|_{W_2^2(\Sigma^\infty)}^2) \\
\leqslant c\delta^2 \sum_i \|\boldsymbol{v}_i\|_{H_\lambda^{l+2}(\Sigma^\infty)}^2 + |\lambda|^l \|\nabla_z p_i\|_{L^2(\Sigma^\infty)}^2 \leqslant c\delta^2 \|(\boldsymbol{d}, g)\|_\lambda^2.
\end{aligned}
$$

For the W_2^l norm, we can suppose that $l > 0$. We use (73), obtaining:

$$
\begin{aligned}
&\sum_{i,k,j,l} \|\eta_i \frac{\partial \phi_i^*}{\partial z_k} \boldsymbol{m}_{kj}^i \frac{\partial p_i}{\partial z_j}\|_{W_2^l(\Sigma^\infty)}^2 + \|\eta_i \frac{\partial \phi_i^*}{\partial z_k} m_{kjl}^i \frac{\partial^2 \boldsymbol{v}_i}{\partial z_j \partial z_l}\|_{W_2^l(\Sigma^\infty)}^2 \\
&\leqslant \sum_i \delta \|\boldsymbol{v}_i\|_{W_2^{l+2}(\Sigma^\infty)}^2 + c(\delta)\|\boldsymbol{v}_i\|_{W_2^2(\Sigma^\infty)}^2 + \delta \|\nabla_z p_i\|_{W_2^l(\Sigma^\infty)}^2 + c(\delta)\|\nabla_z p_i\|_{L^2(\Sigma^\infty)}^2 \\
&\leqslant \sum_i \delta \Big(\|\boldsymbol{v}_i\|_{W_2^{l+2}(\Sigma^\infty)}^2 + \|\nabla_z p_i\|_{W_2^l(\Sigma^\infty)}^2 \Big) + c(\delta) \Big(\|\boldsymbol{v}_i\|_{L^2(\Sigma^\infty)}^2 + \|\nabla_z p_i\|_{L^2(\Sigma^\infty)}^2 \Big) \\
&\leqslant \left(\delta + \frac{c(\delta)}{|\lambda|^l} \right) \|(\boldsymbol{d}, g)\|_\lambda^2,
\end{aligned}
$$

by interpolation inequality and (42). To estimate the term $\sum_i p_i \nabla \eta_i$, we let

$$
\overline{d}_i = -\int_\Sigma \varphi_i (C_i \boldsymbol{d})^3 (x(z))\, ds,
$$

and use (43) and (44) to obtain

$$
\begin{aligned}
\|\sum_i p_i \nabla_x \eta_i\|_{L^2(\Omega)}^2 \leqslant c(\delta) \sum_i \|p_i - \overline{d}_i\|_{L^2(\Sigma^\infty)}^2 + \|\varphi_i \boldsymbol{d}\|_{L^2(\mathcal{G})}^2 \\
\leqslant c(\delta) \sum_i \|\varphi_i \boldsymbol{d}\|_{L^2(\mathcal{G})}^2 + \|\varphi_i g\|_{W_2^1(\mathcal{G})}^2,
\end{aligned}
\tag{77}
$$

and thus

$$
|\lambda|^l \|\sum_i p_i \nabla_x \eta_i\|_{L^2(\Omega)}^2 \leqslant \frac{c(\delta)}{\sqrt{|\lambda|}} \|(\boldsymbol{d}, g)\|_\lambda^2.
$$

Moreover, by interpolation and estimates (42), (77) for the model problem

$$\|\sum_i p_i \nabla_x \eta_i\|^2_{W^l_2(\Omega)} \leqslant c(\delta) \sum_i \|p_i\|^2_{W^l_2(\Sigma^\infty)}$$

$$\leqslant \delta \sum_i \|\nabla_z p_i\|^2_{W^l_2(\Sigma^\infty)} + c(\delta) \sum_i \|p_i\|^2_{L^2(\Sigma^\infty)} \leqslant \left(\delta + \frac{c(\delta)}{\sqrt{|\lambda|}}\right) \|(\boldsymbol{d}, g)\|^2_\lambda,$$

All in all we have obtained

$$\|\widehat{\boldsymbol{f}}\|^2_{H^l_\lambda(\Omega)} \leqslant \big(\delta + \frac{c(\delta)}{|\lambda|^\theta}\big) \|(\boldsymbol{d}, g)\|^2_\lambda, \tag{78}$$

where $\theta > 0$ (equal to l if $0 < l < \frac{1}{2}$, and $\frac{1}{2}$ otherwise).

2. *Construction of* $\widehat{\boldsymbol{F}}$ *and* h'.
We now prove that $\widehat{h}$ can be written as the sum $\widehat{h} = \nabla_x \cdot \widehat{\boldsymbol{F}} + \widehat{h}'$ in a satisfactory way. More precisely we claim that for some tensors M^i_r, $r = 0, \dots, 3$ and functions m_i, smooth and depending only on ϕ_b and $\{\eta_i\}$

$$\widehat{\boldsymbol{F}} = \sum_i M^i_0 \boldsymbol{v}_i + \frac{1}{\lambda} \sum_i (M^i_1 \nabla_y \boldsymbol{v}_i - M^i_2 p_i), \tag{79}$$

$$\widehat{h}' = \sum_i \frac{1}{\lambda} (M^i_3 \nabla_y \boldsymbol{v}_i - m_i p_i), \tag{80}$$

where, in the z coordinates,

$$|M^i_0| \leqslant c |\nabla \phi_i|, \tag{81}$$

for some constant depending only on ϕ_b. To prove this representation, first notice that from (71) we get

$$\nabla_x \cdot C^T_i \boldsymbol{w} = C^T_i \nabla_y \cdot C^T_i \boldsymbol{w} = \nabla_y \boldsymbol{w}, \tag{82}$$

for any vector $\boldsymbol{w}$, being C_i an isometry. Thus, by the solenoidality (in the z coordinates) of $\boldsymbol{v}_i$, it holds

$$\widehat{h} = \nabla_x \cdot (\eta_i C^T_i \boldsymbol{v}_i) = \nabla_y \eta_i \cdot \boldsymbol{v}_i + \eta_i (\nabla_y - \nabla_z) \cdot \boldsymbol{v}_i. \tag{83}$$

Calling l^i_{jk} the entries of $\mathcal{L}_i$ and using summation convention on repeated indexes except i, we can write

$$\Delta_z v^m_i = l^i_{kj} l^i_{sj,k} v^m_{i,s} + l^i_{kj} l^i_{sj} v^m_{i,sk}, \quad \nabla_z p_i = (l^i_{mk} p_{i,k}),$$

and since $\boldsymbol{v}_i = (\Delta_z \boldsymbol{v}_i - \nabla_z p_i)/\lambda$, we have

$$\begin{aligned}\nabla_y \eta_i \cdot \boldsymbol{v}_i &= \frac{1}{\lambda}\eta_{i,m}(l^i_{kj} l^i_{sj,k} v^m_{i,s} + l^i_{kj} l_{sj} v^m_{i,sk} - l^i_{mk} p_{i,k}) \\ &= \frac{1}{\lambda}\big(\eta_{i,m}(l^i_{kj} l^i_{sj} v^m_{i,s} - l^i_{mk} p_i)\big)_{,k} \\ &+ \frac{1}{\lambda}\left[-(\eta_{i,m} l^i_{kj} l_{sj})_{,k} v^m_{i,s} + (\eta_{i,m} l^i_{mk})_{,k} p_i + \eta_{i,m} l^i_{kj} l_{sj,k} v^m_{i,s}\right]\end{aligned}$$

One can proceed in a similar way for the term $\eta_i(\nabla_y - \nabla_z)\cdot \boldsymbol{v}_i$. We define the matrices a^i whose entries are $a^i_{hk} = (I - \mathcal{L}^T_i)_{hk}$, and, using (15), we have

$$|a^i_{hk}| \leqslant c(|\nabla' \phi_i| + |\phi_i|). \tag{84}$$

Proceeding as before, in the y coordinates we have

$$\begin{aligned}\eta_i(\nabla_y - \nabla_z)\cdot \boldsymbol{v}_i &= \eta_i a^i_{mt} v^m_{i,t} = (\eta_i a^i_{mt} v^m_i)_{,t} - \eta_i a^i_{mt,t} v^m \\ &= (\eta_i a^i_{mk} v^m_i)_{,k} - \frac{\eta_i a^i_{mt,t}}{\lambda}(l^i_{kj} l^i_{sj,k} v^m_{i,s} + l^i_{kj} l_{sj} v^m_{i,sk} - l^i_{mk} p_{i,k}) \\ &+ \left(\eta_i a^i_{mk} v^m_i - \frac{1}{\lambda}\eta_i a^i_{mt,t}(l^i_{kj} l^i_{sj} v^m_{i,s} - l^i_{mk} p_i)\right)_{,k} \\ &+ \frac{1}{\lambda}\left[(\eta_i a^i_{mt,t} l^i_{kj} l_{sj})_{,k} v^m_{i,s} - (\eta_i a^i_{mt,t} l^i_{mk})_{,k} p_i - \eta_i a^i_{mt,t} l^i_{kj} l_{sj,k} v^m_{i,s}\right].\end{aligned}$$

If the hk-entry of C_i is denoted by C^{hk}_i, we now define for each i the tensors M^i_r, $r = 0,\dots,3$ as follows

$$\begin{aligned}(M^i_0)^h_m &= C^{kh}_i \eta_i a^i_{mk} \\ (M^i_1)^h_{sm} &= C^{kh}_i(\eta_{i,m} - \eta_i a^i_{mt,t}) l^i_{kj} l^i_{sj} \\ (M^i_2)^h &= C^{kh}_i(\eta_{i,m} - \eta_i a^i_{mt,t}) l^i_{mk} \\ (M^i_3)_{sm} &= (\eta_{i,m} - \eta_i a^i_{mt,t}) l^i_{kj} l^i_{sj,k} - ((\eta_{i,m} - \eta_i a^i_{mt,t}) l^i_{kj} l^i_{sj})_{,k}\end{aligned}$$

and the functions

$$m_i = \big((\eta_{i,m} - \eta_i a^i_{mt,t}) l^i_{mk}\big)_{,k}.$$

Applying (82) for the terms in the divergence and gathering the previous equalities, we obtain

$$\nabla_x \cdot \boldsymbol{v} = \nabla_x \cdot \left[\sum_i M^i_0 \boldsymbol{v}_i + \frac{M^i_1 \nabla \boldsymbol{v}_i - M^i_2 p_i}{\lambda}\right] + \sum_i \frac{M^i_3 \nabla \boldsymbol{v}_i - m_i p_i}{\lambda},$$

which gives (79) and (80), while (84) and (72) give (81) for small δ.

3. *Estimate for $\widehat{h}$, $\widehat{\boldsymbol{F}}$ and $\widehat{h}'$.*
For $\widehat{h}$, using formula (83), we can split the estimate in local coordinates:

$$\|\widehat{h}\|^2_{W_2^{l+1}(\Omega)} \leqslant c(\delta)\sum_i \|\boldsymbol{v}_i\|^2_{W_2^{l+1}(\Sigma^\infty)} + c\sum_i \|\eta_i(\nabla_y - \nabla_z)\boldsymbol{v}_i\|^2_{W_2^{l+1}(\Sigma^\infty)}.$$

The first sum has only lower order terms which can be estimated through interpolation inequality, while by (16) the second one has addends of the form

$$\eta_i \frac{\partial \phi_i^*}{\partial z_k} m^i_{kj} \frac{\partial \boldsymbol{v}_i}{\partial z_j}, \tag{85}$$

for some smooth functions m^i_{jk} depending only on ϕ_b. As before, its W_2^{l+1} square norm is estimated through (73) and interpolation, giving

$$\sum_i \|\eta_i(\nabla_y - \nabla_z)\boldsymbol{v}_i\|^2_{W_2^{l+1}(\Sigma^\infty)} \leqslant c\sum_i \delta\|\nabla_z \boldsymbol{v}_i\|^2_{W_2^{l+1}(\Sigma^\infty)} + c(\delta)\|\boldsymbol{v}_i\|^2_{W_2^1(\Sigma^\infty)},$$

$$\leqslant c\sum_i \delta\|\boldsymbol{v}_i\|^2_{W_2^{l+2}(\Sigma^\infty)} + c(\delta)\|\boldsymbol{v}_i\|^2_{L^2(\Sigma^\infty)} \leqslant c\left(\delta + \frac{c(\delta)}{|\lambda|^{l+2}}\right)\|(\boldsymbol{d}, g)\|^2_\lambda.$$

All in all, we get

$$\|\widehat{h}\|^2_{W_2^{l+1}(\Omega)} \leqslant c\left(\delta + \frac{c(\delta)}{|\lambda|}\right)\|(\boldsymbol{d}, g)\|^2_\lambda.$$

To estimate $\widehat{\boldsymbol{F}}$, using the expression (79), (81) and (72), we have

$$\|\widehat{\boldsymbol{F}}\|^2_{L^2(\Omega)} \leqslant c\sum_i \frac{1}{|\lambda|^2}\left(\|\boldsymbol{v}_i\|^2_{W_2^1(\Sigma^\infty)} + \|p_i\|^2_{L^2(\Sigma^\infty)}\right) + \delta^2\|\boldsymbol{v}_i\|_{L^2(\Sigma^\infty)},$$

and proceeding as in (77) for the pressure term, one obtains

$$|\lambda|^{l+2}\|\widehat{\boldsymbol{F}}\|^2_{L^2(\Omega)} \leqslant c\left(\delta^2 + \frac{c(\delta)}{\sqrt{|\lambda|}}\right)\|(\boldsymbol{d}, g)\|^2_\lambda.$$

The estimate for $\widehat{h}'$, due to the form (80), is even simpler, and is omitted. The full estimate then reads

$$\|\widehat{h}\|^2_{W_2^{l+1}(\Omega)} + |\lambda|^{l+2}(\|\widehat{\boldsymbol{F}}\|^2_{L^2(\Omega)} + \|\widehat{h}'\|^2_{L^2(\Omega)}) \leqslant c\left(\delta + \frac{c(\delta)}{\sqrt{|\lambda|}}\right)\|(\boldsymbol{d}, g)\|^2_\lambda. \tag{86}$$

4. Estimate of $\widehat{\boldsymbol{A}}$

The term $\mathbb{D}_x(\overline{\boldsymbol{v}}_1)\boldsymbol{N}$ is readily estimated through (75), the continuity of the restriction operator and interpolation inequality, giving

$$\begin{aligned}\|\mathbb{D}_x(\overline{\boldsymbol{v}}_1)\boldsymbol{N}\|^2_{H^{l+\frac{1}{2}}_\lambda(\mathcal{G})} &\leqslant c\|\overline{\boldsymbol{v}}_1\|^2_{H^{l+2}_\lambda(\Omega)}\\ &\leqslant c\big(\|\widehat{\boldsymbol{f}}\|^2_{H^l_\lambda(\Omega)} + \|\widehat{h}\|^2_{W^{l+1}_2(\Omega)} + |\lambda|^{l+2}(\|\widehat{\boldsymbol{F}}\|^2_{L^2(\Omega)} + \|\widehat{h}'\|^2_{L^2(\Omega)})\big).\end{aligned}$$

From the previous estimates for $\widehat{\boldsymbol{f}}$, $\widehat{h}$, $\widehat{\boldsymbol{F}}$ and $\widehat{h}'$ one thus obtains

$$\|\mathbb{D}_x(\overline{\boldsymbol{v}}_1)\boldsymbol{N}\|^2_{H^{l+\frac{1}{2}}_\lambda(\mathcal{G})} \leqslant c\Big(\delta + \frac{c(\delta)}{|\lambda|^\theta}\Big)\|(\boldsymbol{d}, g)\|^2_\lambda.$$

Regarding $\boldsymbol{A}$ we have, using $\boldsymbol{d} = \sum_i \varphi_i\eta_i\boldsymbol{d}$,

$$\begin{aligned}\mathbb{T}_x(\boldsymbol{v}, p)\boldsymbol{N} + \sigma L\rho\boldsymbol{N} - \boldsymbol{d} &= \sum_i \eta_i\big(-p_i\boldsymbol{N} + C_i^{-1}\mathbb{D}_x(\boldsymbol{v}_i)\boldsymbol{N} + \sigma L\rho_i\boldsymbol{N} - \varphi_i\boldsymbol{d}\big)\\ &+ \sum_i C_i^{-1}\nabla_x\varphi_i\otimes\boldsymbol{v}_i\cdot\boldsymbol{N} + \sigma(L(\eta_i\rho_i) - \eta_i L\rho_i)\boldsymbol{N},\end{aligned}$$

in the x coordinates. The second sum is a lower order term, and is bounded via interpolation by $\frac{c(\delta)}{|\lambda|}\|(\boldsymbol{d}, g)\|^2_\lambda$. We transform the first sum in the z coordinates and use the boundary conditions for (74), to obtain a sum whose terms are

$$\begin{aligned}\eta_i\big(-p_i(\boldsymbol{N}' - \boldsymbol{e}_3) + (\mathbb{D}_y - \mathbb{D}_z)(\boldsymbol{v}_i)\boldsymbol{N}' + \mathbb{D}_z(\boldsymbol{v}_i)(\boldsymbol{N}' - \boldsymbol{e}_3)\\ + \sigma L\rho_i(\boldsymbol{N}' - \boldsymbol{e}_3) + \sigma(L\rho_i + \Delta'\rho_i)\boldsymbol{e}_3\big).\end{aligned}$$

Notice that $\eta_i(\mathbb{D}_y - \mathbb{D}_z)(\boldsymbol{v}_i)$ and $\eta_i\mathbb{D}_z(\boldsymbol{v}_i)(\boldsymbol{N}' - \boldsymbol{e}_3)$ can be computed explicitly using (16) and both are linear combination of terms of the form (85). Similarly

$$p_i(\boldsymbol{N}' - \boldsymbol{e}_3) = p_i\frac{\partial\phi_i}{\partial z_j}\boldsymbol{m}^i_j, \quad L\rho_i(\boldsymbol{N}' - \boldsymbol{e}_3) = \frac{\partial\phi_i}{\partial z_j}\boldsymbol{m}^i_{jkl}\frac{\partial^2\rho_i}{\partial z_k\partial z_l} + \frac{\partial\phi_i}{\partial z_j}\boldsymbol{m}^i_{jk}\frac{\partial\rho_i}{\partial z_k},$$

for some smooth vectors $\boldsymbol{m}^j$, $\boldsymbol{m}^i_{jk}$, and $\boldsymbol{m}^i_{jkl}$ depending only on ϕ_b. Moreover, letting $g_i = 1 + |\nabla'\phi_i|^2$, we have

$$L\rho_i + \Delta'\rho_i = \frac{|\nabla'\phi_i|^2}{(1+\sqrt{g_i})\sqrt{g_i}}\Delta'\rho_i + \frac{\phi_{i,\alpha}\phi_{i,\beta}}{g_i^{\frac{5}{2}}}\rho_{i,\alpha\beta} + m^i_{\alpha\beta}\phi_{i,\alpha}\rho_{i,\beta},$$

which has the same structure as $L\rho_i(\boldsymbol{N}' - \boldsymbol{e}_3)$. These terms are thus estimated using (73) and interpolation inequalities as follows: for the pressure term

$$\sum_i \|\eta_i p_i(\boldsymbol{N}' - \boldsymbol{e}_3)\|^2_{H_\lambda^{l+\frac{1}{2}}(\Sigma)} \leqslant \sum_i \delta\|p_i\|^2_{H_\lambda^{l+\frac{1}{2}}(\Sigma)} + c(\delta)\|p_i\|^2_{L^2(\Sigma)}$$
$$\leqslant \left(\delta + \frac{c(\delta)}{|\lambda|^{l+\frac{1}{2}}}\right)\|(\boldsymbol{d}, g)\|^2_\lambda,$$

reasoning as in (77), while

$$\sum_i \|\eta_i[(\mathbb{D}_y - \mathbb{D}_z)(\boldsymbol{v}_i)\boldsymbol{N}' + \mathbb{D}_z(\boldsymbol{v}_i)(\boldsymbol{N}' - \boldsymbol{e}_3)]\|^2_{H_\lambda^{l+\frac{1}{2}}(\Sigma)}$$
$$\leqslant \sum_i \delta\|\nabla_z \boldsymbol{v}_i\|^2_{H_\lambda^{l+\frac{1}{2}}(\Sigma)} + c(\delta)\|\nabla_z \boldsymbol{v}_i\|^2_{L^2(\Sigma)}$$
$$\leqslant c\left(\delta + \frac{c(\delta)}{|\lambda|^{l+2}}\right)\sum_i \|\boldsymbol{v}_i\|^2_{H_\lambda^{l+2}(\Sigma^\infty)} \leqslant c\left(\delta + \frac{c(\delta)}{|\lambda|^{l+2}}\right)\|(\boldsymbol{d}, g)\|^2_\lambda,$$

by interpolation inequalities (11) and (12). Finally,

$$\sum_i \|\eta_i L\rho_i(\boldsymbol{N}' - \boldsymbol{e}_3)\|^2_{H_\lambda^{l+\frac{1}{2}}(\Sigma)} + \|\eta_i(L\rho_i + \Delta'\rho_i)\boldsymbol{e}_3\|^2_{H_\lambda^{l+\frac{1}{2}}(\Sigma)}$$

$$\leqslant \sum_i \delta\Big(\|\rho_i\|^2_{W_2^{l+\frac{5}{2}}(\Sigma)} + |\lambda|^{l+\frac{1}{2}}\|\rho_i\|^2_{W_2^2(\Sigma)}\big) + c(\delta)(\|\rho_i\|^2_{W_2^2(\Sigma)} + \|\rho_i\|^2_{W_2^{l+\frac{3}{2}}(\Sigma)}\Big)$$

$$\leqslant c\sum_i \delta\|\rho_i\|^2_{H_\lambda^{l+\frac{5}{2}}(\Sigma)} + c(\delta)\|\rho_i\|^2_{L^2(\Sigma)} \leqslant c\Big(\delta + \frac{c(\delta)}{|\lambda|^{l+\frac{5}{2}}}\Big)\|(\boldsymbol{d}, g)\|^2_\lambda.$$

All in all we have

$$\|\widehat{\boldsymbol{A}}(\boldsymbol{d}, g)\|^2_{H_\lambda^{l+\frac{1}{2}}(\mathcal{G})} \leqslant c\left(\delta + \frac{c(\delta)}{|\lambda|^{l+\frac{1}{2}}}\right)\|(\boldsymbol{d}, g)\|^2_\lambda. \tag{87}$$

5. *Estimate of* $\widehat{A}$

The estimate for $\nabla'_x\phi_b\cdot(\overline{\boldsymbol{v}}_1+\overline{\boldsymbol{v}}_2)-\overline{v}_1^3-\overline{v}_2^3$ follows from (75) for $\overline{\boldsymbol{v}}_1$ and (66) for $\overline{v}_2$. The argument is very similar to those given above, and is omitted. For the estimate of A we localize in the z coordinates, obtaining, via

$$(\nabla'_x\phi_b, -1)C_i^{-1}\boldsymbol{v}_i = (-\nabla'_z\phi_i, 1)\boldsymbol{v}_i, \quad \boldsymbol{v}_b\cdot\nabla'_x\rho_i = C_i\boldsymbol{v}_b\cdot\nabla'_z\rho_i,$$

the explicit representation

$$\begin{aligned}&\lambda\rho+\nabla'_x\phi_b\cdot\boldsymbol{v}-v^3+\boldsymbol{v}_b\cdot\nabla_x\rho-g\\&=\sum_i\eta_i(\lambda\rho_i+(\nabla'_x\phi_b,-1)\cdot C_i^{-1}\boldsymbol{v}_i+\boldsymbol{v}_b\cdot\nabla'_x\rho_i-\varphi_i g)+\sum_i\rho_i\boldsymbol{v}_b\cdot\nabla_x\eta_i\\&=\sum_i\eta_i\left[C_i(\boldsymbol{v}_b-\boldsymbol{v}_b(x_i))\nabla'_z\rho_i-\nabla'_z\phi_i\cdot\boldsymbol{v}_i\right]+\sum_i C_i\rho_i\boldsymbol{v}_b\cdot\nabla_z\eta_i.\end{aligned}$$

The second summand is a lower order term, which as usual is bounded by $\frac{c(\delta)}{|\lambda|}\|(\boldsymbol{d},g)\|^2_\lambda$. The higher order term $\eta_i\nabla'\phi_i\cdot\boldsymbol{v}_i$ is estimated through (73) as

$$\begin{aligned}\|\sum_i\eta_i\nabla'\phi_i\cdot\boldsymbol{v}_i\|^2_{H_\lambda^{l+\frac32}(\Sigma)}&\leqslant\sum_i\delta\|\boldsymbol{v}_i\|^2_{H_\lambda^{l+\frac32}(\Sigma)}+c(\delta)\|\boldsymbol{v}_i\|^2_{L^2(\Sigma)}\\&\leqslant\left(\delta+\frac{c(\delta)}{|\lambda|^{l+\frac32}}\right)\sum_i\|\boldsymbol{v}_i\|^2_{H_\lambda^{l+2}(\Sigma^\infty)}\leqslant c\left(\delta+\frac{c(\delta)}{|\lambda|}\right)\|(\boldsymbol{d},g)\|^2_\lambda,\end{aligned}$$

by interpolation inequality (12). For the remaining terms of the form $\eta_i(\boldsymbol{v}_b-\boldsymbol{v}_b(x_i))\nabla'_z\rho_i$, by the smoothness of $\boldsymbol{v}_b$ we can assume that $|v_b^k-v_b^k(x_i)|\leqslant\delta$ on the support of η_i and thus apply Lemma 4.3 with $\psi=\boldsymbol{v}_b-\boldsymbol{v}_b(x_i)$ to obtain

$$\begin{aligned}\|\sum_i\eta_iC_i(\boldsymbol{v}_b-\boldsymbol{v}_b(x_i))\nabla'_z\rho_i\|^2_{H_\lambda^{l+\frac32}(\Sigma)}&\leqslant\sum_i\delta\|\nabla\rho\|^2_{H_\lambda^{l+\frac32}(\Sigma)}+c(\delta)\|\nabla\rho\|^2_{L^2(\Sigma)}\\&\leqslant\sum_i\delta\|\rho\|^2_{H_\lambda^{l+\frac52}(\Sigma)}+\frac{c(\delta)}{|\lambda|^{l+\frac52}}\|\lambda\rho\|^2_{H_\lambda^{l+\frac32}(\Sigma)}\leqslant c\left(\delta+\frac{c(\delta)}{|\lambda|^{l+\frac52}}\right)\|(\boldsymbol{d},g)\|^2_\lambda,\end{aligned}$$

which completes the proof of the inequality

$$\|\widehat{A}(\boldsymbol{d},g)\|^2_{H_\lambda^{l+\frac32}(\mathcal{G})}\leqslant c\left(\delta+\frac{c(\delta)}{|\lambda|}\right)\|(\boldsymbol{d},g)\|^2_\lambda. \tag{88}$$

6. *Existence and estimates for the solution*

The estimates (87) and (88) yield

$$\|(\widehat{\boldsymbol{A}}(\boldsymbol{d},g),\widehat{A}(\boldsymbol{d},g))\|_\lambda\leqslant c\left(\delta+\frac{c(\delta)}{|\lambda|^\theta}\right)\|(\boldsymbol{d},g)\|_\lambda.$$

Therefore, fixing δ sufficiently small, for any sufficiently large $\operatorname{Re}\lambda$ $(\widehat{\boldsymbol{A}},\widehat{A})$ is a contraction on $W_2^{l+\frac12}(\mathcal{G})\times W_2^{l+\frac32}(\mathcal{G})$ normed with $\|\ \ \|_\lambda$, and thus

$R(I+(\widehat{\boldsymbol{A}},\widehat{A})^{-1}(\boldsymbol{d},g)$ is a solution. To obtain (68), it suffices to prove the continuity of the operator R defined in (76) with respect to the norm

$$\|(\boldsymbol{u},p,\rho)\|_\lambda^2 := \|\boldsymbol{u}\|^2_{H^{l+2}_\lambda(\Omega)} + \|\nabla q\|^2_{H^l_\lambda(\Omega)} + \|q\|^2_{H_\lambda^{l+\frac12}(\mathcal{G})} + \|\rho\|^2_{H_\lambda^{l+\frac52}(\mathcal{G})} + \|\lambda\rho\|^2_{H_\lambda^{l+\frac32}(\mathcal{G})}.$$

Now from (75) and (66) we have that

$$\|\overline{\boldsymbol{v}}_1+\overline{\boldsymbol{v}}_2\|^2_{H^{l+2}_\lambda(\Omega)} + \|\nabla p_2\|^2_{H^l_\lambda(\Omega)} + \|p_2\|^2_{H_\lambda^{l+\frac12}(\mathcal{G})} \leqslant c\big(\|\widehat{\boldsymbol{f}}\|^2_{H^l_\lambda(\Omega)} + \|\widehat{h}\|^2_{W_2^{l+1}(\Omega)} + |\lambda|^{l+2}(\|\widehat{\boldsymbol{F}}\|^2_{L^2(\Omega)} + \|\widehat{h}'\|^2_{L^2(\Omega)})\big),$$

and the right-hand side is bounded by $\|(\boldsymbol{d},g)\|^2_\lambda$ by (78), (86). By the definition of $(\boldsymbol{v},p,\rho)$ we have, through estimate (42), the inequality $\|(\boldsymbol{v},p,\rho)\|_\lambda \leqslant c(\delta)\|(\boldsymbol{d},g)\|_\lambda$, which is the desired continuity estimate.

7. *Uniqueness*

Let $(\boldsymbol{u},p,\rho)$ a solution to (39) with vanishing right-hand sides. Taking the scalar product with $\boldsymbol{u}$ in the first equation and integrating by parts gives

$$\lambda\int_\Omega |\boldsymbol{u}|^2\,dx + \frac{\nu}{2}\int_\Omega |\mathbb{D}(\boldsymbol{u})|^2\,dx = -\sigma\int_{\mathcal{G}} L\rho\boldsymbol{N}\cdot\boldsymbol{u}\,ds, \tag{89}$$

(the boundary terms on Σ vanish due to $\boldsymbol{u}\lfloor_\Sigma = 0$). The right-hand side can be rewritten using the equation for ρ and

$$\boldsymbol{N} = (-\nabla'\phi_b,1)/\sqrt{g_b},\quad g_b = 1+|\nabla\phi_b|^2,\quad ds = \sqrt{g_b}\,dx',$$

giving

$$\int_{\mathcal{G}} L\rho\boldsymbol{N}\cdot\boldsymbol{u}\,ds = \int_\Sigma L\rho(-\nabla'\rho\cdot\boldsymbol{u}+u^3)\,dx' = \int_\Sigma L\rho(\lambda\rho+\boldsymbol{v}_b\cdot\nabla'\rho)\,dx'.$$

If $\lambda = s+it$, taking the real part in (89) thus gives

$$s\int_\Omega |\boldsymbol{u}|^2\,dx + \frac{\nu}{2}\int_\Omega |\mathbb{D}(\boldsymbol{u})|^2\,dx = -\sigma B_s(\rho) < 0,$$

by Lemma 2.1, for s sufficiently large. Therefore, for $s = \operatorname{Re}\lambda$ sufficiently large we get $\boldsymbol{u}=0$, and $\nabla\rho = 0$ from $B_s(\rho)=0$ and (38). From the equation for ρ we thus get $\rho = 0$ and from the boundary condition on the stress tensor, $p=0$ on $\mathcal{G}$. Since the velocity equation now reads $\nabla p = 0$, we conclude that p vanishes in the whole Ω, and thus $(\boldsymbol{u},p,\rho)=(0,0,0)$. □

§5. Time dependent linear problem

In this section, we prove through Laplace transform methods (see again [1]), the solvability of the time-dependent linear problem (36).

We first need to consider the perturbed version of problem (39), i.e.,

$$\begin{cases} \lambda \boldsymbol{u} - \nu\Delta \boldsymbol{u} + \nabla q - \widehat{\boldsymbol{\Phi}}_1(\boldsymbol{u},\rho) = \boldsymbol{f} & \text{in } \Omega_b, \\ \nabla\cdot \boldsymbol{u} - \Phi_2(\rho) = h = \nabla\cdot \boldsymbol{F} & \text{in } \Omega_b, \\ \mathbb{T}(\boldsymbol{u},q)\boldsymbol{N} + \sigma L\rho \boldsymbol{N} - \boldsymbol{\Phi}_3(\rho) = \boldsymbol{d} & \text{on } \mathcal{G}, \\ \lambda\rho + \nabla'\phi_b\cdot \boldsymbol{u} - u^3 + \nabla'\rho\cdot \boldsymbol{v}_b = g & \text{on } \mathcal{G}, \\ \boldsymbol{u} = \boldsymbol{a} & \text{on } \Sigma, \end{cases} \tag{90}$$

where $\widehat{\boldsymbol{\Phi}}_1$ is as in (23) with $\lambda\rho^*$ instead of $\rho^*_{,t}$, and $a^3 = F^3 = 0$ on Σ.

Theorem 5.1. *Let $l \geqslant 0$. For any sufficiently large $\operatorname{Re}\lambda$, there exists a unique periodic solution of (90), for any choice of periodic $\boldsymbol{f} \in W_2^l(\Omega_b)$, $\boldsymbol{d} \in W_2^{l+\frac{1}{2}}(\mathcal{G})$, $g \in W_2^{l+\frac{3}{2}}(\mathcal{G})$, $h \in W_2^{l+1}(\Omega_b)$, and $\boldsymbol{F} \in W_2^1(\Omega_b)$ with $F^3\lfloor_\Sigma = 0$ and this solution satisfies (68).*

Proof. We start estimating the λ-weighted norm of the various Φ_i. For $\widehat{\boldsymbol{\Phi}}_1$, we see from definition (23) that all its terms are of the form (24). Each of these terms can be estimated in the $W_2^l(\Omega_b)$ norm through Proposition 1.3. One considers separately the terms containing the spatial derivatives of ρ and those containing the derivatives of θ to obtain, for $\operatorname{Re}\lambda \gg 1$,

$$\|\widehat{\boldsymbol{\Phi}}_1\|_{W_2^l(\Omega_b)} \leqslant c\Big(\|\nabla\rho\|_{W_2^{l+1}(\mathcal{G})} + |\lambda|\|\rho\|_{W_2^l(\mathcal{G})} + \|\boldsymbol{u}\|_{W_2^{l+1}(\Omega_b)}\Big),$$

where c is a constant depending on the higher order norms of $\boldsymbol{v}_b$, p_b and θ. Applying interpolation inequality one then obtains, again for $\operatorname{Re}\lambda >> 1$,

$$\|\widehat{\boldsymbol{\Phi}}_1\|^2_{W_2^l(\Omega_b)} \leqslant \frac{c}{\sqrt{|\lambda|}}\Big(\|\rho\|^2_{H_\lambda^{l+\frac{5}{2}}(\mathcal{G})} + \|\lambda\rho\|^2_{H_\lambda^{l+\frac{3}{2}}(\mathcal{G})} + \|\boldsymbol{u}\|^2_{H_\lambda^{l+2}(\Omega_b)}\Big).$$

The L^2 norm of $\widehat{\Phi}_1$ is estimated as

$$\|\widehat{\boldsymbol{\Phi}}_1\|_{L^2(\Omega_b)} \leqslant c\Big(\|\nabla\rho\|_{W_2^1(\mathcal{G})} + \|\lambda\rho\|_{L^2(\mathcal{G})} + \|\boldsymbol{u}\|_{W_2^1(\Omega_b)}\Big),$$

thus, bounding $\|\rho\|_{L^2(\mathcal{G})}$ with $\|\rho\|_{W_2^1(\mathcal{G})}$ and using interpolation inequalities,

$$|\lambda|^l\|\widehat{\boldsymbol{\Phi}}_1\|^2_{L^2(\Omega_b)} \leqslant \frac{c}{\sqrt{|\lambda|}}\Big(\|\rho\|^2_{H_\lambda^{l+\frac{5}{2}}(\mathcal{G})} + \|\lambda\rho\|^2_{H_\lambda^{l+\frac{3}{2}}(\mathcal{G})} + \|\boldsymbol{u}\|^2_{H_\lambda^{l+2}(\Omega_b)}\Big).$$

Therefore, for $\operatorname{Re}\lambda > 1$, we have

$$\|\widehat{\mathbf{\Phi}}_1\|^2_{H^l_\lambda(\Omega_b)} \leqslant \frac{c}{\sqrt{|\lambda|}}\Big(\|\rho\|^2_{H_\lambda^{l+\frac{5}{2}}(\mathcal{G})} + \|\lambda\rho\|^2_{H_\lambda^{l+\frac{3}{2}}(\mathcal{G})} + \|\boldsymbol{u}\|^2_{H_\lambda^{l+2}(\Omega_b)}\Big).$$

For Φ_2, recall by (29), that it is a linear combination of terms of the form $\boldsymbol{h}_\rho\rho^*_{,x_i}$ and $\boldsymbol{h}_\rho\rho$, therefore,

$$\|\Phi_2(\rho)\|_{W_2^{l+1}(\Omega_b)} \leqslant c\Big(\|\nabla\rho\|_{W_2^{l+1}(\mathcal{G})} + \|\rho\|_{W_2^{l+1}(\mathcal{G})}\Big),$$

giving

$$\|\Phi_2(\rho)\|^2_{W_2^{l+1}(\Omega_b)} \leqslant \frac{c}{\sqrt{|\lambda|}}\Big(\|\rho\|^2_{H_\lambda^{l+\frac{5}{2}}(\mathcal{G})} + \|\lambda\rho\|^2_{H_\lambda^{l+\frac{3}{2}}(\mathcal{G})}\Big).$$

Moreover, recalling that $\Phi_2(\rho) = \nabla\cdot(I-\widehat{\mathcal{L}})\boldsymbol{v}_b$, we have for $(I-\widehat{\mathcal{L}})\boldsymbol{v}_b$,

$$|\lambda|^{l+2}\|(I-\widehat{\mathcal{L}})\boldsymbol{v}_b\|^2_{L^2(\Omega_b)} \leqslant c|\lambda|^{l+2}\|\rho\|^2_{W_2^1(\mathcal{G})} \leqslant \frac{c}{\sqrt{|\lambda|}}\|\lambda\rho\|^2_{W_\lambda^{l+\frac{3}{2}}(\mathcal{G})}.$$

Finally $\mathbf{\Phi}_3$ is made up of terms of the form $\boldsymbol{h}\rho$ and $\boldsymbol{h}\rho_{,x_i}$ therefore, for $\operatorname{Re}\lambda > 1$,

$$\|\mathbf{\Phi}_3\|^2_{H_\lambda^{l+\frac{1}{2}}(\mathcal{G})} \leqslant c\|\rho\|^2_{H_\lambda^{l+\frac{3}{2}}(\mathcal{G})} \leqslant \frac{c}{|\lambda|}\|\rho\|^2_{H_\lambda^{l+\frac{5}{2}}(\mathcal{G})},$$

with a constant depending on p_b, $\boldsymbol{v}_b$, ϕ_b and p_e. A standard iteration argument now gives existence and uniqueness of a solution satisfying (68) for sufficiently large $\operatorname{Re}\lambda$. □

Theorem 5.2. *Let $l \in (\frac{1}{2}, 1)$ and $T < +\infty$. For any Σ-periodic choice of $\boldsymbol{f} \in K^l(Q_T)$, $h \in W_2^{l+1,0}(Q_T)$, $\boldsymbol{F} \in W_2^{0,\frac{l}{2}+1}(Q_T)$ with $F^3\lfloor_\Sigma = 0$, $\boldsymbol{d} \in K^{l+\frac{1}{2}}(G_T)$, $g \in K^{l+\frac{3}{2}}(G_T)$, $\boldsymbol{a} \in K^{l+\frac{3}{2}}(\Sigma)$ with $a^3 \equiv 0$, $\rho_0 \in W_2^{l+2}(\mathcal{G})$, and $\boldsymbol{u}_0 \in W_2^{l+1}(\Omega)$ such that*

$$\begin{cases} \nabla\cdot\boldsymbol{u}_0 = \Phi_2(\rho_0) + \nabla\cdot\boldsymbol{F}(\cdot,0) & \text{in } \Omega_b, \\ \nu\Pi_b\mathbb{D}(\boldsymbol{u}_0)\boldsymbol{N} = \Pi_b\big(\mathbf{\Phi}_3(\rho_0) + \boldsymbol{d}(\cdot,0)\big) & \text{on } \mathcal{G}, \\ \boldsymbol{u}_0 = \boldsymbol{a}(\cdot,0) & \text{on } \Sigma, \end{cases} \tag{91}$$

there exists a unique solution to (36), and it holds the estimate (see (7))

$$\|(\boldsymbol{u},q,\rho)\|_{l,T} \leqslant c_T\Big(\|\boldsymbol{v}_0\|_{W_2^{l+1}(\Omega_b)} + \|\rho_0\|_{W_2^{l+2}(\mathcal{G})} + \|\boldsymbol{a}\|_{K^{l+\frac{3}{2}}(\Sigma_T)} + \|\boldsymbol{f}\|_{K^l(Q_T)}$$
$$+\|h\|_{W_2^{l+1,0}(Q_T)} + \|\boldsymbol{F}\|_{W_2^{0,\frac{l}{2}+1}(Q_T)} + \|\boldsymbol{d}\|_{K^{l+\frac{1}{2}}(G_T)} + \|g\|_{K^{l+\frac{3}{2}}(G_T)}\Big). \tag{92}$$

Moreover, if $T \geqslant 1$, it holds

$$\begin{aligned} \|(\boldsymbol{u}, q,\rho)\|_{l,T} \leqslant c\Big(&\|\boldsymbol{f}\|_{K^l(Q_T)} + \|h\|_{W_2^{l+1,0}(Q_T)} + \|\boldsymbol{F}\|_{W_2^{0,\frac{l}{2}+1}(Q_T)} \\ &+ \|\boldsymbol{d}\|_{K^{l+\frac{1}{2}}(G_T)} + \|g\|_{K^{l+\frac{3}{2}}(G_T)} + \|\boldsymbol{a}\|_{K^{l+\frac{3}{2}}(\Sigma_T)} \\ &+ \|\boldsymbol{v}_0\|_{W_2^{l+1}(\Omega_b)} + \|\rho_0\|_{W_2^{l+2}(\mathcal{G})} + \|\boldsymbol{u}\|_{L^2(Q_T)} + \|\rho\|_{L^2(G_T)}\Big). \end{aligned} \tag{93}$$

with constant independent of T.

Proof. We will reduce problem (36) to a similar one with homogeneous initial data in order to apply Laplace transform and use Theorem 5.1 to get the solution. First of all we fix $T_0 \geqslant T+1$ and extend all the right-hand terms except h and $\boldsymbol{F}$ (keeping the notation unchanged) to Q_∞ and G_∞ with controlled norm, supposing furthermore that all the terms vanish for $t > T_0$. For $T \geqslant 1$, this can be done with constants independent of T, i.e.,

$$\begin{aligned} &\|\boldsymbol{f}\|_{K^l(Q_\infty)} + \|\boldsymbol{d}\|_{K^{l+\frac{1}{2}}(G_\infty)} + \|g\|_{K^{l+\frac{3}{2}}(G_\infty)} + \|\boldsymbol{a}\|_{K^{l+\frac{3}{2}}(\Sigma_\infty)} \\ &\leqslant c\Big(\|\boldsymbol{f}\|_{K^l(Q_T)} + \|\boldsymbol{d}\|_{K^{l+\frac{1}{2}}(G_T)} + \|g\|_{K^{l+\frac{3}{2}}(G_T)} + \|\boldsymbol{a}\|_{K^{l+\frac{3}{2}}(\Sigma_T)}\Big). \end{aligned}$$

To construct the extensions of h and $\boldsymbol{F}$, we define, for all $t \leqslant T$, $\boldsymbol{w}_0 = \nabla\psi$, where ψ is the solution of

$$\begin{cases} \Delta\psi = h = \nabla\cdot\boldsymbol{F} & \text{in} \quad \Omega_b, \\ \psi = 0 & \text{on} \quad \mathcal{G}, \\ \frac{\partial\psi}{\partial\boldsymbol{n}} = \boldsymbol{F}\cdot\boldsymbol{n} = 0 & \text{on} \quad \Sigma. \end{cases} \tag{94}$$

From standard elliptic estimates valid for any $t \leqslant T$, we have

$$\|\boldsymbol{w}_0\|_{W_2^{l+2,0}(Q_T)} \leqslant c\|h\|_{W_2^{l+1,0}(Q_T)}$$

and

$$\sup_{t\leqslant T}\|\boldsymbol{w}_0\|_{W_2^{l+1}(\Omega_b)} \leqslant c\sup_{t\leqslant T}\|h\|_{W_2^l(\Omega)}.$$

Differentiating in t the weak formulation of (94), one gets

$$\int_{\Omega_b}\boldsymbol{w}_{0,t}\cdot\nabla\eta\,dx = \int_{\Omega_b}\boldsymbol{F}_{,t}\cdot\nabla\eta\,dx, \quad \forall\eta\in C^\infty, \quad \eta\lfloor_{\mathcal{G}} = 0,$$

and a similar identity for the finite differences in time of $\boldsymbol{w}_{0,t}$, which implies

$$\|\boldsymbol{w}_0\|_{W_2^{0,\frac{l}{2}+1}(Q_T)} \leqslant c\|\boldsymbol{F}\|_{W_2^{0,\frac{l}{2}+1}(Q_T)},$$

Therefore, it holds

$$\|\boldsymbol{w}_0\|_{K^{l+2}(Q_T)} \leqslant c\Big(\|h\|_{W_2^{l+1,0}(Q_T)} + \|\boldsymbol{F}\|_{W_2^{0,\frac{l}{2}+1}(Q_T)}\Big).$$

We extend $\boldsymbol{w}_0$ in such a way that $\boldsymbol{w}_0 = 0$ for $t \geqslant T_0$, $w_0^3(x,t) = 0$ for all $t \geqslant 0$ and $x \in \Sigma$, and with $K^{l+2}(Q_\infty)$ norm controlled by the $K^{l+2}(Q_T)$-norm of $\boldsymbol{w}_0$ if $T \geqslant 1$ with a constant independent of T. Then, for all $t \geqslant 0$, we define

$$\boldsymbol{F}_0 = \boldsymbol{w}_0, \quad h = \nabla \cdot \boldsymbol{w}_0,$$

and $\boldsymbol{f}_0 = \boldsymbol{w}_{0,t} - \nu\Delta\boldsymbol{w}_0$. It is clear that problem (36) is equivalent to the same problem with $\boldsymbol{F}_0$ instead of $\boldsymbol{F}$. It holds

$$\|h\|_{W_2^{l+1,0}(Q_\infty)} + \|\boldsymbol{F}_0\|_{W_2^{0,\frac{l}{2}+1}(Q_\infty)} \leqslant c\Big(\|h\|_{W_2^{l+1,0}(Q_T)} + \|\boldsymbol{F}\|_{W_2^{0,\frac{l}{2}+1}(Q_T)}\Big),$$

with constants independent of T, and

$$\begin{aligned}\|\boldsymbol{f}_0\|_{K^{l+2}(Q_\infty)} &\leqslant \|\boldsymbol{w}_0\|_{K^{l+2}(Q_\infty)} \\ &\leqslant c\Big(\|h\|_{W_2^{l+1,0}(Q_\infty)} + \|\boldsymbol{F}_0\|_{W_2^{0,\frac{l}{2}+1}(Q_\infty)}\Big).\end{aligned} \tag{95}$$

We now let

$$\sigma_0 = g(0) - \boldsymbol{v}_b \cdot \nabla'\rho_0 - \nabla'\phi_b \cdot \boldsymbol{u}_0 + u_0^3,$$

and we construct ρ_1 in such a way that

$$\rho_1\lfloor_{t=0} = \rho_0, \quad \rho_{1,t}\lfloor_{t=0} = \sigma_0,$$

$$\begin{aligned}&\|\rho_1\|_{K^{l+\frac{5}{2}}(G_\infty)} + \|\rho_{1,t}\|_{K^{l+\frac{3}{2}}(G_\infty)} \leqslant c\Big(\|\rho_0\|_{W_2^{l+2}(\mathcal{G})} + \|\sigma_0\|_{W_2^{l+\frac{1}{2}}(\mathcal{G})}\Big) \\ &\leqslant c\Big(\|g\|_{K^{l+\frac{3}{2}}(G_\infty)} + \|\rho_0\|_{W_2^{l+2}(\mathcal{G})} + \|\boldsymbol{u}_0\|_{W_2^{l+1}(\Omega_b)}\Big).\end{aligned} \tag{96}$$

We first consider $r_1 \in W_2^{l+\frac{5}{2}}(G_\infty)$ such that $r_1\lfloor_{t=0} = \rho_0$, $r_{1,t}\lfloor_{t=0} = 0$ and

$$\|r_1\|_{W_2^{l+\frac{5}{2}}(G_\infty)} \leqslant c\|\rho_0\|_{W_2^{l+2}(\mathcal{G})}.$$

Then, we construct $r_2 \in K^{l+\frac{7}{2}}(G_\infty)$ such that $r_2\lfloor_{t=0} = 0$, $r_{2,t}\lfloor_{t=0} = \sigma_0$ and

$$\begin{aligned}\|r_2\|_{K^{l+\frac{7}{2}}(G_\infty)} &\leqslant \|\sigma_0\|_{W_2^{l+\frac{1}{2}}(\mathcal{G})} \\ &\leqslant c\Big(\|g(0)\|_{W_2^{l+\frac{1}{2}}(\mathcal{G})} + \|\rho_0\|_{W_2^{l+\frac{3}{2}}(\mathcal{G})} + \|\boldsymbol{u}_0\|_{W_2^{l+\frac{1}{2}}(\mathcal{G})}\Big) \\ &\leqslant c\Big(\|g\|_{K^{l+\frac{3}{2}}(G_\infty)} + \|\rho_0\|_{W_2^{l+2}(\mathcal{G})} + \|\boldsymbol{u}_0\|_{W_2^{l+1}(\Omega_b)}\Big),\end{aligned}$$

for a constant c depending only on $\boldsymbol{v}_b$ and ϕ_b. The sum $\rho_1 = r_1 + r_2$ clearly satisfies the initial conditions and

$$\|\rho_1\|_{W_2^{l+\frac{5}{2},0}(G_\infty)} + \|\rho_{1,t}\|_{K^{l+\frac{3}{2}}(G_\infty)} \leqslant c\Big(\|\rho_0\|_{W_2^{l+2}(\mathcal{G})} + \|\sigma_0\|_{W_2^{l+\frac{1}{2}}(\mathcal{G})}\Big).$$

Finally, from the inequality

$$(1+|\xi_0|+|\xi|^2)^{l+\frac{5}{2}} \leqslant c_l\Big((1+|\xi|^2)^{l+\frac{5}{2}} + (1+|\xi_0|+|\xi|^2)^{l+\frac{1}{2}}|\xi_0|^2\Big),$$

we get, through local coordinates, Fourier transform and Parceval identity

$$\|\rho_1\|_{K^{l+\frac{5}{2}}(G_\infty)} \leqslant c\Big(\|\rho_1\|_{W_2^{l+\frac{5}{2},0}(G_\infty)} + \|\rho_{1,t}\|_{K^{l+\frac{1}{2}}(G_\infty)}\Big),$$

and thus (96). Clearly we can modify ρ_1 so as to obtain $\rho_1 = 0$ for $t \geqslant T_0$, without affecting the latter inequality. Then we take out a part of the divergence, considering $\boldsymbol{w}_1 = \nabla\psi$ where ψ is the periodic solution of

$$\begin{cases} \Delta\psi = \Phi_2(\rho_1) = \nabla\cdot\boldsymbol{F}_1 & \text{in} \quad \Omega_b, \\ \psi = 0 & \text{on} \quad \mathcal{G}, \\ \frac{\partial\psi}{\partial\boldsymbol{n}} = \boldsymbol{F}_1\cdot\boldsymbol{n} = 0 & \text{on} \quad \Sigma, \end{cases}$$

where $\boldsymbol{F}_1 = (I - \widehat{\mathcal{L}}(\rho_1))\boldsymbol{v}_b$, which vanishes in a neighborhood of Σ. Notice that since ρ_1 vanishes for $t \geqslant T_0$ this is also true for $\boldsymbol{w}_1$. We set $\boldsymbol{f}_1 = \boldsymbol{w}_{1,t} - \nu\Delta\boldsymbol{w}_1$. With the same argument as for problem (94), we obtain

$$\|\boldsymbol{f}_1\|_{K^l(Q_\infty)} \leqslant c\|\boldsymbol{w}_1\|_{K^{l+2}(Q_\infty)} \leqslant c\Big(\|\Phi_2(\rho_1)\|_{W_2^{l+1}(Q_\infty)} + \|\boldsymbol{F}_1\|_{W_2^{0,\frac{l}{2}+1}(Q_\infty)}\Big)$$

and looking at the explicit form (29), (28) of Φ_2 and $\boldsymbol{F}_1$, we find that

$$\begin{aligned} \|\boldsymbol{w}_1\|_{K^{l+2}(Q_\infty)} &\leqslant c\Big(\|\rho_1\|_{W_2^{l+2,0}(Q_\infty)} + \|\nabla\rho_1\|_{W_2^{0,\frac{l}{2}+1}(Q_\infty)} + \|\rho_1\|_{W_2^{0,\frac{l}{2}+1}(Q_\infty)}\Big) \\ &\leqslant c\Big(\|\rho_1\|_{K^{l+\frac{5}{2}}(Q_\infty)} + \|\rho_{1,t}\|_{K^{l+\frac{3}{2}}(Q_\infty)}\Big), \end{aligned}$$

which (by (96)) yields the estimate

$$\|\boldsymbol{f}_1\|_{K^l(Q_\infty)} \leqslant c\Big(\|g\|_{K^{l+\frac{3}{2}}(G_\infty)} + \|\rho_0\|_{W_2^{l+2}(\mathcal{G})} + \|\boldsymbol{u}_0\|_{W_2^{l+1}(\Omega_b)}\Big).$$

We choose $\boldsymbol{w}_2 \in K^{l+2}(Q_\infty)$ in such a way that

$$\nabla\cdot\boldsymbol{w}_2 = 0, \quad \forall t \geqslant 0, \quad \boldsymbol{w}_2(\cdot,0) = \boldsymbol{u}_0(\cdot) - \boldsymbol{w}_0(\cdot) - \boldsymbol{w}_1(\cdot,0),$$

with $\boldsymbol{w}_2 = 0$ for $t > T_0$, and optimal regularity estimates: by a result of Bogowskii [5], $\boldsymbol{u}_0(x) - \boldsymbol{w}_0(x,0) - \boldsymbol{w}_1(x,0)$ can be extended with preservation of class and solenoidality for all $x \in \mathbb{R}^3$, as a vector $\boldsymbol{w}_2^*$. We set

$$\boldsymbol{w}_2(x,t) = \phi(t) \int\limits_{\mathbb{R}^3} \Gamma(x-y,t)\boldsymbol{w}_2^*(y)\,dy,$$

where $\phi(t)$ is a smooth function equal to one for small t and vanishing for $t \geqslant T_0$, and $\Gamma(x,t)$ is the fundamental solution of the heat equation. Well known estimates of the heat potential give

$$\|\boldsymbol{w}_2\|_{K^{l+2}(Q_T)} \leqslant c\|\boldsymbol{w}_2^*\|_{W_2^{l+1}(\mathbb{R}^3)} \leqslant c\|\boldsymbol{u}_0 - \boldsymbol{w}_1\|_{W_2^{l+1}(\Omega_b)}.$$

Letting $\boldsymbol{f}_2 = \boldsymbol{w}_{2,t} - \nu\Delta\boldsymbol{w}_2$, we obtain

$$\begin{aligned}\|\boldsymbol{f}_2\|_{K^l(Q_\infty)} &\leqslant c\|\boldsymbol{w}_2\|_{K^{l+2}(Q_\infty)} \\ &\leqslant c\Big(\|\boldsymbol{u}_0\|_{W_2^{l+1}(\Omega_b)} + \|\boldsymbol{w}_0\|_{K^{l+2}(Q_\infty)} + \|\boldsymbol{w}_1\|_{K^{l+2}(Q_\infty)}\Big).\end{aligned}$$

Finally, we set $\boldsymbol{f}_3 = -\boldsymbol{\Phi}_1(\boldsymbol{w}_0+\boldsymbol{w}_1+\boldsymbol{w}_2,\rho_1)$. From the explicit structure of $\boldsymbol{\Phi}_1$ given in (23) and applying (13), one sees that it holds

$$\|\boldsymbol{f}_3\|_{K^l(Q_\infty)} \leqslant c\Big(\sum_{i=0}^{3}\|\boldsymbol{w}_i\|_{K^{l+2}(Q_\infty)} + \|\rho_1\|_{K^{l+\frac{5}{2}}(G_\infty)} + \|\rho_{1,t}\|_{K^{l+\frac{3}{2}}(G_\infty)}\Big).$$

Letting

$$\sigma_2 = \nu\boldsymbol{N}\cdot\mathbb{D}(\boldsymbol{u}_0)\boldsymbol{N} + \sigma L\rho_0 - \boldsymbol{d}(0)\cdot\boldsymbol{N} - \boldsymbol{\Phi}_3(\rho_0)\cdot\boldsymbol{N},$$

we have $\sigma_2 \in W_2^{l-\frac{1}{2}}(\mathcal{G})$ (the four terms have regularity, respectively, l, l, $l-\frac{1}{2}$ and $l+1$) and thus we can extend it to the whole Ω_b as $\widehat{\sigma}_2 \in W_2^l(\Omega_b)$ with controlled norm, and subsequently define $p_1 \in K^{l+1}(Q_\infty)$ as an extension of $\widehat{\sigma}_2$ to Q_∞, also with controlled norm. Therefore p_1 satisfies

$$p_1(0)\lfloor_{\mathcal{G}} = \nu\boldsymbol{N}\cdot\mathbb{D}(\boldsymbol{u}_0)\boldsymbol{N} + \sigma L\rho_0 - \boldsymbol{d}(0)\cdot\boldsymbol{N} - \boldsymbol{\Phi}_3(\rho_0)\cdot\boldsymbol{N},$$

$$\|p_1\|_{K^{l+1}(Q_\infty)} \leqslant c\Big(\|\boldsymbol{v}_0\|_{W_2^{l+1}(\Omega_b)} + \|\rho_0\|_{W_2^{l+2}(\mathcal{G})} + \|\boldsymbol{d}\|_{K^{l+\frac{1}{2}}(G_\infty)}\Big), \quad (97)$$

where we used that $\boldsymbol{\Phi}_3(\rho_0)$ is of the type $\boldsymbol{M}\nabla\rho_0 + \boldsymbol{m}\rho$ for regular $\boldsymbol{M}$ and $\boldsymbol{m}$ depending on the data. We finally define $\boldsymbol{w} = \boldsymbol{w}_0 + \boldsymbol{w}_1 + \boldsymbol{w}_2$ and

$$\begin{aligned}
&(\widehat{\boldsymbol{u}}, \widehat{q}, \widehat{\rho}) = (\boldsymbol{u} - \boldsymbol{w}, q - p_1, \rho - \rho_1),\\
&\widehat{\boldsymbol{f}} = \boldsymbol{f} - \nabla p_1 - \sum_{i=0}^{3} \boldsymbol{f}_i,\\
&\widehat{\boldsymbol{d}} = \boldsymbol{d} - \nu\mathbb{D}(\boldsymbol{w})\boldsymbol{N} - \sigma L\rho_1\boldsymbol{N} + p_1\boldsymbol{N} + \boldsymbol{\Phi}_3(\rho_1),\\
&\widehat{g} = g - \rho_{1,t} - \nabla'\phi_b \cdot \boldsymbol{w} + w^3 - \boldsymbol{v}_b \cdot \nabla'\rho_1,\\
&\widehat{\boldsymbol{a}} = \boldsymbol{a} - \boldsymbol{w}.
\end{aligned}$$

Problem (36) is then reduced to

$$\begin{cases}
\widehat{\boldsymbol{u}}_{,t} - \nu\Delta\widehat{\boldsymbol{u}} + \nabla\widehat{q} - \boldsymbol{\Phi}_1(\widehat{\boldsymbol{u}}, \widehat{\rho}) = \widehat{\boldsymbol{f}} & \text{in} \quad \Omega_b,\\
\nabla \cdot \widehat{\boldsymbol{u}} - \Phi_2(\widehat{\rho}) = 0 & \text{in} \quad \Omega_b,\\
\mathbb{T}(\widehat{\boldsymbol{u}}, \widehat{q})\boldsymbol{N} + \sigma L\widehat{\rho}\boldsymbol{N} - \boldsymbol{\Phi}_3(\widehat{\rho}) = \widehat{\boldsymbol{d}} & \text{on} \quad \mathcal{G},\\
\widehat{\rho}_t + \nabla'\phi_b \cdot \widehat{\boldsymbol{u}} - \widehat{u}^3 + \nabla'\widehat{\rho} \cdot \boldsymbol{v}_b = \widehat{g} & \text{on} \quad \mathcal{G},\\
\widehat{\boldsymbol{u}} = \widehat{\boldsymbol{a}} \text{ on } \Sigma \quad \text{for all} \quad t \geqslant 0, &\\
\widehat{\boldsymbol{u}}(x,0) = 0, \quad x \in \Omega_b, \quad \widehat{\rho}(x',0) = 0, \quad x \in \Sigma, &
\end{cases} \tag{98}$$

where $\widehat{g}(0) = 0$ by the definition of ρ_1, $\widehat{\boldsymbol{a}}(0) = 0$ by the third condition in (91) and the definition of $\boldsymbol{w}_2$, and $\widehat{\boldsymbol{d}}(0) = 0$ by the second condition in (91) and the definition of $p_1(0)$. Since $l < 1$, $\widehat{\boldsymbol{f}}$, $\widehat{\boldsymbol{d}}$, $\widehat{g}$, and $\widehat{\boldsymbol{a}}$ can be extended with 0 for $t < 0$ preserving regularity, and we can apply the Laplace transform to convert (98) to a problem of the form (90). The latter is solvable for $\operatorname{Re}\lambda \geqslant \gamma > 0$ for γ sufficiently large by Theorem 5.1. Inverting the Laplace transform gives a solution in weighted Sobolev–Slobodetskii space $W_{2,\gamma'}^{\eta,\eta/2}$ for $\gamma' > \gamma$, defined for all t and vanishing for $t < 0$. We obtain a weighted estimated, which can be localised in $[0,T)$ on the left-hand side, while the-right one is controlled by the non weighted norm since all the terms vanish for $t \geqslant T_0$. Localization gives rise to the constant $C(T)$ in the bound (92).

Moreover, from (95)–(97), it follows

$$\begin{aligned}
&\|\widehat{\boldsymbol{f}}\|_{K^l(Q_\infty)} + \|\widehat{\boldsymbol{d}}\|_{K^{l+\frac{1}{2}}(G_\infty)} + \|\widehat{g}\|_{K^{l+\frac{3}{2}}(G_\infty)} + \|\widehat{\boldsymbol{a}}\|_{K^{l+\frac{3}{2}}(\Sigma_\infty)} \\
&\leqslant c\Big(\|\boldsymbol{f}\|_{K^l(Q_\infty)} + \|h\|_{W_2^{l+1,0}(Q_\infty)} + \|\boldsymbol{F}_0\|_{W_2^{0,\frac{l}{2}+1}(Q_\infty)} + \|\boldsymbol{d}\|_{K^{l+\frac{1}{2}}(G_\infty)} \\
&\quad + \|g\|_{K^{l+\frac{3}{2}}(G_\infty)} + \|\boldsymbol{a}\|_{K^{l+\frac{3}{2}}(\Sigma_\infty)} + \|\boldsymbol{u}_0\|_{W_2^{l+1}(\Omega_b)} + \|\rho_0\|_{W_2^{l+2}(\mathcal{G})}\Big) \\
&\leqslant c\Big(\|\boldsymbol{f}\|_{K^l(Q_T)} + \|h\|_{W_2^{l+1,0}(Q_T)} + \|\boldsymbol{F}_0\|_{W_2^{0,\frac{l}{2}+1}(Q_T)} + \|\boldsymbol{d}\|_{K^{l+\frac{1}{2}}(G_T)} \\
&\quad + \|g\|_{K^{l+\frac{3}{2}}(G_T)} + \|\boldsymbol{a}\|_{K^{l+\frac{3}{2}}(\Sigma_T)} + \|\boldsymbol{u}_0\|_{W_2^{l+1}(\Omega_b)} + \|\rho_0\|_{W_2^{l+2}(\mathcal{G})}\Big),
\end{aligned}$$

with c independent of T. The same estimate also hold for $\|(\boldsymbol{w}, p_1, \rho_1)\|_{l,T}$ and summing back those term to $(\widehat{\boldsymbol{u}}, \widehat{q}, \widehat{\rho})$ does not affect (92). Uniqueness is obtained through standard properties of Laplace Transform. To obtain (93) one splits the solution through a partition of unity of $[0, T]$ with functions having support of length 1, in order to apply (92) with $T \equiv 1$ on each such support. Lower order terms appearing due to this splitting are dealt with interpolation inequality, and give rise to the L^2 terms in (93). $\square$

§6. Estimate of the nonlinear terms

In this section, our main assumption will be that the transformation (4) is well defined, and thus we require that $\sup|\rho| \leqslant \mu(\theta) \ll 1$. This ensures that all the nonlinear terms are polynomials in the derivatives of $\boldsymbol{u}$, p, and ρ multiplied by a nonlinear term which is of the form $h_\rho = h(x, \rho, \nabla\rho)$ for smooth f's. Indeed the only singularity in the nonlinear terms appears in the Jacobian of Hanzawa transformation, where

$$\det \mathcal{L}^{-1} = \frac{1}{1+\theta'\rho}.$$

In the following, we will call μ any positive number such that

$$\mu \leqslant \frac{c}{\sup_\Sigma |\theta'|}. \tag{99}$$

for sufficiently small constant, so that for example the condition

$$\|\rho\|_{W_2^{l+1}(\mathcal{G})} \leqslant \mu$$

will ensure $|\rho\theta'| < \frac{1}{2}$ and the smoothness of the nonlinear terms. We prove the following theorem.

Theorem 6.1. *Let $l \in (\frac{1}{2}, 1)$. Suppose that $\|(\boldsymbol{u}, p, \rho)\|_{l,T} \leqslant \mu$, where μ is such that* (99) *holds. There exists $c(\mu)$, bounded for bounded μ such that*

$$\|\boldsymbol{l}_0(\boldsymbol{u},\rho)+\boldsymbol{l}_1(\boldsymbol{u},p,\rho)\|_{K^l(Q_T)}+\|l_2(\boldsymbol{u},\rho)\|_{W_2^{l+1,0}(Q_T)}+\|\boldsymbol{G}(\boldsymbol{u},\rho)\|_{W_2^{0,\frac{l}{2}+1}(Q_T)}$$
$$+\|\boldsymbol{l}_3(\boldsymbol{u},\rho)\|_{K^{l+\frac{1}{2}}(G_T)}+\|l_4(\boldsymbol{u},\rho)\|_{K^{l+\frac{3}{2}}(G_T)} \leqslant c_\mu \|(\boldsymbol{u},p,\rho)\|_{l,T}^2.$$

The constant c_μ also depends on $\boldsymbol{v}_b$, p_b, ϕ_b, $\boldsymbol{f}$, and does not depend on T as long as $T \geqslant 1$.

In the rest of this section, we will thus always suppose $l \in (\frac{1}{2}, 1)$, $\|(\boldsymbol{u}, p, \rho)\|_l \leqslant \mu$. Moreover, for any given function $g : X \times Y \to \mathbb{R}$ and $\eta \geqslant 0$ we denote by $\|g\|_{W_2^\eta(X)}$ the function $y \to \|g(\cdot, y)\|_{W_2^\eta(X)}$ and similarly for $\|g\|_{W_2^\eta(Y)}$.

Since $\rho^* = \theta\rho$ and θ is C^∞, any norm of ρ^* in Ω_b or Q_T is bounded by the same norm of ρ in $\mathcal{G}$ or G_T. Notice that, letting from now on

$$\|\rho\|_l = \|\rho\|_{K^{l+\frac{5}{2}}(G_T)} + \|\rho_{,t}\|_{K^{l+\frac{3}{2}}(G_T)},$$

proposition 1.4 gives

$$\sup_{t<T} \|\rho\|_{W_2^{l+2}(\mathcal{G})} \leqslant c\big(\|\rho\|_{W_2^{l+\frac{5}{2},0}(G_T)} + \|\rho_{,t}\|_{W_2^{l+\frac{3}{2},0}(G_T)}\big) \leqslant c\|\rho\|_l. \tag{100}$$

From standard embedding theorems, it follows

$$\sup_{Q_T} |\rho^*| + |\nabla\rho^*| \leqslant c(\sup_{G_T} |\rho| + |\nabla'\rho|) \leqslant c\|\rho\|_l. \tag{101}$$

We will also frequently use the following bounds:

$$\sup_{\mathcal{G}} \|\rho\|_{W_2^{\frac{l}{2}+\frac{5}{4}}(0,T)} + \|\nabla\rho\|_{W_2^{\frac{l}{2}+\frac{3}{4}}(0,T)} \leqslant c\Big(\|\rho\|_{W_2^{\frac{l}{2}+\frac{3}{4}}(0,T)} + \|\nabla\rho\|_{W_2^{\frac{l}{2}+\frac{1}{4}}(0,T)}$$
$$+\sup_{\mathcal{G}} \|\rho_{,t}\|_{W_2^{\frac{l}{2}+\frac{1}{4}}(0,T)} + \|\nabla\rho_{,t}\|_{W_2^{\frac{l}{2}-\frac{1}{4}}(0,T)}\Big) \tag{102}$$
$$\leqslant c\Big(\|\rho\|_{K^{l+\frac{5}{2}}(G_T)}+\|\nabla\rho\|_{K^{l+\frac{3}{2}}(G_T)}+\|\rho_{,t}\|_{K^{l+\frac{3}{2}}(G_T)}+\|\nabla\rho_{,t}\|_{K^{l+\frac{1}{2}}(G_T)}\Big)\leqslant c\|\rho\|_l,$$

$$\sup_{\Omega_b} \|\boldsymbol{u}\|_{W_2^{\frac{l}{2}+\frac{1}{4}}(0,T)} \leqslant c\|\boldsymbol{u}\|_{W_2^{l+2,\frac{l}{2}+1}(Q_T)}. \tag{103}$$

Indeed (102) and (103) follow from repeated application (twice for ρ and thrice for $\boldsymbol{u}$) of the standard restriction estimates in anisotropic Sobolev–Slobodetskii spaces. The previous constants depend only on $\mathcal{G}$ and Ω_b, and not on T, as long as $T \geqslant 1$, which will always be assumed.

Lemma 6.2. *Let $l \in (\frac{1}{2}, 1)$, and suppose $\|\rho\|_l \leqslant \mu$. Given a smooth function $f : \mathcal{G} \times \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}$, there exists a constant $c_f(\mu)$, depending also on $\mathcal{G}$ and bounded for bounded μ, such that for any function $g = g(x,t)$ and any $\eta \leqslant 1 + l$, $\eta' \leqslant \frac{l}{2} + \frac{3}{4}$, it holds*

$$\|f(x,\rho,\nabla\rho)g\|_{W_2^{\eta,0}(G_T)} \leqslant c_f(\mu)\|g\|_{W_2^{\eta,0}(G_T)},$$
$$\|f(x,\rho,\nabla\rho)g\|_{W_2^{0,\eta'}(G_T)} \leqslant c_f(\mu)\|g\|_{W_2^{0,\eta'}(G_T)}.$$

Proof. Letting $f = f(x,s,p)$, we will omit the explicit dependence of $f(x,\rho,\nabla\rho)$ on its arguments, letting for simplicity

$$f_\rho(x,t) = f(x,\rho(x,t),\nabla\rho(x,t)), \qquad f_{\rho,x}(x,t) = f_{,x}(x,\rho(x,t),\nabla\rho(x,t)),$$
$$f_{\rho,s}(x,t) = f_{,s}(x,\rho(x,t),\nabla\rho(x,t)), \qquad f_{\rho,p}(x,t) = f_{,p}(x,\rho(x,t),\nabla\rho(x,t)),$$

and similarly for other derivatives. We claim that if $\|\rho\|_l \leqslant \mu$, then

$$\sup_{t<T} \|f_\rho\|_{W_2^{1+l}(\mathcal{G})} \leqslant c_f(\mu). \tag{104}$$

To prove this fix $t \in [0,T]$ (all the norms will be calculated at time t) and notice that by (101), f_ρ, $f_{\rho,x}$ and $f_{\rho,s}\nabla\rho$ are bounded by a constant $c_f(\mu)$ independent of t, as well as, thus, their $L^2(\mathcal{G})$-norms. The principal part of the $W_2^{1+l}(\mathcal{G})$-norm has square bounded by

$$\|f_{\rho,x}\|^2_{W_2^l(\mathcal{G})} + \|f_{\rho,s}\nabla\rho\|^2_{W_2^l(\mathcal{G})} + \|f_{\rho,p}D^2\rho\|^2_{W_2^l(\mathcal{G})}, \tag{105}$$

The first term is readily estimated as

$$\|f_{\rho,x}\|^2_{W_2^l(\mathcal{G})} \leqslant \|f_{\rho,x}\|^2_{L^2(\mathcal{G})} + \|f_{\rho,xx}\|^2_{L^2(\mathcal{G})} + \|f_{\rho,xs}\nabla\rho\|^2_{L^2(\mathcal{G})} + \|f_{\rho,xp}D^2\rho\|^2_{L^2(\mathcal{G})},$$

since the first three addends are bounded, and the fourth is estimated through (100). For the second term in (105) we apply proposition 1.3, to get

$$\|f_{\rho,s}\nabla\rho\|_{W_2^l(\mathcal{G})} \leqslant c\|f_{\rho,s}\|_{W_2^l(\mathcal{G})}\|\nabla\rho\|_{W_2^{1+l}(\mathcal{G})},$$

and the first factor can be estimated as $f_{\rho,x}$ above, while the second is less than μ by (100). For the third term in (105) we apply the mean value theorem:

$$\|f_{\rho,p}D^2\rho\|^2_{W_2^l(\mathcal{G})} \leqslant c_f(\mu) + c\int\limits_{|z|\leqslant 1} \|f_{\rho,p}\Delta_{-z}D^2\rho\|^2_{L^2(\mathcal{G})}\frac{dz}{|z|^{2+2l}}$$
$$+ c\int\limits_{|z|\leqslant 1} \|D^2\rho D f_{\rho,p}(\boldsymbol{\xi}_z)(z,\Delta_{-z}\rho,\Delta_{-z}\nabla\rho)\|^2_{L^2(\mathcal{G})}\frac{dz}{|z|^{2+2l}},$$

for some uniformly bounded function $\boldsymbol{\xi}_z$. Since $f_{\rho,p}$ is bounded, the first integral is less than $c_f(\mu)\|\rho\|_{W_2^{l+2}(\mathcal{G})}$. Moreover, by proposition 1.3,

$$\int\limits_{|z|\leqslant 1} \|D^2\rho D f_{\rho,p}(\boldsymbol{\xi}_z)(z,\Delta_{-z}\rho,\Delta_{-z}\nabla\rho)\|^2_{L^2(\mathcal{G})}\frac{dz}{|z|^{2+2l}}$$
$$\leqslant c_f(\mu)\int\limits_{|z|\leqslant 1} \|D^2\rho(|z|+\Delta_{-z}\rho+\Delta_{-z}\nabla\rho)\|^2_{L^2(\mathcal{G})}\frac{dz}{|z|^{2+2l}}$$
$$\leqslant c_f(\mu)\|D^2\rho\|^2_{W_2^l(\mathcal{G})}\big(1+\int\limits_{|z|\leqslant 1} \|\Delta_{-z}\rho\|^2_{W_2^{2-l}(\mathcal{G})}\frac{dz}{|z|^{2+2l}}\big).$$

The last integral is bounded by $\|\rho\|^2_{W_2^{l+2}(\mathcal{G})}$, and using (100), we obtain a bound depending only on f and μ. Taking the supremum in $t<T$ gives (104).

One can proceed in the same way for the time derivative, to obtain

$$\sup_{\mathcal{G}}\|f_\rho\|_{W_2^{\frac{l}{2}+\frac{3}{4}}(0,T)} \leqslant c_f(\mu). \tag{106}$$

To provide an example, suppose $f_\rho = f(\nabla\rho)$, and let us prove

$$\sup_{\mathcal{G}}\|f_{\rho,p}\nabla\rho_{,t}\|_{W_2^{\frac{l}{2}-\frac{1}{4}}(0,T)} \leqslant c_f(\mu),$$

which is the higher order term. We have

$$\|f_{\rho,p}\nabla\rho_{,t}\|^2_{\mathring{W}_2^{\frac{l}{2}-\frac{1}{4}}(0,T)}$$
$$\leqslant 2\int\limits_0^T \frac{dh}{h^{\frac{l+1}{2}}}\|f_{\rho,p}\Delta_{-h}\nabla\rho_{,t}\|^2_{L^2(0,T)} + \|\nabla\rho_{,t}\Delta_{-h}f_{\rho,p}\|^2_{L^2(0,T)},$$

and the first term in the integral is bounded by $c_f(\mu)$ indipendently of $x \in \mathcal{G}$ due to (102). For the second one, by the mean value theorem and (102)

$$\begin{aligned}\int_0^T \frac{dh}{h^{\frac{l+1}{2}}} \|\nabla\rho_{,t}\Delta_h f_{\rho,p}\|^2_{L^2(0,T)} &\leqslant c_f \int_0^T \frac{dh}{h^{\frac{l+1}{2}}} \|\nabla\rho_{,t} f_{\rho,pp}(\boldsymbol{\xi}_h)\Delta_{-h}\nabla\rho\|^2_{L^2(0,T)} \\ &\leqslant c_f(\mu)\|\nabla\rho_{,t}\|^2_{W_2^{\frac{l}{2}-\frac{1}{4}}(0,T)} \int_0^T \frac{dh}{h^{\frac{l+1}{2}}} \|\Delta_{-h}\nabla\rho\|^2_{W_2^{\frac{3}{4}-\frac{l}{2}}(0,T)} \\ &\leqslant c_f(\mu) \int_0^T \frac{dh}{h^{\frac{l+1}{2}}} \|\Delta_{-h}\nabla\rho\|^2_{L^2(0,T)} + \|\Delta_{-h}\nabla\rho_{,t}\|^2_{L^2(0,T)} \\ &\leqslant c_f(\mu)(\|\nabla\rho\|^2_{W_2^{\frac{l}{2}-\frac{1}{4}}(0,T)} + \|\nabla\rho_{,t}\|^2_{W_2^{\frac{l}{2}-\frac{1}{4}}(0,T)}) \leqslant c_f(\mu).\end{aligned}$$

Now we can apply proposition 1.3, noting that

$$\|f_\rho g\|^2_{W_2^{\eta,0}(G_T)} = \int_0^T \|f_\rho g\|^2_{W_2^{\eta}(\mathcal{G})}\,dt \leqslant \sup_{t<T}\|f_\rho\|^2_{W_2^{1+l}(\mathcal{G})} \int_0^T \|g\|^2_{W_2^{\eta}(\mathcal{G})}\,dt,$$

$$\|f_\rho g\|^2_{W_2^{0,\eta'}(G_T)} = \int_{\mathcal{G}} \|f_\rho g\|^2_{W_2^{\eta'}(0,T)}\,dx \leqslant \sup_{\mathcal{G}}\|f_\rho\|^2_{W_2^{\frac{l}{2}+\frac{3}{4}}(0,T)} \int_{\mathcal{G}} \|g\|^2_{W_2^{\eta'}(0,T)}\,dt,$$

and thus (104) and (106) for $f = f_\rho$ justify the claim of the lemma. □

Remark 6.3. *We will estimate also functions of the form $f(x, \rho^*(x,t), \nabla\rho^*(x,t))$ for $x \in \Omega_b$. The proof of* (106) *and* (104) *carries over in this case, using the fact that any norm of ρ^* on Ω_b is bounded by the same norm of ρ on $\mathcal{G}$. For the final step, we recall that $1 + l > \frac{3}{2}$ and thus Proposition* 1.3 *still applies.*

From now on c will denote a constant depending on μ, the base state of the system $(\Omega_b, \boldsymbol{v}_b, p_b)$ and a finite set of functions $f = f(x,s,p)$, which can change from line to line but will be anyway denoted by c. Moreover, f_ρ will denote a smooth function evaluated at $(x, \rho, \nabla\rho)$.

Estimate of $\|\boldsymbol{l}_0 + \boldsymbol{l}_1\|_{K^l(Q_T)}$.

We will describe the estimates of the higher order terms in (26), since the other ones follow with easier arguments. For the $W_2^{l,0}(Q_T)$ norm one can simply use (100) and the algebra property (9) with $s = l + 1 > \frac{3}{2}$, toghether with Remark 6.3 to get rid of the nonlinear factors on all the following terms:

$$\rho^{*2}, \quad \rho^*\rho^*_{,x_i}, \quad \rho^*_{,x_i}\rho^*_{,x_j}, \quad \rho^*\rho^*_{,t}, \; \rho^*\rho^*_{,x_ix_j}, \; \rho^*_{,x_i}u^k_{,x_j}, \; \rho^*_{,x_i}u^k_{,x_jx_m}, \quad \rho^*_{,x_i}q_{,x_j}.$$

For example,

$$\|f_\rho\rho^*_{,x_j}\rho^*_{,t}\|_{W_2^{l,0}(Q_T)} \leqslant c\sup_{t<T}\|\nabla\rho\|_{W_2^{l+1}(\Omega_b)}\|\rho_{,t}\|_{W_2^{l,0}(Q_T)} \leqslant c\|\rho\|_l^2,$$

and similarly for the other ones. For the term $u^k u^h_{,x_i}$, we use standard restriction estimates on u^k instead of (100), and for $\rho^*_{,t}u^k_{,x_i}$ we have

$$\|\rho^*_{,t}u^j_{,x_i}\|_{W_2^{l,0}(Q_T)} \leqslant c\sup_{t<T}\|\rho_{,t}\|_{W_2^l(\mathcal{G})}\|\nabla\boldsymbol{u}\|_{W_2^{l+1,0}(Q_T)} \leqslant c\|\boldsymbol{u}\|_{K^{l+2}(Q_T)}\|\rho_{,t}\|_l,$$

and similarly result for $\rho^*_{,x_ix_j}u^k_{,x_h}$. Now we estimate the $W_2^{0,\frac{l}{2}}(Q_T)$ norm. All the terms

$$\rho^{*2}, \quad \rho^*\rho^*_{,x_i}, \quad \rho^*_{,x_i}\rho^*_{,x_j}, \quad \rho^*\rho^*_{,t}, \quad \rho^*\rho^*_{,x_ix_j},$$
$$\rho^*_{,x_i}u^k_{,x_j}, \quad \rho^*_{,x_i}u^k_{,x_jx_m}, \quad \rho^*_{,x_i}q_{,x_j}, \quad \rho^*_{,t}u^k_{,x_j},$$

are estimated as before through (9) with $s = \frac{l}{2} + \frac{1}{4} > \frac{1}{2}$ and (102). For example,

$$\begin{aligned}\|\rho^*_{,t}u^k_{,x_i}\|_{W_2^{0,\frac{l}{2}}(Q_T)} &\leqslant c\sup_{\mathcal{G}}\|\rho_{,t}\|_{W_2^{\frac{l}{2}+\frac{1}{4}}(0,T)}\|\nabla\boldsymbol{u}\|_{W_2^{0,\frac{l}{2}}(Q_T)}\\ &\leqslant c\sup_{\mathcal{G}}\|\rho\|_{W_2^{\frac{l}{2}+\frac{5}{4}}(0,T)}\|\boldsymbol{u}\|_{K^l(Q_T)} \leqslant c\|\rho\|_l\|\boldsymbol{u}\|_{K^l(Q_T)}.\end{aligned}$$

For the term $u^k u^h_{,x_i}$ one uses (103) instead of (102), and it remains to estimate $\rho^*_{,x_ix_j}\boldsymbol{u}_{,x_m}$. Let us focus on the higer order term. Since

$$\begin{aligned}\|\rho^*_{,x_ix_j}\boldsymbol{u}_{,x_m}\|^2_{\mathring{W}_2^{0,\frac{l}{2}}(Q_T)} &= \int_0^T \frac{dh}{h^{1+l}}\int_h^T \|\Delta_{-h}(\rho^*_{,x_ix_j}\boldsymbol{u}_{,x_m})\|^2_{L^2(\Omega_b)}\,dt\\ \leqslant c\int_0^T \frac{dh}{h^{1+l}}\int_h^T &\|\boldsymbol{u}_{,x_m}\Delta_{-h}\rho^*_{,x_ix_j}\|^2_{L^2(\Omega_b)} + \|\rho^*_{,x_ix_j}\Delta_{-h}\boldsymbol{u}_{,x_m}\|^2_{L^2(\Omega_b)}\,dt,\end{aligned}$$

we split the estimate into two parts. By Proposition 1.3, we have

$$\int_0^T \frac{dh}{h^{1+l}} \int_h^T \|\rho^*_{,x_i x_j} \Delta_{-h} \boldsymbol{u}_{,x_m}\|^2_{L^2(\Omega_b)}\, dt$$
$$\leqslant \int_0^T \frac{dh}{h^{1+l}} \int_h^T \|\rho_{,x_i x_j}\|^2_{W_2^l(\mathcal{G})} \|\Delta_{-h} \boldsymbol{u}_{,x_m}\|^2_{W_2^{\frac{3}{2}-l}(\Omega_b)}\, dt$$
$$\leqslant \sup_{t<T} \|\rho\|^2_{W_2^{2+l}(\mathcal{G})} \int_0^T \frac{dh}{h^{1+l}} \int_h^T \|\Delta_{-h} \boldsymbol{u}_{,x_m}\|^2_{L^2(\Omega_b)} + \|\Delta_{-h} \nabla \boldsymbol{u}_{,x_m}\|^2_{L^2(\Omega_b)}\, dt$$
$$\leqslant \sup_{t<T} \|\rho\|^2_{W_2^{2+l}(\mathcal{G})} \|\boldsymbol{u}\|^2_{K^{l+2}(Q_T)},$$

since $\frac{3}{2} - l < 1$. To estimate the other term, we consider extensions of $\boldsymbol{u}$, ρ and θ (still denoted with the same symbols) to the whole $\mathbb{R}^3$, $\mathbb{R}^2$ and $\mathbb{R}$ respectively, with controlled K^{l+2}, $K^{l+5/2}$, and C^∞-norms, respectively. Hölder inequality for mixed norms gives

$$\|\boldsymbol{u}_{,x_m} \Delta_{-h} \rho^*_{,x_i x_j}\|^2_{L^2(\mathbb{R}^3)} \leqslant \|\boldsymbol{u}_{,x_m}\|^2_{L^{p,p,2}(\mathbb{R}^3)} \|\Delta_{-h} \rho^*_{,x_i x_j}\|^2_{L^{q,q,\infty}(\mathbb{R}^3)}$$
$$\leqslant c \|\boldsymbol{u}_{,x_m}\|^2_{L^{p,p,2}(\mathbb{R}^3)} \|\Delta_{-h} \rho_{,x_i x_j}\|^2_{L^q(\mathbb{R}^2)},$$

where $1/p + 1/q = 1$ and

$$\|f\|_{L^{p,p,2}(\mathbb{R}^3)} = \|f\|_{L^2(dx_3; L^p(dx_1 dx_2))},$$
$$\|f\|_{L^{q,q,\infty}(\mathbb{R}^3)} = \|f\|_{L^\infty(dx_3; L^q(dx_1 dx_2))}.$$

The anisotropic Sobolev embedding theorem ensures that, given $\boldsymbol{p} = (p_1, \dots, p_N)$ with $p_i > 1$, $i = 1, \dots, N$, it holds

$$W_2^s(\mathbb{R}^N) \hookrightarrow L^{\boldsymbol{p}}(\mathbb{R}^N) \quad \text{if} \quad s + \sum_{i=1}^N \frac{1}{p_i} \geqslant \frac{N}{2}.$$

Chosing p in such a way that

$$l + \frac{2}{p} + \frac{1}{2} = \frac{3}{2},$$

we get $p > 1$ and for the corresponding q the relation $(1 - l) + 2/q = 1$. Therefore, being $\theta \in C^\infty$, we obtain

$$\|\boldsymbol{u}_{,x_m}\|^2_{L^{p,p,2}(\mathbb{R}^3)} \|\Delta_{-h} \rho^*_{,x_i x_j}\|^2_{L^{q,q,\infty}(\mathbb{R}^3)} \leqslant c \|\boldsymbol{u}_{,x_m}\|^2_{W_2^l(\Omega_b)} \|\Delta_{-h} \rho_{,x_i x_j}\|^2_{W_2^{1-l}(\mathcal{G})},$$

which yields

$$\int_0^T \frac{dh}{h^{1+l}} \int_h^T \|\boldsymbol{u}_{,x_m} \Delta_{-h} \rho^*_{,x_i x_j}\|^2_{L^2(\Omega_b)}\, dt$$

$$\leqslant c \int_0^T \frac{dh}{h^{1+l}} \int_h^T \|\boldsymbol{u}(\cdot,t)\|^2_{W_2^{l+1}(\Omega_b)} \|\Delta_{-h}\rho_{,x_i x_j}\|^2_{W^{1-l}(\mathcal{G})}\, dt$$

$$\leqslant c \sup_{t<T} \|\boldsymbol{u}\|^2_{W_2^{l+1}(\Omega)} |\rho|^2_{\frac{l}{2},3-l} \leqslant c\|u\|^2_{K^{l+2}(Q_T)} \|\rho\|_{K^{l+\frac{5}{2}}(G_T)},$$

by (14) and noting that $3 < \frac{5}{2} + l$.

Estimates of $\|l_2(\boldsymbol{u},\rho)\|_{W_2^{l+1,0}(Q_T)}$ *and* $\|\boldsymbol{G}(\boldsymbol{u},\rho)\|_{W_2^{0,\frac{l}{2}+1}(Q_T)}$.
Note that l_2 (cf. (30)) is a linear combination of terms of the form $\rho^*_{,x_i} u^k_{,x_3}$ and $\rho^* u^k_{,x_i}$ and thus its $W_2^{l+1,0}(Q_T)$ norm is estimated as before through (9) and (100). For the time derivative of $\boldsymbol{G}$, also given in (30), notice that its $W_2^{0,\frac{l}{2}}(Q_T)$ can also be estimated through (9) and (102), Therefore it suffice to estimate the $W_2^{0,\frac{l}{2}}(Q_T)$ of its time derivative, i.e.,

$$(\nabla\rho^*_{,t} \cdot \boldsymbol{u})\boldsymbol{e}_3 + (\nabla\rho \cdot \boldsymbol{u}_{,t})\boldsymbol{e}_3 - \theta'(\rho_{,t}\boldsymbol{u} + \rho\boldsymbol{u}_{,t}).$$

To this end notice that, applying (103), one gets

$$\|\nabla\rho^*_{,t} \cdot \boldsymbol{u}\|_{W_2^{0,\frac{l}{2}}(Q_T)} \leqslant c \sup_{\Omega_b} \|\boldsymbol{u}\|_{W_2^{\frac{l}{2}+\frac{1}{4}}([0,T])} \|\nabla\rho_{,t}\|_{W_2^{0,\frac{l}{2}}(G_T)}$$

$$\leqslant c\|\boldsymbol{u}\|_{K^{l+2}(Q_T)} \|\rho_{,t}\|_{K^{l+1}(G_T)} \leqslant c\|\boldsymbol{u}\|_{K^{l+2}(Q_T)} \|\rho\|_l,$$

since $\frac{l}{2} + \frac{1}{4} > \frac{1}{2}$. Furthermore, by (102),

$$\|\nabla\rho^* \boldsymbol{u}_{,t}\|_{W_2^{0,\frac{l}{2}}(Q_T)} \leqslant c \sup_{\mathcal{G}} \big(\|\rho\|_{W_2^1([0,T])} + \|\nabla\rho\|_{W_2^1([0,T])}\big) \|\boldsymbol{u}_{,t}\|_{W_2^{0,\frac{l}{2}}(Q_T)}$$

$$\leqslant c\|\rho\|_l \|\boldsymbol{u}\|_{W_2^{0,\frac{l}{2}+1}(Q_T)}.$$

The same estimates holds for the terms in $\rho_{,t}\boldsymbol{u}$ and $\rho\boldsymbol{u}_{,t}$, and thus we obtain

$$\|l_2(\boldsymbol{u},\rho)\|_{W_2^{l+1,0}(Q_T)} + \|\boldsymbol{G}(\boldsymbol{u},\rho)\|_{W_2^{0,\frac{l}{2}+1}(Q_T)} \leqslant c\|\rho\|_l \|\boldsymbol{u}\|_{K^{l+2}(Q_T)}.$$

Estimate of $\|\boldsymbol{l}_3\|_{K^{l+\frac{1}{2}}(G_T)}$.

Since $l+\frac{1}{2}>1$, all the terms in (35) are estimated in $W_2^{l+\frac{1}{2},0}(G_T)$ through the algebra property (10), (100) and standard restiction theorem for Sobolev–Slobodetskii spaces. For example,

$$\|\rho_{,x_k}\rho_{,x_ix_j}\|_{W_2^{l+\frac{1}{2}}(G_T)} \leqslant \sup_{t<T}\|\nabla\rho\|_{W_2^{l+1}(\mathcal{G})}\|\rho\|_{W_2^{l+\frac{5}{2}}(G_T)} \leqslant c\|\rho\|_l^2.$$

Similarly one can proceed for the $W_2^{0,\frac{l}{2}+\frac{1}{4}}$, since $\frac{l}{2}+\frac{1}{4}>\frac{1}{2}$, this time using (102) on the terms containing ρ or $\rho_{,x_i}$.

Estimate of $\|l_4\|_{K^{l+\frac{3}{2}}(G_T)}$.

The explicit form of l_4 is given in (20). For the spatial derivative, we use (10) and find that

$$\|\nabla\rho\cdot\boldsymbol{u}\|_{W_2^{l+\frac{3}{2}}(\mathcal{G})} \leqslant c\Big(\|\nabla\rho\|_{W_2^{l+\frac{1}{2}}(\mathcal{G})}\|\boldsymbol{u}\|_{W_2^{l+\frac{3}{2}}(\mathcal{G})} + \|\nabla\rho\|_{W_2^{l+\frac{3}{2}}(\mathcal{G})}\|\boldsymbol{u}\|_{W_2^{l+\frac{1}{2}}(\mathcal{G})}\Big),$$

since $l+\frac{1}{2}>1$. Now (100) and the standard restriction estimates for anisotropic Sobolev–Slobodetskii spaces imply

$$\begin{aligned}\|\nabla\rho\cdot\boldsymbol{u}\|_{W_2^{l+\frac{3}{2},0}(G_T)} &\leqslant c\Big(\sup_{t<T}\|\rho\|_{W_2^{l+\frac{3}{2}}(\mathcal{G})}\|\boldsymbol{u}\|_{W_2^{l+\frac{3}{2},0}(G_T)} + \sup_{t<T}\|\boldsymbol{u}\|_{W_2^{l+\frac{1}{2}}(\mathcal{G})}\|\rho\|_{W_2^{l+\frac{5}{2},0}(\mathcal{G})}\Big)\\ &\leqslant c\|\rho\|_{K^{l+\frac{5}{2}}(G_T)}\|\boldsymbol{u}\|_{K^{l+2}(Q_T)}.\end{aligned}$$

For the time derivative, we use (102) and deduce the estimate

$$\|\nabla\rho\cdot\boldsymbol{u}\|_{W_2^{0,\frac{l}{2}+\frac{3}{4}}(G_T)} \leqslant c\sup_{\mathcal{G}}\|\nabla\rho\|_{W_2^{\frac{l}{2}+\frac{3}{4}}(0,T)}\|\boldsymbol{u}\|_{W_2^{0,\frac{l}{2}+\frac{3}{4}}(G_T)} \leqslant c\|\rho\|_l\|\boldsymbol{u}\|_{K^{l+2}(Q_T)},$$

which is the last estimate needed for the proof of 6.1.

Finally, we prove a continuity estimate for the nonlinear terms.

Theorem 6.4. *Let $l \in (\frac{1}{2}, 1)$, and $\|\rho\|_l$, $\|\rho'\|_l \leqslant \mu$ such that* (99) *holds. There exists $c(\mu)$, bounded for bounded μ, such that*

$$\begin{aligned}
&\|\boldsymbol{l}_0(\boldsymbol{u},\rho) - \boldsymbol{l}_0(\boldsymbol{u}',\rho')\|_{K^l(Q_T)} + \|\boldsymbol{l}_1(\boldsymbol{u},p,\rho) - \boldsymbol{l}_1(\boldsymbol{u}',p',\rho')\|_{K^l(Q_T)} \\
&+ \|l_2(\boldsymbol{u},\rho) - l_2(\boldsymbol{u}',\rho')\|_{W_2^{l+1,0}(Q_T)} + \|\boldsymbol{G}(\boldsymbol{u},\rho) - \boldsymbol{G}(\boldsymbol{u}',\rho')\|_{W_2^{0,\frac{l}{2}+1}(Q_T)} \\
&+ \|\boldsymbol{l}_3(\boldsymbol{u},\rho) - \boldsymbol{l}_3(\boldsymbol{u}',\rho')\|_{K^{l+\frac{1}{2}}(G_T)} + \|l_4(\boldsymbol{u},\rho) - l_4(\boldsymbol{u}',\rho')\|_{K^{l+\frac{3}{2}}(G_T)} \\
&\leqslant c(\mu)(\|(\boldsymbol{u},p,\rho)\|_{l,T} + \|(\boldsymbol{u}',p',\rho')\|_{l,T})\|(\boldsymbol{u}-\boldsymbol{u}',p-p',\rho-\rho')\|_{l,T}.
\end{aligned}$$

Proof. This is a consequence of the structure of the nonlinear terms: as noted in the previous estimates, each nonlinear term is a linear combinations of products of the form

$$h_\rho \pi(\boldsymbol{u}, \nabla \boldsymbol{u}, \rho, \rho_{,t}, \nabla\rho, \nabla^2\rho, p, \nabla p),$$

where π stands for a monomial of total degree at least 1 in a certain subset of the arguments. Therefore, except for the term h_ρ, each of these terms is separately linear in its arguments, and can be estimated as above, provided one can prove an estimate of the form

$$\begin{aligned}
&\|(h_\rho - h_{\rho'})g\|_{W_2^{\eta,0}(G_T)} \leqslant c_h(\mu)\|\rho-\rho'\|_l \|g\|_{W_2^{\eta,0}(G_T)}, \\
&\|(h_\rho - h_{\rho'})g\|_{W_2^{0,\eta'}(G_T)} \leqslant c_h(\mu)\|\rho-\rho'\|_l \|g\|_{W_2^{0,\eta'}(G_T)},
\end{aligned} \tag{107}$$

(similar estimates can be obtained for ρ^* on Q_T), with $\eta \leqslant 1+l$ and $\eta' \leqslant 1$. Indeed, it suffices to split the difference of the products using the algebraic formula

$$\Pi_{i=1}^N x_i - \Pi_{i=1}^N y_i = \sum_{i=1}^N y_1 \dots \cdot y_{i-1}(x_i - y_i)x_{i+1}\dots x_N,$$

and exploit estimates (107) for the terms containing the differences of the nonlinear terms h's. The particular structure of the various products π (for example, being of degree at most one in $\boldsymbol{u}$ and $\nabla\boldsymbol{u}$) ensures that the estimates of the previous proof carry over in this case. To prove (107), we notice that, as in the proof of Lemma 6.2, it suffices to show that

$$\sup_{t<T} \|h_\rho - h_{\rho'}\|_{W_2^{1+l}(\mathcal{G})} \leqslant c_h(\mu)\|\rho-\rho'\|_l, \tag{108}$$

$$\sup_{\mathcal{G}} \|h_\rho - h_{\rho'}\|_{W_2^1(0,T)} \leqslant c_h(\mu)\|\rho-\rho'\|_l. \tag{109}$$

(this also implies estimates of the form (107) involving ρ^* on Q_T; see Remark 6.3). We sketch the proof of these two estimate, supposing for simplicity that $h = h(\nabla\rho)$, which is the higher order term. Smoothness of h, together with (101), means that $h(\nabla\rho)$ is Lipschitz continuous w.r.t. $\nabla\rho$ in C^0 and the inequality

$$\sup_{G_T} |h(\nabla\rho) - h(\nabla\rho')| \leqslant sc_h(\mu) \sup_{G_T} |\nabla\rho - \nabla q| \|\rho - \rho'\|_l \tag{110}$$

holds. Thus, it remains to estimate the norm

$$\begin{aligned} &\|h_{,p}(\nabla\rho)D^2\rho - h_{,p}(\nabla\rho')D^2\rho'\|_{W_2^l(\mathcal{G})} \\ &\quad \leqslant \|(h_{,p}(\nabla\rho) - h_{,p}(\nabla\rho'))D^2\rho\|_{W_2^l(\mathcal{G})} + \|h_{,p}(\nabla\rho')D^2(\rho' - \rho)\|_{W_2^l(\mathcal{G})}. \end{aligned}$$

The second term in the right-hand side is treated through Lemma 6.2 and (100), while for the first one we apply Proposition 1.3, point 2 and obtain

$$\begin{aligned} &\|(h_{,p}(\nabla\rho) - h_{,p}(\nabla\rho'))D^2\rho\|_{W_2^l(\mathcal{G})} \\ &\qquad \leqslant c\mu(\|h_{,p}(\nabla\rho) - h_{,p}(\nabla\rho')\|_{W_2^1(\mathcal{G})} + \sup_{\mathcal{G}} |h_{,p}(\nabla\rho) - h_{,p}(\nabla\rho')|). \end{aligned}$$

Property (110) takes care of the second term and the L^2 part of the first norm. Therefore, it remains to estimate $h_{,p}(\nabla\rho)D^2\rho - h_{,p}(\nabla\rho')D^2\rho$ in L^2, which can be splitted as before. Thus,

$$\begin{aligned} &\|h_{,p}(\nabla\rho)D^2\rho - h_{,p}(\nabla\rho')D^2\rho\|_{L^2(\mathcal{G})} \\ &\qquad \leqslant c_h(\mu)\|\rho - \rho'\|_{W_2^2(\mathcal{G})} + \sup_{\mathcal{G}} |h_{,p}(\nabla\rho) - h_{,p}(\nabla\rho')| \|\rho\|_{W_2^2(\mathcal{G})}. \end{aligned}$$

Applying once again (110) proves (108). To prove of (109) it suffice again to estimate $h_{,p}(\nabla\rho)\nabla\rho_{,t} - h_{,p}(\nabla\rho')\nabla\rho'_{,t}$ in L^2, which can be done as before:

$$\begin{aligned} &\|h_{,p}(\nabla\rho)\nabla\rho_{,t} - h_{,p}(\nabla\rho')\nabla\rho'_{,t}\|_{L^2(0,T)} \\ &\leqslant \|(h_{,p}(\nabla\rho) - h_{,p}(\nabla\rho'))\nabla\rho_{,t}\|_{L^2(0,T)} + \|h_{,p}(\nabla\rho')\nabla(\rho_{,t} - \rho'_{,t})\|_{L^2(0,T)} \\ &\leqslant c_h(\mu)(\|\rho - \rho'\|_l \|\nabla\rho\|_{W_2^1(0,T)} + \|\nabla(\rho - \rho')\|_{W_2^1(0,T)}) \\ &\leqslant c_h(\mu)\|\rho - \rho'\|_l. \end{aligned}$$

□

§7. The abstract linearization principle

In this section, we prove Theorem 1.1. Thus $(\boldsymbol{v}_b, p_b, \phi_b)$ is a fixed smooth, stationary solution of problem (1), which is supposed to be linearly exponentially stable.

Proof of Theorem 1.1. We construct the solution as a sum

$$(\boldsymbol{u}, p, \rho) = (\boldsymbol{u}_1 + \boldsymbol{u}_2, p_1 + p_2, \rho_1 + \rho_2),$$

where $(\boldsymbol{u}_1, p_1, \rho_1)$ is a solution of (5) for some initial data $\boldsymbol{u}_1\lfloor_{t=0} := \boldsymbol{u}_1^0$, $\rho_1\lfloor_{t=0} = \rho_0$, and $(\boldsymbol{u}_2, p_2, \rho_2)$ solves a nonlinear problem with initial data $\boldsymbol{u}_2\lfloor_{t=0} = \boldsymbol{u}_0 - \boldsymbol{u}_1^0$, $\rho_2\lfloor_{t=0} = 0$. We split the proof into three steps.

Step 1: construction of $(\boldsymbol{u}_1, p_1, \rho_1)$.
We start by constructing $\boldsymbol{w}_0$. We solve the problem

$$\begin{cases} \nabla \cdot \boldsymbol{w}_0 = l_2(\boldsymbol{u}_0, \rho_0) & \text{in} \quad \Omega_b, \\ \nu \Pi_b \mathbb{D}(\boldsymbol{w}_0)\boldsymbol{N} = \Pi_b \boldsymbol{l}_3(\boldsymbol{u}_0, \rho_0) & \text{on} \quad \mathcal{G}, \\ \boldsymbol{w}_0 = 0 & \text{on} \quad \Sigma, \end{cases} \tag{111}$$

and use the estimate

$$\|\boldsymbol{w}_0\|_{W_2^{l+1}(\Omega_b)} \leqslant c\Big(\|l_2(\boldsymbol{u}_0, \rho_0)\|_{W_2^l(\Omega_b)} + \|\boldsymbol{l}_3(\boldsymbol{u}_0, \rho_0)\|_{W_2^{l-\frac{1}{2}}(\mathcal{G})}\Big).$$

Recall that $l_2(\boldsymbol{u}, \rho)$ is a linear combination of terms of the form $\rho^*_{x_i} u^k_{x_3}$ and $\rho^* u^k_{x_i}$, and, therefore, Proposition 1.3 yields the estimate

$$\|l_2(\boldsymbol{u}_0, \rho_0)\|_{W_2^l(\Omega_b)} \leqslant c\|\rho_0\|_{W_2^{l+2}(\mathcal{G})} \|\boldsymbol{u}_0\|_{W_2^{l+1}(\Omega_b)}.$$

For the term $\boldsymbol{l}_3$, one has to consider addends of the following kind:

$$\rho_{,x_i} u^k_{,x_j} h_\rho, \quad \rho_{x_i} \rho_{,x_j} h_\rho, \quad \text{or} \quad \rho \rho_{,x_j} h_\rho.$$

As in the proof of Lemma 6.2 for $\rho \equiv \rho_0$, we obtain

$$\|\boldsymbol{l}_3(\boldsymbol{u}_0, \rho_0)\|_{W_2^{l-\frac{1}{2}}(\mathcal{G})} \leqslant c\Big(\|\rho_0\|_{W_2^{l+\frac{3}{2}}(\mathcal{G})} \|\boldsymbol{u}_0\|_{W_2^{l+1}(\Omega_b)} + \|\rho_0\|^2_{W_2^{l+\frac{3}{2}}(\mathcal{G})}\Big).$$

Finally, we deduce the estimate

$$\|\boldsymbol{w}_0\|_{W_2^{l+1}(\Omega_b)} \leqslant c\Big(\|\rho_0\|_{W_2^{l+2}(\mathcal{G})} + \|\boldsymbol{u}_0\|_{W_2^{l+1}(\Omega_b)}\Big)^2, \tag{112}$$

and define the quantity

$$U_0 := \|\rho_0\|_{W_2^{l+2}(\mathcal{G})} + \|\boldsymbol{u}_0\|_{W_2^{l+1}(\Omega_b)},$$

supposing it is sufficiently small, in a sense to be specified later. Now, since $(\boldsymbol{u}_0, \rho_0)$ satisfies (19), it is clear that the couple $(\boldsymbol{u}_1^0, \rho_0)$, where $\boldsymbol{u}_1^0 = \boldsymbol{u}_0 - \boldsymbol{w}_0$, satisfies the compatibility conditions (6) for the homogeneous linear problem (5). We will then let $(\boldsymbol{u}_1, p_1, \rho_1)$ be solution of (5) with such initial data. The stability hypothesis gives for $\gamma > 0$ and $T \geqslant 1$

$$\begin{aligned} \|(\boldsymbol{u}_1, p_1, \rho_1)\|_{l,\infty} &\leqslant \|e^{\gamma t}(\boldsymbol{u}_1, p_1, \rho_1)\|_{l,\infty} \\ &\leqslant c(\|\boldsymbol{u}_1^0\|_{W_2^{l+1}} + \|\rho_0\|_{W_2^{l+2}(\mathcal{G})}) \leqslant c_1 U_0, \end{aligned} \tag{113}$$

and by standard restriction estimates for unbounded intervals

$$e^{\gamma T}\big(\|\boldsymbol{u}_1(\cdot,T)\|_{W_2^{l+1}(\Omega_b)}+\|\rho_1(\cdot,T)\|_{W_2^{l+2}(\mathcal{G})}\big) \leqslant \|e^{\gamma t}(\boldsymbol{u}_1,p_1,\rho_1)\|_{l,\infty}\leqslant c_1U_0, \tag{114}$$

with a constant $c_1 \geqslant 1$ independent of $T \geqslant 1$.

Step 2: construction of $(\boldsymbol{u}_2, p_2, \rho_2)$.
We seek for a solution of the nonlinear problem

$$\begin{cases} \boldsymbol{u}_{2,t}-\nu\Delta\boldsymbol{u}_2+\nabla q_2-\boldsymbol{\Phi}_1(\boldsymbol{u}_2,\rho_2)=(\boldsymbol{l}_0+\boldsymbol{l}_1)(\boldsymbol{u}_1+\boldsymbol{u}_2,q_1+q_2,\rho_1+\rho_2) & \text{in } \Omega_b,\\ \nabla\cdot\boldsymbol{u}_2-\Phi_2(\rho_2)=l_2(\boldsymbol{u}_1+\boldsymbol{u}_2,\rho_1+\rho_2)=\nabla\cdot\boldsymbol{G}(\boldsymbol{u}_1+\boldsymbol{u}_2,\rho_1+\rho_2) & \text{in } \Omega_b,\\ \mathbb{T}(\boldsymbol{u}_2,q)\boldsymbol{N}-\boldsymbol{\Phi}_3(\rho_2)=\boldsymbol{l}_3(\boldsymbol{u}_1+\boldsymbol{u}_2,\rho_1+\rho_2) & \text{on } \mathcal{G},\\ \rho_{2,t}+\nabla'\phi_b\cdot\boldsymbol{u}_2-u_2^3+\nabla'\rho_2\cdot\boldsymbol{v}_b=l_4(\boldsymbol{u}_1+\boldsymbol{u}_2,\rho_1+\rho_2) & \text{on } \mathcal{G},\\ \boldsymbol{u}_2=0 \quad \text{on } \Sigma, \text{ for all } t\geqslant 0, \\ \boldsymbol{u}_2(x,0)=\boldsymbol{w}_0(x), \quad \text{for } x\in\Omega_b, \quad \rho_2(x',0)=0 \quad \text{for } x'\in\Sigma. \end{cases} \tag{115}$$

We apply the standard iteration scheme, defining a sequence of solutions of linear problems and consider an extension $\boldsymbol{v}_0$ for $t \geqslant 0$ of $\boldsymbol{w}_0$ such that

$$\|\boldsymbol{v}_0\|_{K^{l+2}(Q_\infty)}\leqslant c\|\boldsymbol{w}_0\|_{W_2^{l+1}(\Omega_b)}\leqslant c_3U_0^2.$$

We start with the triple $(\boldsymbol{v}_0, 0, 0)$, assuming that $c_3U_0^2 \leqslant 1$. Then, we define $(\boldsymbol{v}_{n+1}, p_{n+1}, \rho_{n+1})$ as the solution to problem (36) with the right-hand sides

$$\boldsymbol{f}_n := (\boldsymbol{l}_0+\boldsymbol{l}_1)(\boldsymbol{u}_1+\boldsymbol{v}_n,q_1+q_n,\rho_1+\rho_n), \quad h_n := l_2(\boldsymbol{u}_1+\boldsymbol{u}_n,\rho_1+\rho_n),$$

$$\boldsymbol{d}_n := \boldsymbol{l}_3(\boldsymbol{u}_1+\boldsymbol{u}_n,\rho_1+\rho_n), \quad g_n := l_4(\boldsymbol{u}_1+\boldsymbol{u}_n,\rho_1+\rho_n),$$

and initial data $\boldsymbol{v}_{n+1}(0) = \boldsymbol{w}_0$ and $\rho_{n+1}(0) = 0$ (notice that (31) holds at each stage). For this problem, the compatibility conditions (91) are satisfied at each stage by (111). The coercive estimate (92), together with Theorems 6.1, (112), and (113), yields

$$\begin{aligned}\|(\boldsymbol{v}_{n+1},q_{n+1},\rho_{n+1})\|_{l,T} &\leqslant c_Tc(\mu)(\|(\boldsymbol{v}_n,q_n,\rho_n)\|_{l,T}^2+\|(\boldsymbol{u}_1,p_1,\rho_1)\|_{l,T}^2+U_0^2)\\ &\leqslant c_2(\mu,T)(\|(\boldsymbol{v}_n,q_n,\rho_n)\|_{l,T}^2+U_0^2)\end{aligned}$$

if

$$\max\{\|(\boldsymbol{v}_n,q_n,\rho_n)\|_{l,T},\|(\boldsymbol{u}_1,p_1,\rho_1)\|_{l,T}\}\leqslant\mu, \tag{116}$$

for μ satisfying (99). We fix $\mu = \mu(\theta) < 1$ so that (99) holds, set $c_2(1,T) = c_2(T)$ supposing $c_2(T) \geqslant \max\{\mu^{-1}, c_1, c_3\}$, then U_0 so small that

$$U_0 \leqslant \frac{1}{8c_1c_2(T)} =: \varepsilon(T) \leqslant \frac{1}{8c_2(T)}. \qquad (117)$$

With such U_0, one can prove by induction that (116) holds. More precisely,

$$\|(\boldsymbol{v}_n, q_n, \rho_n)\|_{l,T} \leqslant \frac{2c_2(T)U_0^2}{1+\sqrt{1-4c_2(T)^2U_0^2}} \leqslant 2c_2(T)U_0^2 \leqslant \frac{U_0}{4} \leqslant \mu. \quad (118)$$

It remains to prove that $(\boldsymbol{v}_n, q_n, \rho_n)$ strongly converges to a solution of (115). To this end, consider

$$(\widehat{\boldsymbol{v}}_n, \widehat{p}_n, \widehat{\rho}_n) := (\boldsymbol{v}_{n+1} - \boldsymbol{v}_n, p_{n+1} - p_n, \rho_{n+1} - \rho_n).$$

They satisfy a linear system of the type (36) with the right-hand sides defined by the relations

$$\begin{aligned}
\boldsymbol{f}_n &:= (\boldsymbol{l}_0 + \boldsymbol{l}_1)(\boldsymbol{u}_1 + \boldsymbol{v}_n, q_1 + q_n, \rho_1 + \rho_n) \\
&\quad - (\boldsymbol{l}_0 + \boldsymbol{l}_1)(\boldsymbol{u}_1 + \boldsymbol{v}_{n-1}, q_1 + q_{n-1}, \rho_1 + \rho_{n-1}), \\
h_n &:= l_2(\boldsymbol{u}_1 + \boldsymbol{u}_n, \rho_1 + \rho_n) - l_2(\boldsymbol{u}_1 + \boldsymbol{u}_{n-1}, \rho_1 + \rho_{n-1}) \\
&\quad = \nabla \cdot \boldsymbol{G}(\boldsymbol{u}_1 + \boldsymbol{u}_n, \rho_1 + \rho_n), \\
\boldsymbol{d}_n &:= \boldsymbol{l}_3(\boldsymbol{u}_1 + \boldsymbol{u}_n, \rho_1 + \rho_n) - \boldsymbol{l}_3(\boldsymbol{u}_1 + \boldsymbol{u}_{n-1}, \rho_1 + \rho_{n-1}), \\
g_n &:= l_4(\boldsymbol{u}_1 + \boldsymbol{u}_n, \rho_1 + \rho_n) - l_4(\boldsymbol{u}_1 + \boldsymbol{u}_{n-1}, \rho_1 + \rho_{n-1}),
\end{aligned}$$

and zero initial data. We can assume that the constant in Theorem 6.4 is equal to $c_2(T)$. Hence, from (92), (113), (117), and (118), and Theorem 6.4, we obtain

$$\begin{aligned}
\|(\widehat{\boldsymbol{v}}_{n+1}, \widehat{p}_{n+1}, \widehat{\rho}_{n+1})\|_{l,T} &\leqslant c_2(T)\big(\|(\boldsymbol{v}_{n+1}, p_{n+1}, \rho_{n+1})\|_{l,T} \\
&\quad + \|(\boldsymbol{v}_n, p_n, \rho_n)\|_{l,T} + 2\|(\boldsymbol{u}_1, p_1, \rho_1)\|_{l,T}\big)\|(\widehat{\boldsymbol{v}}_n, \widehat{p}_n, \widehat{\rho}_n)\|_{l,T} \\
&\leqslant \frac{1}{2}\|(\widehat{\boldsymbol{v}}_n, \widehat{p}_n, \widehat{\rho}_n)\|_{l,T},
\end{aligned}$$

This in turn gives strong convergence of the sequence $(\widehat{\boldsymbol{v}}_n, \widehat{p}_n, \widehat{\rho}_n)$. By Theorem 6.4, the nonlinear terms converge too, and, thus, the limit solves (115). Clearly (118) holds for the solution.

Step 3: construction of the global solution.
We chose T_0 so large that $c_1e^{-\gamma T_0} < \frac{1}{4}$ in (113), then choose $\varepsilon_0 := \varepsilon(T_0)$ as in (117). If $U_0 \leqslant \varepsilon_0$ we have a global solution in $[0, T_0]$ of (18) defined

as the sum $(\boldsymbol{u}, p, \rho) := (\boldsymbol{u}_1 + \boldsymbol{u}_2, p_1 + p_2, \rho_1 + \rho_2)$. From (114) and (118), we obtain

$$\begin{aligned}
&\|\boldsymbol{u}(T_0)\|_{W_2^{l+1}(\Omega_b)} + \|\rho(T_0)\|_{W_2^{l+2}(\mathcal{G})} \leqslant \|\boldsymbol{u}_1(T_0)\|_{W_2^{l+1}(\Omega_b)} \\
&+ \|\rho_1(T_0)\|_{W_2^{l+2}(\mathcal{G})} + \|\boldsymbol{u}_2(T_0)\|_{W_2^{l+1}(\Omega_b)} + \|\rho_2(T_0)\|_{W_2^{l+2}(\mathcal{G})} \\
&\leqslant \frac{U_0}{4} + c_1\|(\boldsymbol{u}_2, p_2, \rho_2)\|_{l,T} \leqslant \frac{U_0}{4} + 2c_1c_2(T_0)U_0^2 \leqslant \frac{U_0}{2} \leqslant \frac{\varepsilon_0}{2}.
\end{aligned}$$

Setting $U_1 = \|\boldsymbol{u}(T_0)\|_{W_2^{l+1}(\Omega_b)} + \|\rho(T_0)\|_{W_2^{l+2}(\mathcal{G})}$, we see that (117) holds for U_1 with $\varepsilon_1 = \varepsilon_0/2$, and we can solve system (18) in $[T_0, 2T_0]$ with the same procedure, as above, and initial data $\boldsymbol{u}(T_0)$, $\rho(T_0)$. Proceeding in this way, we obtain a global solution $(\boldsymbol{u}, p, \rho)$, which satisfies

$$U_k := \|\boldsymbol{u}(kT_0)\|_{W_2^{l+1}(\Omega_b)} + \|\rho(kT_0)\|_{W_2^{l+2}(\mathcal{G})} \leqslant \frac{U_0}{2^k}.$$

If between kT_0 and $(k+1)T_0$, we denote the solution of the linear system as $(\boldsymbol{u}_1^{(k)}, p_1^{(k)}, \rho_1^{(k)})$ and the solution of the nonlinear one as $(\boldsymbol{u}_2^{(k)}, p_2^{(k)}, \rho_2^{(k)})$, then (114) and (118) hold with U_k at each step. We have

$$\begin{aligned}
\|(\boldsymbol{u}_1^{(k)}, p_1^{(k)}, \rho_1^{(k)})\|_{\{l,[kT_0,(k+1)T_0]\}} + \|(\boldsymbol{u}_2^{(k)}, p_2^{(k)}, \rho_2^{(k)})\|_{\{l,[kT_0,(k+1)T_0]\}} \\
\leqslant c_1U_k + \frac{U_k}{4} \leqslant c\frac{U_0}{2^k}.
\end{aligned}$$

Since T_0 is bounded away from zero we can split the norms over $[0, +\infty)$ as a sum of the norms over $[kT_0, (k+1)T_0)$, and from the previous inequality and

$$\|e^{\gamma' t}(\boldsymbol{u}, p, \rho)\|_{\{W,l,[kT_0,(k+1)T_0]\}} \leqslant ce^{\gamma'(k+1)T_0}\|(\boldsymbol{u}, p, \rho)\|_{\{W,l,[kT_0,(k+1)T_0]\}},$$

we get (8) for $0 < \gamma' < \log 2/T_0$. The fact that $\int_\Sigma \rho\, dx' = 0$ for all times follows from the fact that $\int_\Sigma \rho\, dx'$ is a motion's integral, as noted after (19). □

§8. Exponential stability of the rest state

In this section, we prove Theorem 1.2. We apply the linearization principle to obtain nonlinear stability of the rest state $\boldsymbol{v}_b \equiv 0$ in a layer of fluid defined by $\phi_b \equiv h$ for some $h > 0$, subjected to a potential force $\boldsymbol{f} = \nabla V$ and an external pressure p_e defined in $\{x_3 \geqslant 0\}$. In order for the layer of fluid at rest to be a stationary solution, the pressure p_b must satisfy

$$\begin{cases} \nabla p_b = \nabla V & \text{in} \quad \Omega_b, \\ p_b = p_e & \text{on} \quad \mathcal{G} := \{x_3 = h\}, \end{cases}$$

and thus it must hold $V = p_e + c$ on $\mathcal{G}$ for some constant c. In this case, in view of (23), (29), and (34), the corresponding linearized problem for the pertubation has the form

$$\begin{cases} \boldsymbol{u}_{,t} - \nu\Delta\boldsymbol{u} + \nabla q - \nabla\rho^* p_{b,x_3} - \boldsymbol{f}_{,x_3}\rho^* = 0 & \text{in } \Omega_b, \\ \nabla\cdot\boldsymbol{u} = 0 & \text{in} \quad \Omega_b, \\ \mathbb{T}(\boldsymbol{u}, q)\boldsymbol{N} - \sigma\Delta'\rho\boldsymbol{N} + p_{e,x_3}\rho\boldsymbol{N} = 0 & \text{on} \quad \mathcal{G}, \\ \rho_{,t} - u^3 = 0 & \text{on} \quad \mathcal{G}, \\ \boldsymbol{u} = 0 & \text{on} \quad \Sigma \quad \text{for all } t \geqslant 0, \\ \boldsymbol{u}(x,0) = \boldsymbol{u}_0(x), \quad \rho(x',0) = \rho_0(x'), & \text{for } x \in \Omega_b, \quad x' \in \Sigma, \end{cases} \tag{119}$$

where $\boldsymbol{N} \equiv \boldsymbol{e}_3$, with the compatibility conditions

$$\nabla\cdot\boldsymbol{u}_0 = 0, \quad \Pi_b\mathbb{D}(\boldsymbol{u}_0) = 0, \quad \int_\Sigma \rho\, ds = 0.$$

Notice that since $\boldsymbol{f} = \nabla V = \nabla p_b$ in Ω_b,

$$\nabla\rho^* p_{b,x_3} + \boldsymbol{f}_{,x_3}\rho^* = \nabla(\rho^* p_{b,x_3}).$$

Let $\mathcal{J}$ be the set of Σ-periodic, square summable solenoidal vector fields with zero normal component on Σ. More precisely, letting $L^2(\Omega_{b\#})$ be the set of Σ-periodic vector fields on the Σ periodic extension $\Omega_{b\#}$ of Ω_b, we have

$$\mathcal{J} := \Big\{\boldsymbol{v} \in L^2_{\text{loc}}(\Omega_{b\#}) : \int_{\Omega_b} \boldsymbol{v}\cdot\nabla\eta\, dx = 0,\ \forall\eta \in W^1_2(\Omega_{b\#})\ s.t.\ \eta\lfloor_{\mathcal{G}} = 0\Big\}.$$

We denote by P the orthogonal projection of $L^2(\Omega_{b\#})$ onto $\mathcal{J}$. Given a periodic vector field $\boldsymbol{w}$, it can be splitted as $\boldsymbol{w} = P\boldsymbol{w} + (I-P)\boldsymbol{w}$, where $(I-P)\boldsymbol{w} = \nabla\varphi_{\boldsymbol{w}}$ and $\varphi_{\boldsymbol{w}}$ is the periodic weak solution to

$$\begin{cases} \Delta\varphi_{\boldsymbol{w}} = \nabla\cdot\boldsymbol{w} & \text{in} \quad \Omega_b, \\ \varphi_{\boldsymbol{w}} = 0 & \text{on} \quad \mathcal{G}, \\ \frac{\partial\varphi_{\boldsymbol{w}}}{\partial x_3} = w^3 & \text{on} \quad \Sigma. \end{cases} \tag{120}$$

The operator P is continuous in $W_2^{\eta}(\Omega_b)$, $\eta \geqslant 0$. Projecting the first equation of (119) onto $\mathcal{J}$ gives

$$\boldsymbol{u}_{,t} - \nu P \Delta \boldsymbol{u} + \nabla \chi = 0,$$

where, using (120), χ is a Σ-periodic function such that

$$\begin{cases} \Delta \chi = 0 & \text{in} \quad \Omega_b, \\ \chi = \nu \boldsymbol{N} \cdot \mathbb{D}(\boldsymbol{u}) \boldsymbol{N} - \sigma \Delta' \rho + \rho (p_{e,x_3} - p_{b,x_3}) & \text{on} \quad \mathcal{G}, \\ \frac{\partial \chi}{\partial x_3} = 0 & \text{on} \quad \Sigma. \end{cases}$$

It can be splitted as $\chi = \chi_{\boldsymbol{u}} + \chi_\rho$, where $\chi_{\boldsymbol{u}}$ and χ_ρ are two Σ-periodic harmonic functions with vanishing normal derivative on Σ and

$$\chi_{\boldsymbol{u}} \lfloor_{\mathcal{G}} = \boldsymbol{N} \cdot \mathbb{D}(\boldsymbol{u}) \boldsymbol{N}, \quad \chi_\rho \lfloor_{\mathcal{G}} = -\sigma \Delta' \rho + (p_e - p_b)_{,x_3} \rho.$$

We then define a linear operator $\mathcal{A} : D(\mathcal{A}) \subseteq X \to X$ on the Hilbert space,

$$X := \mathcal{J} \times \left\{ \rho \in L^2(\Sigma) : \int_\Sigma \rho \, ds = 0 \right\}$$

equipped with the norm

$$\|(\boldsymbol{u}, \rho)\|_X = \left(\|\boldsymbol{u}\|^2_{L^2(\Omega_b)} + \|\rho\|^2_{L^2(\Sigma)} \right)^{\frac{1}{2}},$$

and corresponding standard inner product. We let $\mathcal{A} = (A_{ij})_{i,j=1,2}$, where

$$A_{11}(\boldsymbol{u}) = \nu P \Delta \boldsymbol{u} - \nabla \chi_{\boldsymbol{u}}, \quad A_{12}(\rho) = -\nabla \chi_\rho, \quad A_{21} = u^3, \quad A_{22} = 0.$$

The linear operator $\mathcal{A}$ will have domain $D(\mathcal{A})$ defined as

$$D(\mathcal{A}) := X \cap \{ \boldsymbol{u} \in W_2^2(\Omega_b) : \boldsymbol{u} \lfloor_\Sigma = \Pi_b \mathbb{D}(\boldsymbol{u}) \boldsymbol{N} \lfloor_{\mathcal{G}} = 0 \} \times W_2^{\frac{5}{2}}(\Sigma),$$

which is a Banach space w.r.t. the norm

$$\|(\boldsymbol{u}, \rho)\|_{D(\mathcal{A})} = \left(\|\boldsymbol{u}\|^2_{W_2^2(\Omega_b)} + \|\rho\|^2_{W_2^{\frac{5}{2}}(\Sigma)} \right)^{\frac{1}{2}}.$$

A resolvent estimate for $\mathcal{A}$ where $\operatorname{Re} \lambda$ is sufficiently large has been proved in Theorem 4.4 (for $l = 0$), and this gives that $\lambda - \mathcal{A}$ is coercive (and thus closed) for $\operatorname{Re} \lambda$ sufficiently large. Thus, $\mathcal{A}$ is closed and since $D(\mathcal{A})$ is compactly embedded in X, its spectrum consists of a countable number of eigenvalues with the only accumulation point at $-\infty$.

We can look at problem (119) as the evolutionary problem

$$\boldsymbol{U}_{,t} - \mathcal{A} \boldsymbol{U} = 0, \quad \boldsymbol{U}(0) = \boldsymbol{U}_0 = (\boldsymbol{u}_0, \rho_0),$$

whose exponential stability follows from classical results once one can show negativity of the real part of the spectrum of $\mathcal{A}$. To this end, suppose $(\boldsymbol{u}, \rho) \in D(\mathcal{A})$ is a solution of the complex eigenvalue problem

$$\begin{cases} \lambda \boldsymbol{u} - \nu P \Delta \boldsymbol{u} + \nabla \chi = 0 & \text{in} \quad \Omega_b, \\ \nabla \cdot \boldsymbol{u} = 0 & \text{in} \quad \Omega_b, \\ \mathbb{T}(\boldsymbol{u}, \chi)\boldsymbol{N} - \sigma \Delta' \rho \boldsymbol{N} = 0 & \text{on} \quad \mathcal{G}, \\ \lambda \rho - u^3 = 0 & \text{on} \quad \mathcal{G}, \\ \boldsymbol{u} = 0 & \text{on} \quad \Sigma \end{cases}$$

for $\lambda \in \mathbb{C}$. Multiply by $\boldsymbol{u}$ the first equation and integrate by parts to get

$$\begin{aligned} 0 &= \int\limits_{\Omega_b} \lambda |\boldsymbol{u}|^2 + \frac{\nu}{2} |\mathbb{D}(\boldsymbol{u})|^2 \, dx - \nu \int\limits_{\mathcal{G}} \mathbb{D}(\boldsymbol{u}) \boldsymbol{N} \cdot \boldsymbol{u} \, ds + \int\limits_{\mathcal{G}} \chi \boldsymbol{u} \cdot \boldsymbol{N} \, ds \\ &= \int\limits_{\Omega_b} \lambda |\boldsymbol{u}|^2 + \frac{\nu}{2} |\mathbb{D}(\boldsymbol{u})|^2 \, dx - \nu \int\limits_{\mathcal{G}} \mathbb{D}(\boldsymbol{u}) \boldsymbol{N} \cdot \boldsymbol{u} - \boldsymbol{N} \cdot \mathbb{D}(\boldsymbol{u}) \boldsymbol{N} \boldsymbol{u} \cdot \boldsymbol{N} \, ds \\ &\quad - \sigma \int\limits_{\Sigma} \Delta' \rho \lambda \rho \, ds + \lambda \int\limits_{\Sigma} \rho^2 (p_{e,x_3} - p_{b,x_3}) \, ds \\ &= \lambda \left(\int\limits_{\Omega_b} |\boldsymbol{u}|^2 \, dx + \int\limits_{\Sigma} \sigma |\nabla' \rho|^2 + (p_{e,x_3} - p_{b,x_3}) \rho^2 \, ds \right) + \frac{\nu}{2} \int\limits_{\Omega_b} |\mathbb{D}(\boldsymbol{u})|^2 \, dx, \end{aligned}$$

where we used the fact that $\boldsymbol{u} \cdot \boldsymbol{N} = \lambda \rho$ and $\Pi_b \mathbb{D}(\boldsymbol{u}) \boldsymbol{N} = 0$ to cancel out the boundary terms containing $\mathbb{D}(\boldsymbol{u})$. Since

$$\mathcal{B}(\rho) := \int\limits_{\Sigma} \sigma |\nabla' \rho|^2 + (p_{e,x_3} - p_{b,x_3}) \rho^2 \, ds > 0, \quad \forall \rho \neq 0, \tag{121}$$

the spectrum is real and $\lambda \leqslant 0$ for any eigenvalue of $\mathcal{A}$. Assume that $0 \in \sigma(\mathcal{A})$ to obtain $\boldsymbol{u} \equiv 0$ from the previous equality, and thus $\nabla \chi_\rho \equiv 0$ from the equation. Therefore, for some $c \in \mathbb{R}$,

$$-\sigma \Delta' \rho + (p_e - p_b)_{,x_3} \rho \equiv c.$$

Multiplying by ρ, integrating and using the periodicity, we find that

$$\mathcal{B}(\rho) = c \int\limits_{\Sigma} \rho \, dS = 0,$$

since ρ has zero mean value, forcing $\rho = 0$ by (121). Thus,

$$\sigma(\mathcal{A}) \subset \{\operatorname{Re}\lambda < 0\}$$

and the linear problem is exponentially stable. More precisely, by standard results (for example, see [6]), it holds

$$\|\boldsymbol{U}(t)\|_X \leqslant c(\gamma)e^{\gamma t}\|\boldsymbol{U}_0\|_X,$$
$$\int\limits_0^T e^{-2\gamma t}\|\boldsymbol{U}(t)\|_X^2\,dt \leqslant c(\gamma)\|\boldsymbol{U}_0\|_X^2 \tag{122}$$

for $0 > \gamma > \sup\{\lambda : \lambda \in \sigma(\mathcal{A})\}$ with $c(\gamma)$ independent of T. Now if $(\boldsymbol{u}, p, \rho)$ solves (5), the triple $e^{-\gamma t}(\boldsymbol{u}, p, \rho) =: (\boldsymbol{u}_\gamma, p_\gamma, \rho_\gamma)$ solves the same system except for a forcing term $-\gamma\boldsymbol{u}_\gamma$ in the equation for $\boldsymbol{u}_{\gamma,t}$ and $-\gamma\rho_\gamma$ in the one for $\rho_{\gamma,t}$. Applying (93) to $(\boldsymbol{u}_\gamma, p_\gamma, \rho_\gamma)$ we obtain

$$\|e^{-\gamma t}(\boldsymbol{u}, p, \rho)\|_{l,\infty} \leqslant c\Big(\|\boldsymbol{u}_\gamma\|_{K^l(Q_\infty)} + \|\rho_\gamma\|_{K^{l+\frac{3}{2}}(G_\infty)} + \|\boldsymbol{u}_0\|_{W_2^{l+1}(\Omega_b)} + \|\rho_0\|_{W_2^{l+2}(\mathcal{G})} + \|\boldsymbol{u}_\gamma\|_{L^2(Q_\infty)} + \|\rho_\gamma\|_{L^2(G_\infty)}\Big).$$

Using interpolation inequalities and (122), we finally obtain

$$\|(\boldsymbol{u}_\gamma, p_\gamma, \rho_\gamma)\|_{l,\infty} \leqslant c(\|\boldsymbol{u}_0\|_{W_2^{l+1}(\Omega_b)} + \|\rho_0\|_{W_2^{l+2}(\mathcal{G})}),$$

which is (8). Applying Theorem 1.1, thus concludes the proof of the nonlinear exponential stability of the rest state, and in particular of Theorem 1.2.

Condition (121) has an appealing physical meaning: the term

$$p_{e,x_3} - p_{b,x_3} = (p_e - V)_{,x_3} = (\nabla p_e - \boldsymbol{f}) \cdot \boldsymbol{N}$$

represents the total volume force acting on a fluid particle on the free boundary in the direction $-\boldsymbol{N}$, and thus its positivity implies that volume elements at the free boundary are subjected to a force pointing *inside* the layer. More generally, the surface tension allows volume forces to point outwards, as long as the wavelength ℓ of the perturbation is sufficiently small compared to the capillarity. More precisely, by the explicit constant of the Poincaré inequality for periodic functions, we obtain stability as long as it holds the inequality

$$\inf_{\Sigma}(p_e(x', h) - V(x', h))_{,x_3} > -\frac{\sigma}{\ell^2}. \tag{123}$$

For $p_e \equiv p_{atm}$ and $\boldsymbol{f} = (0, 0, -g)$, (121) is certainly satisfied, thus giving exponential stability of the rest state in the small scale model. Stability is also obtained for the more refined model in which the fluid has density d_1, the gas filling $\{x_3 > h\}$ density d_2 and both are subjected to a gravitational force directed along $(0, 0, -1)$ with modulus $-d_i V'(x_3)$, respectively. In this case,

$$\boldsymbol{f} = (0, 0, -d_1 V'(x_3)), \quad p_e = d_2 V(x_3), \quad V' \leqslant 0$$

and (121) holds true as long as $d_1 > d_2$, i.e., the gas filling $\{x_3 > h\}$ is less dense than the fluid. For small wavelength ℓ, the case $d_1 \leqslant d_2$ may also lead to stable solutions. In this case, we need to require (123).

Finally, we consider the case in which $\boldsymbol{f} = (0, 0, g)$ and $p_e = p_{atm}$, which models a layer of fluid on a roof. To get a meaningful stable solution, we consider

$$p_b = p_{atm} + g d_1 (x_3 - h), \quad h \leqslant \frac{p_{atm}}{g d_1},$$

where h, as usual, is the height of the layer and d_1 is the density of the fluid. Condition (123), which implies (121) holds as long as the wavelength of the perturbation satisfies $\ell < \sqrt{\frac{\sigma}{g d_1}}$.

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