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A LIOUVILLE THEOREM FOR THE STOKES SYSTEM IN HALF-SPACE

Abstract. In this note, we describe all non-trivial bounded ancient solutions to the Stokes system in half space with non-slip boundary conditions.

§1. Main result

In the present paper, we address the following question. We are looking for non-trivial bounded ancient solutions to the Stokes systems satisfying non-slip boundary conditions on the boundary. To be precise, we are looking for non-trivial solutions $u \in L_\infty(Q^+_-)$ of the following boundary value problem

$$\partial_t u - \Delta u + \nabla p = 0, \quad \operatorname{div} u = 0 \qquad \text{in } Q^+_-, \tag{1.1}$$

$$u(x',0,t) = 0 \qquad x' \in \mathbb{R}^{n-1}, \quad t \in \mathbb{R}_-. \tag{1.2}$$

Here, $Q^+_- = \mathbb{R}^n_+ \times \mathbb{R}_-$, $\mathbb{R}^n_+ = \{x = (x', x_n) : \ x' \in \mathbb{R}^{n-1}, \ x_n \in \mathbb{R}_+\}$, $\mathbb{R}_+ = \{s \in \mathbb{R} : \ s > 0\}$, and $\mathbb{R}_- = \{s \in \mathbb{R} : \ s < 0\}$.

We shall understand solutions $u \in L_\infty(Q^+_-)$ to (1.1) and (1.2) in the following weak sense:

$$\int\limits_{Q^+_-} (u \cdot \partial_t w + u \cdot \Delta\, w) dz = 0 \tag{1.3}$$

for any $w \in W = \{w \in C^\infty_0(\mathbb{R}^{n+1}) : \ \operatorname{div} w = 0, \ w(x',0,t) = 0 \ \forall (x',t) \in \mathbb{R}^{n-1} \times \mathbb{R}_-, \ w(x,0) = 0 \ \forall x \in \mathbb{R}^n\}$ and for a.a. $t < 0$

$$\int\limits_{\mathbb{R}^n_+} u(x,t) \cdot \nabla q(x) dx = 0 \tag{1.4}$$

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for all $q \in C_0^\infty(\mathbb{R}^n)$.

Obviously, smooth solutions to (1.1) and (1.2) satisfy (1.3) and (1.4). As it was shown in [4], the above problem may have non-trivial solutions that are not necessary smooth in spatial variables (in contrast to the case of the whole space $\mathbb{R}^3$, where non-smoothness may occur but only in time). Those non-trivial solutions have the form $u = (u_\alpha(x_3,t),0)$, $\alpha = 1,2$, and describe a bounded shear flow. This kind of non-smoothness in spatial variables, caused by the presence of the boundary, has been observed earlier in [3]. The aim of this note is to give an alternative proof of that all bounded not-trivial solutions to (1.1) and (1.2) have the above form, compare with [2].

Theorem 1.1. *Let $n = 3$ and $u \in L_\infty(Q^+_-)$ satisfy identities* (1.3) *and* (1.4). *Then $u(x,t) = (u_\alpha(x_3,t),0)$ with $\alpha = 1,2$. Functions u_α obey the identity*

$$\int\limits_{-\infty}^{0}\int\limits_{0}^{\infty} u_\alpha(\partial_t w_{\alpha,3} + w_{\alpha,333})dx_n dt = 0 \tag{1.5}$$

for any functions $w_\alpha \in \widetilde{W}$, $\alpha = 1,2$, where $\widetilde{W} = \{h \in C_0^\infty(\mathbb{R}^2): \ h(0,t) = h_{,3}(0,t) = 0, \quad h(x_3,0) = 0\}$.

§2. Proof of Theorem 1.1

Proof Take two arbitrary vector-valued functions $\psi^{(\alpha)} \in C_0^\infty(Q^+_+)$, $\alpha = 1,2$, where $Q^+_+ = \mathbb{R}^3_+ \times \mathbb{R}_+$, and consider the following initial boundary value problem

$$\begin{aligned} \partial_t w - \Delta w + \nabla r = f, \quad \operatorname{div} w = 0 \qquad & \text{in } Q^+_+, \\ w(x',0,t) = 0 \qquad & x' \in \mathbb{R}^2,\ t > 0, \\ w(x,0) = 0 \qquad & x \in R^3_+ \end{aligned} \tag{2.1}$$

where

$$f = \psi^{(\alpha)}_{,\alpha} = \psi^{(1)}_{,1} + \psi^{(2)}_{,2}. \tag{2.2}$$

Since $\psi^{(\alpha)} \in L_2(Q^+_+)$, problem (2.1) has a unique energy solution w satisfying the energy estimate

$$\int\limits_{R^3_+} |w(x,t)|^2 dx + \int\limits_0^t \int\limits_{R^3_+} |\nabla w|^2 dx dt' \leqslant \int\limits_0^t \int\limits_{R^3_+} \psi^{(\alpha)} \cdot \psi^{(\alpha)} dx dt' \tag{2.3}$$

for all $t>0$. Using standard multiplicative inequalities, we can state

$$\int\limits_0^\infty\int\limits_{R^3_+}|w|^{\frac{10}{3}}dxdt<\infty$$

and there exists a sequence t_k tending to infinity such that

$$\int\limits_{R^3_+}|w(\cdot,t_k)|^{\frac{10}{3}}dx\to 0. \tag{2.4}$$

On the other hand, Solonnikov, see [5], has constructed the solution w with the help of Green's function. Let us describe his results, which are crucial in our approach. For reader's convenience, we are going to keep his notation. First, we introduce two scalar functions $\Phi^{(\alpha)}$, $\alpha=1,2$, so that

$$\Phi^{(\alpha)}(x,t)=-\int\limits_{\mathbb{R}^3_+}\nabla_y N(x,y)\cdot\Psi^{(\alpha)}(y,t)dy,$$

where $N(x,y)=E(x-y)-E(x-y^*)$, $E(x)$ is the fundamental solution to the Laplace operator in $\mathbb{R}^3$, and $y^*=(y',-y_3)$ for $y=(y',y_3)$. Then we let

$$\Phi(x,t)=\Phi^{(\alpha)}_{,\alpha}(x,t),$$

and

$$f'(x,t)=f(x,t)-\nabla\Phi(x,t),$$

so that $f'=\psi'^{(\alpha)}_{,\alpha}=\psi'^{(1)}_{,1}+\psi'^{(2)}_{,2}$, where

$$\psi'^{(\alpha)}(x,t)=\psi^{(\alpha)}(x,t)-\nabla\Phi^{(\alpha)}(x,t).$$

Now, Solonnikov's solution formula is as follows:

$$w(x,t)=\int\limits_0^t\int\limits_{\mathbb{R}^3_+}G(x,y,t-\tau)\psi'^{(\alpha)}_{,\alpha}(y,\tau)dyd\tau, \tag{2.5}$$

where

$$G(x,y,t)=\mathbb{I}\Gamma(x-y,t)+G^*(x,y,t),$$

$\Gamma(x,t)$ is the fundamental solution to the heat equation, and function G^* and its derivatives obey the estimates, see [5]:

$$|G^*(x,y,t)| \leqslant c(|x-y^*|^2+t)^{-\frac{3}{2}}e^{-\frac{cy_3^2}{t}},$$
$$\Big|\frac{\partial}{\partial y_\alpha}G^*(x,y,)\Big| \leqslant c(|x-y^*|^2+t)^{-2}e^{-\frac{cy_3^2}{t}} \tag{2.6}$$

In the same paper, Solonnikov proved the following estimates of w:

$$|w(x,t)| \leqslant \frac{C_1(t)}{(1+|x|)^{4-\delta}}$$
$$|\nabla w(x,t)| \leqslant \frac{C_2(t)}{(1+|x|)^4} \tag{2.7}$$
$$|\partial_t w(x,t)|+|\nabla^2 w(x,t)| \leqslant \frac{C_3(t)}{(1+|x|)^4 x_3^{2\varepsilon}},$$

where $\delta\in]0,1[$, and $\varepsilon\in]0,1/2[$, and C_i, $i=1,2,3$, are non-decaying (power) function of t. We are going to show that the first estimate can be slightly improved. Namely, $C_1(t)=c$ can be taken independent of t.

According to our assumptions, we have

$$\operatorname{supp}\psi^{(\alpha)}\subset\{(x',x_3,t):\ |x'|<a,\,\delta_1<x_3<A_1,\ \delta_2<t<A_2\}$$

for some positives numbers a, δ_α, and A_α with $\alpha=1,2$. Then

$$|\psi'^{(\alpha)}(x,t)|+|\nabla\Phi^{(\alpha)}(x,t)| \leqslant \frac{c\chi_2(t)}{(|x|+1)^3}, \tag{2.8}$$

where $\chi_2(t)=1$ if $\delta_2<t<A_2$ and $\chi_2(t)=0$ otherwise, and

$$|\psi'^{(\alpha)}_{,\alpha}(x,t)| \leqslant \frac{c\chi_2(t)}{(|x|+1)^4}. \tag{2.9}$$

Let

$$w=w^{(1)}+w^{(2)}, \tag{2.10}$$

where

$$w^{(2)}(x,t)=\int\limits_0^t\int\limits_{\mathbb{R}^3_+} G^*(x,y,t-\tau)\psi'^{(\alpha)}_{,\alpha}(y,\tau)dyd\tau.$$

We are going to estimate $w^{(2)}$ only. Bounds of $w^{(1)}$ are just easier.

First, we let

$$w^{(2)}(x,t)=\overline{w}^{(2)}(x,t)+\widetilde{w}^{(2)}(x,t),$$

where

$$\widetilde{w}^{(2)}(x,t)=\int\limits_0^t\int\limits_{\{|y|\geqslant|x|/2\}\cap\{x_3>0\}}G^*(x,y,t-\tau)\psi'^{(\alpha)}_{,\alpha}(y,\tau)dyd\tau.$$

Here, we are going to use the first estimate in (2.6). So, we have, for $t>2A_2$,

$$|\widetilde{w}^{(2)}(x,t)|\leqslant\frac{c}{(1+|x|)^4}I(x,t),$$

with

$$I(x,t)=\int\limits_{\delta_2}^{A_2}i(x,y,t-\tau)d\tau$$

and

$$i(x,y,t-\tau)=\int\limits_{\{|y|\geqslant|x|/2\}\cap\{x_3>0\}}|G^*(x,y,t-\tau)|dy.$$

After changing variables $z=y/\sqrt{t-\tau}$, we have

$$i(x,y,t-\tau)\leqslant\int\limits_{\mathbb{R}^3}\frac{e^{-cz_3^2}dz'dz_3}{\Big(|\frac{x'}{\sqrt{t-\tau}}-z'|^2+|\frac{x_3}{\sqrt{t-\tau}}+z_3|^2+1\Big)^{\frac{3}{2}}}$$

$$\leqslant\int\limits_{-\infty}^{\infty}e^{-cz_3^2}dz_3\int\limits_{\mathbb{R}^2}\frac{dy'}{(|y'|^2+1)^{\frac{3}{2}}}=c<\infty.$$

So, we have

$$|\widetilde{w}^{(2)}(x,t)|\leqslant\frac{c}{(1+|x|)^4}.$$

In the integral representing $\overline{w}^{(2)}(x,t)$, we do integration by parts. As a result, we find

$$\overline{w}^{(2)}(x,t)=J_1+J_2,$$

where

$$J_1=-\int\limits_0^t\int\limits_{\{|y|<|x|/2\}\cap\{x_3>0\}}\frac{\partial}{\partial y_\alpha}G^*(x,y,t-\tau)\psi'^{(\alpha)}(y,\tau)dyd\tau$$

and

$$J_2=\int\limits_0^t\int\limits_{\{|y|=|x|/2\}\cap\{x_3>0\}}G^*(x,y,t-\tau)\nu_\alpha(y)\psi'^{(\alpha)}(y,\tau)ds_yd\tau.$$

Here, ν is an outward normal to the sphere $|y|=|x|/2$.

Next, if $|y|\leqslant|x|/2$, then $|x-y^*|\geqslant|x|/2$ and thus, using the second estimate in (2.5) we have

$$|J_1|\leqslant c\int\limits_{\delta_2}^{A_2}\frac{d\tau}{(|x|^2+(t-\tau))^2}\int\limits_0^{|x|/2}e^{-\frac{cy_3^2}{t-\tau}}dy_3\int\limits_{|y'|<|x|/2}\frac{dy'}{(|y'|^2+y_3^2+1)^{\frac{3}{2}}}$$

$$\leqslant c\int\limits_{\delta_2}^{A_2}\frac{d\tau}{(|x|^2+(t-\tau))^2}\int\limits_0^{|x|/2}\frac{dy_3}{\sqrt{1+y_3^2}}$$

$$\leqslant c\frac{\ln(|x|+1)}{(|x|+1)^4}$$

for any $t>2A_2$ with a constant c independent of t.

As to J_2, the corresponding bound is easier.

$$|J_2|\leqslant\int\limits_{\delta_2}^{A_2}\int\limits_{\{|y|=|x|/2\}\cap\{x_3>0\}}\frac{e^{-\frac{cy_3^2}{t-\tau}}ds_yd\tau}{(|x-y^*|^2+(t-\tau))^{\frac{3}{2}}(1+|y|)^2}$$

$$\leqslant\frac{c}{(1+|x|)^4}.$$

Now, we proceed with the proof of Theorem 1.1. Take a smooth cut-off function having the following properties:

$$\varphi_R(x)=1\qquad x\in B(5/4R),\qquad\varphi_R(x)=0\qquad x\notin B(7/4R),$$

$$|\nabla^k\varphi_R(x)|\leqslant cR^{-k}\qquad x\in\mathbb{R}^3,\quad k=0,1,2.$$

Apparently,

$$w\cdot\nabla\varphi_R\in\overset{\circ}{W}{}^1_2(T_+(R))$$

with $T_+(R)=B_+(2R)\setminus\overline{B}_+(R)$. By simple scaling arguments and by Bogovskii results, see [1], we can claim that there exists a function

$$w_R(\cdot,t)\in\overset{\circ}{W}{}^2_2(T_+(R))$$

possessing properties:

$$\operatorname{div} w_R(\cdot,t) = \nabla\varphi(x)\cdot w(\cdot,t)$$

in $T_+(R)$ and thus in $\mathbb{R}^3_+$ and

$$\|\nabla^2 w_R(\cdot,t)\|_{2,T_+(R)} \leqslant c\|\nabla(\nabla\varphi(\cdot)\cdot w(\cdot,t))\|_{2,T_+(R)}$$

with a positive constant c being independent of R and t.

Applying bounds (2.7), we can get

$$\|\nabla^2 w_R(\cdot,t)\|^2_{2,T_+(R)} \leqslant \frac{C(t)}{R^7}$$

with perhaps a polynomially growing function $t \mapsto C(t)$. From Bogovskii's construction, it follows two additional estimates:

$$\|\partial_t w_R(\cdot,t)\|_{2,T_+(R)} \leqslant cR\|\nabla\varphi(\cdot)\cdot\partial_t w(\cdot,t)\|_{2,T_+(R)}$$

and

$$\|w_R(\cdot,t)\|_{2,T_+(R)} \leqslant cR\|\nabla\varphi(\cdot)\cdot w(\cdot,t)\|_{2,T_+(R)}$$

with a positive constant c being independent of R and t. The right hand side of both them can be evaluated with the help of bounds (2.7). As a result, we have

$$R^2\|\nabla\varphi(\cdot)\cdot\partial_t w(\cdot,t)\|^2_{2,T_+(R)} \leqslant C(t)\int\limits_{T_+(R)}\frac{dx}{|x|^8 x_3^{4\varepsilon}} \leqslant \frac{C(t)}{R^5}$$

for sufficiently small positive ε. For the second right hand side, we show

$$R^2\|\nabla\varphi(\cdot)\cdot w(\cdot,t)\|^2_{2,T_+(R)} \leqslant C(t)\int\limits_{T_+(R)}\frac{dx}{|x|^{2(4-\delta)}} \leqslant \frac{C(t)}{R^5}$$

for sufficiently small positive δ.

With these three estimates, we can easily show that, for which fixed $T>0$,

$$\int\limits_0^T\int\limits_{T_+(R)}(|\partial_t w_R| + |w_R| + |\nabla^2 w_R|)dxdt \to 0$$

as $R\to\infty$.

Now let $\chi(t)$ be an arbitrary sufficiently smooth function which is equal to zero for $t \leqslant -T$ and then test identity (1.3) with the following test function

$$\chi(t)(\varphi_R(x)\widetilde{w}(x,t) - \widetilde{w}_R(x,t)),$$

where $\widetilde{w}(x,t) = w(x,-t)$ and $\widetilde{w}_R(x,t) = w_R(x,-t)$. It is admissible since the function $w_R(\cdot,t)$ can be extended by zero to the whole $\mathbb{R}^3$. So, as a result, we have the identity

$$I_R^1 + I_R^2 + I_R^3 + I_R^4 = 0,$$

where

$$I_R^1 = \int\limits_{-T}^{0}\int\limits_{\mathbb{R}^3_+} \chi\varphi_R u\cdot(\partial_t\widetilde{w} + \Delta\widetilde{w})dxdt,$$

$$I_R^2 = \int\limits_{-T}^{0}\int\limits_{\mathbb{R}^3_+} \chi u\cdot(\widetilde{w}\Delta\varphi_R + 2\nabla\widetilde{w}\nabla\varphi_R)dxdt,$$

$$I_R^3 = \int\limits_{-T}^{0}\int\limits_{\mathbb{R}^3_+} \varphi_R u\cdot\widetilde{w}\partial_t\chi dxdt,$$

$$I_R^4 = \int\limits_{-T}^{0}\int\limits_{\mathbb{R}^3_+} u\cdot(\chi\partial_t\widetilde{w}_R + \widetilde{w}_R\partial_t\chi + \chi\Delta\widetilde{w}_R)dxdt.$$

As it has been already shown,

$$I_R^4 \to 0$$

as $R\to\infty$. In the same way, using bound (2.7), we can state that

$$I_R^2 \to 0$$

as $R\to\infty$. For the same reasons, we have

$$I_R^1 \to \int\limits_{-T}^{0}\int\limits_{\mathbb{R}^3_+} \chi u\cdot(\partial_t\widetilde{w} + \Delta\widetilde{w})dxdt$$

and

$$I_R^3 \to \int\limits_{-T}^{0}\int\limits_{\mathbb{R}^3_+} u\cdot\widetilde{w}\partial_t\chi dxdt$$

as $R\to\infty$.

So, we have

$$\int_{-T}^{0}\int_{\mathbb{R}^3_+}\chi u\cdot(\partial_t\widetilde{w}+\Delta\widetilde{w})dxdt+\int_{-T}^{0}\int_{\mathbb{R}^3_+}u\cdot\widetilde{w}\partial_t\chi dxdt=0.$$

For the first term on the left hand side of the latter relation we have

$$\int_{-T}^{0}\int_{\mathbb{R}^3_+}\chi u\cdot(\partial_t\widetilde{w}+\Delta\widetilde{w})dxdt=\int_{-T}^{0}\int_{\mathbb{R}^3_+}\chi u\cdot(\nabla\widetilde{r}-\widetilde{f})dxdt,$$

where $\widetilde{r}(x,t)=r(x,-t)$ and $\widetilde{f}(x,t)=f(x,-t)$.

Our aim now is to show that, for all $t<0$,

$$\int_{\mathbb{R}^3_+}u(\cdot,t)\cdot\nabla\widetilde{r}(\cdot,t)dx=0.$$

To this end, we take the same cut-off function φ_R, and then by (1.4)

$$\int_{\mathbb{R}^3_+}u\cdot\nabla(\varphi_R(\widetilde{r}-[\widetilde{r}]_{T_+(R)}))dx=0=H^1_R+H^2_R,$$

where $[\widetilde{r}]_{T_+(R)}(t)$ is the mean value of $\widetilde{r}(\cdot,t)$ over $T_+(R)$ and

$$H^1_R=\int_{\mathbb{R}^3_+}\varphi_R u\cdot\nabla\widetilde{r}dx$$

and

$$H^2_R=\int_{\mathbb{R}^3_+}u\cdot\nabla\varphi_R(\widetilde{r}-[\widetilde{r}]_{T_+(R)})dx.$$

Taking into account the equation $\nabla\widetilde{r}=\partial_t\widetilde{w}+\Delta\widetilde{w}+\widetilde{f}$ and estimates (2.7), we have

$$H^1_R\to\int_{\mathbb{R}^3_+}u\cdot\nabla\widetilde{r}dx$$

as $R\to\infty$ and for all $t<0$. For H^2_R, using the same arguments as above, we have

$$|H^2_R|\leqslant c\|u\|_{\infty,Q^+_-}R^{\frac{3}{2}}\Big(\frac{1}{R^2}\int_{T_+(R)}|\widetilde{r}-[\widetilde{r}]_{T_+(R)}|^2dx\Big)^{\frac{1}{2}}$$

$$\leqslant c\|u\|_{\infty,Q_-^+}R^{\frac{3}{2}}\Big(\int\limits_{T_+(R)}|\nabla\widetilde{r}|^2dx\Big)^{\frac{1}{2}}\leqslant c\|u\|_{\infty,Q_-^+}R^{\frac{3}{2}}C(t)\Big(\int\limits_{T_+(R)}\frac{dx}{|x|^8x_3^{4\varepsilon}}\Big)^{\frac{1}{2}}.$$

The right hand side of the latter inequality goes to zero as $R\to\infty$ for all $t<0$ if $\varepsilon>0$ is sufficiently small.

Finally, assuming that $T>2A_2$ and $\chi(t)=1$ for $-A_2<t<0$, we have

$$\int\limits_{Q_-^+}u\cdot\widetilde{f}dxdt=\int\limits_{-T}^{0}\int\limits_{\mathbb{R}^3_+}u\cdot\widetilde{w}\partial_t\chi dxdt.$$

Taking a sequence of appropriated functions χ, we can derive for the latter that for all $t<-2A_2$

$$\int\limits_{Q_-^+}u\cdot\widetilde{f}dxdt=-\int\limits_{\mathbb{R}^3_+}u(x,t)\cdot\widetilde{w}(x,t)dx.$$

Now, we wish to show that the right hand side of the latter identity at $t=-t_k$, see (2.4) for the definition of t_k, tends to zero as $k\to\infty$. To this end, it is enough to show that

$$\int\limits_{\mathbb{R}^3_+}|w(x,t_k)|dx\to 0$$

as $k\to\infty$. Indeed, according to the improved version of the first estimate in (2.7) with $C_1(t)=c$, given $\varepsilon>0$, one can find $R_*>0$ such that

$$\int\limits_{\mathbb{R}^3_+\setminus B_+(R_*)}|w(x,t)|dx<\varepsilon/2$$

for all $t>A_2$. By (2.4), we can find K_* such that for $k>K_*$

$$\int\limits_{B_+(R_*)}|w(x,t_k)|dx\leqslant cR_*^{\frac{21}{10}}\Big(\int\limits_{\mathbb{R}^3_+}|w(x,t_k)|^{\frac{10}{3}}dx\Big)^{\frac{3}{10}}<\varepsilon/2.$$

So, the required statement is proved.

We have now

$$\int\limits_{Q_-^+}u\cdot\widetilde{f}dxdt=0.$$

Since functions $\psi^{(\alpha)} \in C_0^\infty(Q_-^+)$ are taken arbitrarily, we state that

$$u(x,t) = u(x_3,t).$$

Moreover, from (1.4), it follows that $u_3 = 0$. In order to establish (1.5), take function $\varphi \in C^\infty(\mathbb{R}^2)$ and take two admissible functions $w_\alpha \in \widetilde{W}$, $\alpha = 1, 2$, so that the function

$$w(x,t) = (\varphi w_{1,3}, \varphi w_{2,3}, -\varphi_{,\alpha} w_\alpha)$$

is admissible in (1.3). Then (1.5) follows easily. Theorem 1.1 is proved.

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