Записки научных
семинаров ПОМИ
Том 410, 2013 г.

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ESTIMATES OF SOLUTIONS TO THE PERTURBED STOKES SYSTEM

Abstract. In this paper, we derive local estimates of solutions to the Perturbed Stokes system, which arises as a reduction of the Stokes system near a curved part of the boundary of the domain if a diffeomorphism flatting the boundary is applied. The estimates obtained in the paper play the crucial role in the investigation of partial regularity of weak solutions to the Navier–Stokes system near a curved part of the boundary.

Dedicated to 90-th anniversary of Olga Alexandrovna Ladyzhenskaya

§1. Introduction

Let $B_R^+ := \{x \in \mathbb{R}^n : \ |x| < R, x_n > 0\}$ be a half-ball in $\mathbb{R}^n$, $n \geqslant 2$, and assume $Q_R^+ = B_R^+ \times (-R^2, 0)$. For any $x \in \mathbb{R}^n$, $x = (x_1, \dots, x_{n-1}, x_n)$ we denote by $x' \in \mathbb{R}^{n-1}$ the vector $x' := (x_1, \dots, x_{n-1})$. Denote

$$S_R := \{\ x' \in \mathbb{R}^{n-1} : |x'| < R\ \}$$

and assume $\varphi : \bar{S}_R \to \mathbb{R}$ is a sufficiently smooth function. In this paper we obtain local estimates for the following system which we call the Perturbed Stokes system:

$$\left\{ \begin{array}{rcl} \partial_t v \ - \ \widehat{\Delta}_\varphi v \ + \ \widehat{\nabla}_\varphi p \ = \ f \\ \widehat{\nabla}_\varphi \cdot v \ = \ 0 \end{array} \right. \quad \text{in} \quad Q_R^+. \tag{1.1}$$

Here v, $f : Q_R^+ \to \mathbb{R}^n$ are vector fields, $p : Q_R^+ \to \mathbb{R}$ is a scalar functions, $\widehat{\Delta}_\varphi$ and $\widehat{\nabla}_\varphi$ are the differential operators with variable coefficients defined

Key words and phrases: Stokes system, boundary regularity.
The second author is supported by RFBR, grant 11-01-00324.

via a function φ by formulas

$$\begin{aligned} \widehat{\Delta}_\varphi v &:= \Delta v - 2v_{,\alpha n}\varphi_{,\alpha} + v_{,nn}|\nabla'\varphi|^2 - v_{,n}\Delta'\varphi, \\ \widehat{\nabla}_\varphi \cdot v &:= \operatorname{div} v - v_{\alpha,n}\varphi_{,\alpha}, \\ \widehat{\nabla}_\varphi p &:= \nabla p - p_{,n}\begin{pmatrix} \nabla'\varphi \\ 0 \end{pmatrix}. \end{aligned} \tag{1.2}$$

Here we assume summation from 1 to $n-1$ over repeated Greek indexes and ∇' and Δ' denote the gradient and Laplacian with respect to $(x_1, \dots, x_{n-1})$ variables. We will also make use of the differential operator

$$\widehat{\nabla}_\varphi v = \nabla v - v_{,n} \otimes \begin{pmatrix} \nabla'\varphi \\ 0 \end{pmatrix}, \tag{1.3}$$

where for any a, $b \in \mathbb{R}^n$ the symbol $a \otimes b$ denotes the $n \times n$- matrix with components $(a_i b_j)$, i, $j = 1, \dots, n$.

In this paper we study the problem (1.1) assuming v satisfies the slip boundary condition on the hyperplane $\{x_n = 0\}$:

$$v|_{x_n=0} = 0. \tag{1.4}$$

The Perturbed Stokes system arises as a reduction of the usual Stokes system in a domain near a point belonging to the curved part of the boundary if the latter is a graph of φ. Let $(u, q, \widetilde{f})$ satisfy the Stokes system in $\Omega_R \times (-R^2, 0)$,

$$\begin{cases} \partial_t u - \Delta u + \nabla q = \widetilde{f} \\ \operatorname{div} u = 0 \end{cases} \quad \text{in} \quad \Omega_R \times (-R^2, 0). \tag{1.5}$$

We assume that

$$\Omega_R = \left\{ y \in \mathbb{R}^n : |y'| < R,\ \varphi(y') < y_n < \varphi(y') + \sqrt{R^2 - |y'|^2} \right\},$$

where φ is a certain appropriate coordinate system and impose the slip boundary condition on u:

$$u|_{x_n=\varphi(x')} = 0. \tag{1.6}$$

In this paper, we assume that φ is of class W^3_∞ (i.e., its second derivatives are Lipschitz continuous) and the Cartesian coordinate system is chosen in such a way that the following relations hold:

$$\varphi(0) = 0, \qquad \nabla\varphi(0) = 0, \qquad \|\varphi\|_{W^3_\infty(S_R)} \leqslant \mu. \tag{1.7}$$

Now we apply the diffeomorphism flattering the boundary. In other words, we introduce new coordinates $x = \psi(y)$ by the relations

$$\psi : \Omega_R \to B_R^+, \qquad x = \psi(y) \ = \ \begin{pmatrix} y' \\ y_n - \varphi(y') \end{pmatrix}, \tag{1.8}$$

$$y \in \Omega_R \quad \Longleftrightarrow \quad x \in B_R^+.$$

and denote

$$v := u \circ \psi^{-1}, \qquad p := q \circ \psi^{-1}.$$

Then, for $x = \psi(y)$ we have relations

$$\nabla q(y) = \widehat{\nabla}_\varphi p(x), \quad \Delta u(y) = \widehat{\Delta}_\varphi v(x), \quad \operatorname{div} u(y) = (\widehat{\nabla}_\varphi \cdot v)(x).$$

Hence, the Stokes system (1.5), (1.6) in $\Omega_R \times (-R^2, 0)$ in y-variables transfers to the Perturbed Stokes system (1.1), (1.4) in Q_R^+ in x-variables.

Now we need to introduce some functional spaces: assume $1 \leqslant s, l < +\infty$. Assume $\Omega \subset \mathbb{R}^n$, $Q_T = \Omega \times (0, T)$ and let $L_{s,l}(Q_T)$ be the anisotropic Lebesgue space equipped with the norm

$$\|f\|_{L_{s,l}(Q_T)} := \Big(\int\limits_0^T \Big(\int\limits_\Omega |f(x,t)|^s \, dx \Big)^{l/s} dt \Big)^{1/l},$$

and denote

$$W^{1,0}_{s,l}(Q_T) \equiv L_l(0, T; W^1_s(\Omega)) = \{ \ u \in L_{s,l}(Q_T) : \ \nabla u \in L_{s,l}(Q_T) \ \},$$
$$W^{2,1}_{s,l}(Q_T) = \{ \ u \in W^{1,0}_{s,l}(Q_T) : \ \nabla^2 u, \ \partial_t u \in L_{s,l}(Q_T) \ \}.$$

These spaces are equiped with the following norms:

$$\|u\|_{W^{1,0}_{s,l}(Q_T)} = \|u\|_{L_{s,l}(Q_T)} + \|\nabla u\|_{L_{s,l}(Q_T)},$$
$$\|u\|_{W^{2,1}_{s,l}(Q_T)} = \|u\|_{W^{1,0}_{s,l}(Q_T)} + \|\nabla^2 u\|_{L_{s,l}(Q_T)} + \|\partial_t u\|_{L_{s,l}(Q_T)}.$$

We also denote by $W_s^{-1}(\Omega)$ the conjugate space to $\overset{\circ}{W}{}^1_{s'}(\Omega)$ equipped with the norm

$$\|f\|_{W_s^{-1}(\Omega)} = \sup_{w \in \overset{\circ}{W}{}^1_{s'}(\Omega), \ \|w\|_{W^1_{s'}}(\Omega) \leqslant 1} |\langle f, w \rangle|$$

and denote by $L_l(0,T;W_s^{-1}(\Omega))$ the space of measurable functions $f:[0,T]\to W_s^{-1}(\Omega)$ such that the following norm is finite:

$$\|f\|_{L_l(0,T;W_s^{-1}(\Omega))}=\Big(\int\limits_0^T\|f(\cdot,t)\|^l_{W_s^{-1}(\Omega)}\,dt\Big)^{1/l}.$$

Definition 1.1. *Assume that $1<s,l<+\infty$ and $f\in L_{s,l}(Q_R^+)$. We say that the functions (v,p) are the strong solution of the problem* (1.1), (1.4), *if they belong to the spaces*

$$v\in W^{2,1}_{s,l}(Q_R^+),\qquad p\in W^{1,0}_{s,l}(Q_R^+),$$

satisfy the equations (1.1) *a.e. in Q_R^+, and satisfy the boundary conditions* (1.4) *in the sense of traces.*

Definition 1.2. *Assume $1<s,l<+\infty$ and $f\in L_l(-R^2,0;W_s^{-1}(B_R^+))$. We say that the functions (v,p) are the generalized solution of the problem* (1.1), (1.4), *if they belong to the spaces*

$$v\in W^{1,0}_{s,l}(Q_R^+),\qquad p\in L_{s,l}(Q_R^+),$$

(v,p) satisfy (1.1) *in the sense of distributions and v satisfies the boundary condition* (1.4) *in the sense of traces.*

Note that though $\widehat{\Delta}_\varphi$ and $\widehat{\nabla}_\varphi$ are the operators with variable coefficients, the function φ is independent of x_n and thus these operators possess the properties

$$\int\limits_{B_R^+}\widehat{\Delta}_\varphi v\cdot w\,dx=-\int\limits_{B_R^+}\widehat{\nabla}_\varphi v:\widehat{\nabla}_\varphi w\,dx=\int\limits_{B_R^+}v\cdot\widehat{\Delta}_\varphi w\,dx,$$

$$\int\limits_{B_R^+}\widehat{\nabla}_\varphi p\cdot w\,dx=-\int\limits_{B_R^+}p\widehat{\nabla}_\varphi\cdot w\,dx$$

for any $v\in W_s^2(B_R^+)$, $p\in W_s^1(B_R^+)$, $w\in C_0^\infty(B_R^+)$. Hence, for the equations (1.1) with variable coefficients there it is easy to define solutions "in the sense of distributions" in the usual way (similar to PDEs with constant coefficients) by putting all differential operators $\widehat{\Delta}_\varphi$ and $\widehat{\nabla}_\varphi$ on a smooth test function.

Also, we note that if (v,p) is a generalized solution to (1.1), then the following identity holds in $\mathcal{D}'(Q_R^+)$ (i.e., in the sense of distributions):

$$\partial_t v = f + \operatorname{div}\Big(\nabla v - p\mathbb{I}\Big) + \frac{\partial}{\partial x_n}\Big(-2v_{,\alpha}\varphi_\alpha + v_{,n}|\nabla'\varphi|^2 - v\Delta'\varphi + p\begin{pmatrix}\nabla'\varphi\\ 0\end{pmatrix}\Big).$$

This identity implies that $\partial_t v \in L_l(-R^2,0;W_s^{-1}(B_R^+))$ and the estimate

$$\begin{aligned}&\|\partial_t v\|_{L_l(-R^2,0;W_s^{-1}(B_R^+))}\\ &\leqslant C\,\Big(\|f\|_{L_l(-R^2,0;W_s^{-1}(B_R^+))} + \|v\|_{W^{1,0}_{s,l}(Q_R^+)} + \|p\|_{L_{s,l}(Q_R^+)}\Big)\end{aligned} \tag{1.9}$$

holds. In particular, it is possible to choose a representative of v such that

$$\forall\ w \in \overset{\circ}{W}{}^1_{s'}(\Omega)\quad t \mapsto \int\limits_{B_R^+} v(x,t)\cdot w(x)\ dx \text{ is continuous on } [-R^2,0].$$

Hence we can assume that every generalized solution (v,p) satisfies the integral identity

$$\begin{aligned}&\int\limits_{B_R^+} v(x,t)\cdot\eta(x,t)\ dx\ \Big|_{t=-R^2}^{t=0} + \int\limits_{Q_R^+}(-v\cdot\partial_t\eta + \widehat{\nabla}_\varphi v : \widehat{\nabla}_\varphi\eta)\ dxdt\\ &\qquad = \int\limits_{-R^2}^{0}\langle f(t),\eta(t)\rangle\ dt + \int\limits_{Q_R^+} p\widehat{\nabla}_\varphi\cdot\eta\ dxdt\end{aligned} \tag{1.10}$$

for any $\eta \in C^\infty(\bar{Q}_R^+)$ such that $\eta|_{\partial B_R^+\times(-R^2,0)} = 0$.

In the paper, we use the following notation

- $\partial\Omega$ is a boundary of a domain $\Omega \subset \mathbb{R}^n$.
- $\partial' Q_R^+ = (\partial B_R^+ \times (-R^2,0)) \cup (B_R^+ \times \{t = -R^2\})$.
- We assume summation from 1 to n over repeated Latin indexes and summation from 1 to $n-1$ over repeated Greek indexes.
- The indexes after comma imply the derivatives with respect to the corresponding spatial variables.
- $a\cdot b = a_i b_i$ is the scalar product of vectors a, $b \in \mathbb{R}^n$.
- $A : B = A_{ij}B_{ij}$ is the scalar product of matrices A, $B \in \mathbb{M}^{n\times n}$.

§2. Main Results

In this section, we formulate four theorems which are the main results of the present paper. At the end of this section, we give some comments to these results.

Theorem 2.1. *Assume $\Omega \subset \mathbb{R}^n$ is a bounded domain with smooth boundary. Assume that Ω is diffeomorphic to a ball and denote $Q_T = \Omega \times (0,T)$. Suppose s, $l \in (1,\infty)$. There is a positive constant μ_1 (depending on Ω, T, s, l, n) such that for any function $\varphi \in W^3_\infty(\Omega)$ which is independent on x_n-variable and satisfies the condition*

$$\|\varphi\|_{W^3_\infty(\Omega)} \leqslant \mu_1 \tag{2.1}$$

and for any f and g satisfying conditions

$$f \in L_{s,l}(Q_T), \tag{2.2}$$

$$g \in W^{1,0}_{s,l}(Q_T), \tag{2.3}$$

$$\partial_t g \in L_{s,l}(Q_T), \tag{2.4}$$

$$\int\limits_\Omega g(x,t)\,dx = 0, \qquad \text{a.e. } t \in (0,T), \qquad g(\cdot,0) = 0, \tag{2.5}$$

the problem

$$\begin{cases} \partial_t u - \widehat{\Delta}_\varphi u + \widehat{\nabla}_\varphi q = f \\ \widehat{\nabla}_\varphi \cdot u = g \end{cases} \quad \text{in} \quad Q_T, \quad u|_{\partial\Omega\times(0,T)} = 0, \quad u|_{t=0} = 0, \tag{2.6}$$

has a unique solution $u \in W^{2,1}_{s,l}(Q_T)$, $q \in W^{1,0}_{s,l}(Q_T)$, $\int\limits_\Omega q(x,t)\,dx = 0$, for a.e. $t \in (0,T)$ and the estimate

$$\begin{aligned} &\|u\|_{W^{2,1}_{s,l}(Q_T)} + \|\nabla q\|_{L_{s,l}(Q_T)} \\ &\leqslant C_* \left(\|f\|_{L_{s,l}(Q_T)} + \|g\|_{W^{1,0}_{s,l}(Q_T)} + \|\partial_t g\|^{1/s}_{L_{s,l}(Q_T)} \|\partial_t g\|^{1/s'}_{L_l(0,T;W_s^{-1}(\Omega))} \right) \end{aligned} \tag{2.7}$$

holds with some constant $C_* > 0$ depending only on Ω, T, n, s, and l.

Theorem 2.2. *Assume that s, $l \in (1,\infty)$, and $0 < r < R$ are fixed. There exists a positive constant μ_2 (depending only on n, s, l, r, and R) such that if $\varphi \in W^3_\infty(S_R)$ satisfies (1.7) with $\mu \leqslant \mu_2$, then for any $f \in L_{s,l}(Q^+_R)$,*

and any strong solution $v \in W^{2,1}_{s,l}(Q^+_R)$, $p \in W^{1,0}_{s,l}(Q^+_R)$ *to the system* (1.1), (1.4) *in* Q^+_R, *the following local estimate holds:*

$$\begin{aligned} &\|v\|_{W^{2,1}_{s,l}(Q^+_r)} + \|\nabla p\|_{L_{s,l}(Q^+_r)} \\ &\leqslant C\left(\|f\|_{L_{s,l}(Q^+_R)} + \|\nabla v\|_{L_{s,l}(Q^+_R)} + \inf_{b\in L_l(-R^2,0)} \|p-b\|_{L_{s,l}(Q^+_R)}\right), \end{aligned} \tag{2.8}$$

where b *is a function of* t*-variable and the constant* C *depends only on* n, s, l, r, R.

Theorem 2.3. *Assume that* s, $l \in (1,\infty)$, *and* $0 < r < R$ *are fixed. There exists a positive constant* μ_3 *(depending only on* n, s, l, r, *and* R*) such that if* $\varphi \in W^3_\infty(S_R)$ *satisfies* (1.7) *with* $\mu \leqslant \mu_3$, *then for any* $f \in L_{s,l}(Q^+_R)$ *and any generalized solution* $v \in W^{1,0}_{s,l}(Q^+_R)$, $p \in L_{s,l}(Q^+_R)$ *to the system* (1.1), (1.4) *in* Q^+_R *the following inclusions hold:* $v \in W^{2,1}_{s,l}(Q^+_r)$, $p \in W^{1,0}_{s,l}(Q^+_r)$.

Theorem 2.4. *Assume that* s, l, $m \in (1,\infty)$, $m \geqslant s$ *and* $0 < r < R$ *are fixed. There exists a positive constant* μ_4 *(depending only on* n, s, l, r, m, R*) such that if* $\varphi \in W^3_\infty(S_R)$ *satisfies* (1.7) *with* $\mu \leqslant \mu_4$ *then for any* $f \in L_{m,l}(Q^+_R)$ *and any generalized solution* $v \in W^{2,1}_{s,l}(Q^+_R)$, $p \in W^{1,0}_{s,l}(Q^+_R)$ *to the system* (1.1), (1.4) *in* Q^+_R *we have the inclusions* $v \in W^{2,1}_{m,l}(Q^+_r)$, $\nabla p \in L_{m,l}(Q^+_r)$ *and the following local estimate holds:*

$$\begin{aligned} &\|v\|_{W^{2,1}_{m,l}(Q^+_r)} + \|\nabla p\|_{L_{m,l}(Q^+_r)} \\ &\leqslant C\left(\|f\|_{L_{m,l}(Q^+_R)} + \|\nabla v\|_{L_{s,l}(Q^+_R)} + \inf_{b\in L_l(-R^2,0)} \|p-b\|_{L_{s,l}(Q^+_R)}\right) \end{aligned} \tag{2.9}$$

with some constant C *depending only on* n, s, l, m, r, R.

Remark. The constants μ_i controlling the smallness of the W^3_∞-norm of the function φ in Theorems 2.2–2.4 depend on the domain (or on the size of the half-cylinders Q^+_r and Q^+_R). Nevertheless, for applications to the investigation of the Stokes and the Navier–Stokes systems near the point at the curved part of the boundary this is not a serious obstacle (in contrast to the smoothness assumption that φ is of class W^3_∞). This fact is explained by the following scaling property of the Perturbed Stokes system: if (v, p, f, φ) satisfy (1.1) in the cylinder Q^+_R with φ satisfying (1.7)

then the functions

$$\begin{aligned} &v^R(x,t) = Rv(Rx, R^2t), \quad p^R(x,t) = R^2p(Rx, R^2t), \\ &f^R(x,t) = R^3f(Rx, R^2t), \quad \text{and} \quad \varphi^R(x') = \frac{1}{R}\varphi(Rx') \end{aligned} \tag{2.10}$$

satisfy the Perturbed Stokes system in Q^+ and from Taylor decomposition of the function φ^R one can obtain for $R \leqslant 1$

$$\varphi^R(0) = 0, \qquad \nabla'\varphi^R(0) = 0, \qquad \|\varphi^R\|_{W^3_\infty(S_1)} \leqslant \mu R.$$

Hence, we can take a canonical domain (say, $Q^+_R = Q^+_1$, $Q^+_r = Q^+_{1/2}$) and compute the constants $\mu^*_i = \mu_i$ for these particular domains. We emphasize that μ^*_i are constants depending only on n, s, l, and m. Then, we consider the Stokes system (1.5), (1.6) near a point of the W^3_∞-smooth boundary without any restrictions on the curvature of the boundary (i.e., the constant μ in (1.7) can be arbitrary large). After that, we choose R in (1.5) so small that the following estimates hold:

$$\mu R \leqslant \mu^*_i, \qquad i = 2, 3, 4. \tag{2.11}$$

Making change of variables by means of (1.8), we obtain the functions (v, p, f, φ), which satisfy the Perturbed Stokes system (1.1), (1.4) in Q^+_R. At this step our Perturbed Stokes system is not a small perturbation of the usual Stokes system (i.e., so far smallness conditions of Theorems 2.2–2.4 are not satisfied). Then, we make the scaling (2.10) and obtain functions $(v^R, p^R, f^R, \varphi^R)$, which satisfy the Perturbed Stokes system in Q^+ and also satisfy the smallness assumptions (2.11). Thus, we can apply results of Theorems 2.2–2.4 to the functions $(v^R, p^R, f^R, \varphi^R)$ and recover information about the original functions (v, p).

It is worth appending some comments to Theorems 2.1–Theorem 2.4.

Theorem 2.1 in the case of the Stokes system (i.e. for $\varphi \equiv 0$) was proved in [2]. The generalization in the case of a "small perturbation" of the Stokes system is quite obvious. The proof is presented in Section 3.

Theorem 2.2 presents a local estimate for strong solutions to the Perturbed Stokes system. In the case of the usual Stokes system near a plane part of the boundary, such estimates were originally proved in [3]. In [7], the same estimates were proved for solutions to the Stokes system near curved part of the boundary. In our approach, Theorem 2.2 follows from Theorem 2.1 by arguments presented in [2]. We reproduce these arguments in Section 4 just for the sake of completeness.

In Theorem 2.3, we prove that any generalized solution is actually a strong one. In the case of the Stokes system, this result originally was proved in [5]. In Section 5 we deduce a similar result for the Perturbed Stokes system with the help of a new approach based on the estimates obtained in Theorem 2.1. Probably this section contains main novelty of the present paper.

Finally, in Section 6 we obtain improved local estimate of solution to the Perturbed Stokes system. The estimate in Theorem 2.4 turns out to be the crucial step in investigation of boundary regularity of solutions to the nonlinear Navier–Stokes system, see [4] and [6]. For the Stokes system this estimate was originally obtained in [3] in the case of plane boundary, and after that in [7] in the case of curved boundary. In our approach, we obtain the corresponding estimate for solutions to the Perturbed Stokes system (under certain conditions that guarantee smallness of the "perturbation") as a direct consequence of our Theorems 2.3 and 2.2.

§3. Proof of Theorem 2.1

We derive Theorem 2.1 with help of the following result.

Theorem 3.1. *Assume that s, $l \in (1,\infty)$, a_{ijkl}, b_{ijk}, c_{ij}, $d_{,ij} \in L_\infty(Q_T)$ and consider the problem*

$$\begin{cases} \partial_t w_i - a_{ijkm} w_{j,km} + b_{ijk} w_{j,k} + c_{ij} w_j + d_{ij} q_{,j} = f_i, & \text{in } Q_T, \\ \operatorname{div} w = 0, \\ w|_{t=0} = 0, \qquad w|_{\partial\Omega\times(0,T)} = 0. \end{cases} \tag{3.1}$$

There exists a constant $\mu_0 > 0$ (depending on Ω, T, s, l, n) such that if the coefficients a_{ijkl}, b_{ijk}, c_{ij}, d_{ij} satisfy the estimate

$$\sup_{z\in\bar{Q}_T} \Big(|a_{ijkm}(z) - \delta_{ij}\delta_{km}| + |d_{ij}(z) - \delta_{ij}| + |b_{ijk}(z)| + |c_{ij}(z)| \Big) \leqslant \mu_0, \tag{3.2}$$

for all $i,j,k,m = 1,\dots,n$, then for any f satisfying conditions (2.2) the problem (3.1) has the unique solution $w \in W^{2,1}_{s,l}(Q_T)$, $q \in W^{1,0}_{s,l}(Q_T)$, $\int\limits_\Omega q\,dx = 0$ a.e. $t \in \Omega$, and the estimate

$$\|w\|_{W^{2,1}_{s,l}(Q_T)} + \|q\|_{W^{1,0}_{s,l}(Q_T)} \leqslant C\, \|f\|_{L_{s,l}(Q_T)} \tag{3.3}$$

holds with some constant $C > 0$ depending only on Ω, T, n, s, l.

Proof of Theorem 3.1. Let

$$\mathcal{H} := \Big\{ (w,q) \in W^{2,1}_{s,l}(Q_T) \times W^{1,0}_{s,l}(Q_T) : \ \operatorname{div} w = 0 \text{ a.e. in } Q_T, \\ w|_{\partial\Omega\times(0,T)} = 0, \ w|_{t=0} = 0, \ \int\limits_\Omega q \ dx = 0 \text{ a.e. } t \in (0,T) \Big\}$$

denote the Banach space equipped with the norm

$$\|(w,q)\|_{\mathcal{H}} \ := \ \|w\|_{W^{2,1}_{s,l}(Q_T)} + \|q\|_{W^{1,0}_{s,l}(Q_T)}.$$

For any $f \in L_{s,l}(Q_T)$ denote by $(w,q) \in \mathcal{H}$ the unique strong solution to the Stokes system:

$$\left\{ \begin{array}{c} \partial_t w - \Delta w + \nabla q = f, \\ \operatorname{div} w = 0, \\ w|_{t=0} = 0, \qquad w|_{\partial\Omega\times(0,T)} = 0, \end{array} \right. \tag{3.4}$$

and consider the bijective operator

$$\mathcal{A}_0 : \mathcal{H} \to L_{s,l}(Q_T), \qquad \mathcal{A}_0(w,q) := f.$$

Then, we know (see [8], Theorem 1.1) that there is a positive constant C_* such that

$$C_* \, \|(w,q)\|_{\mathcal{H}} \ \leqslant \ \|\mathcal{A}_0(w,q)\|_{L_{s,l}(Q_T)}$$

for any $(w,q) \in \mathcal{H}$. Hence, the linear operator $\mathcal{A}_0$ is invertible and its inverse operator is bounded from $L_{s,l}(Q_T)$ to $\mathcal{H}$.

Consider the operator

$$\mathcal{A}_1 : \mathcal{H} \to L_{s,l}(Q_T)$$

determined by the system (3.1). The system (3.1) can be reduced to the system (3.4) with the right-hand side $\widetilde{f}$, where

$$\widetilde{f}_i = f_i + (a_{ijkl} - \delta_{ij}\delta_{kl}) w_{j,kl} - b_{ijk} w_{j,k} - c_{ij} w_j - (d_{ij} - \delta_{ij}) q_{,j},$$

and due to conditions (3.2)

$$\|\widetilde{f} - f\|_{L_{s,l}(Q_T)} \ \leqslant C\mu_0 \|(w,q)\|_{\mathcal{H}}, \tag{3.5}$$

Then for every $f \in L_{s,l}(Q_T)$ we have

$$\|(\mathcal{A}_1 - \mathcal{A}_0)\mathcal{A}_0^{-1} f\|_{L_{s,l}(Q_T)} = \|\widetilde{f} - f\|_{L_{s,l}(Q_T)} \\ \leqslant C\mu_0 \|(w,q)\|_{\mathcal{H}} \leqslant \frac{C\mu_0}{C_*} \|f\|_{L_{s,l}(Q_T)}.$$

Choosing $\mu_0 < \frac{C_*}{2C}$ we obtain

$$\|(\mathcal{A}_1 - \mathcal{A}_0)\mathcal{A}_0^{-1}\|_{L_{s,l}(Q_T)\to L_{s,l}(Q_T)} \leqslant \frac{1}{2}$$

and hence there exists $\mathcal{A}_1^{-1} : L_{s,l}(Q_T) \to \mathcal{H}$ which is a bounded operator. Theorem 3.1 is proved. □

Proof of Theorem 2.1. Let $\mathcal{L}_\varphi = \nabla\psi$, where ψ is introduced in (1.8). Note that $\mathcal{L}_\varphi$ is a smooth matrix and it is non-degenerate. Denote $w := \mathcal{L}_\varphi u$. Then

$$\widehat{\nabla}_\varphi \cdot u \;=\; \operatorname{div} w \quad \text{a.e. in} \quad Q_T$$

and the system (2.6) can be reduced to the form

$$\left\{\begin{array}{rcl} \partial_t w - \mathcal{L}_\varphi \widehat{\Delta}_\varphi(\mathcal{L}_\varphi^{-1} w) + \mathcal{L}_\varphi \widehat{\nabla}_\varphi q &=& \mathcal{L}_\varphi f \\ \operatorname{div} w &=& g \\ w|_{t=0} = 0, \qquad w|_{\partial\Omega\times(0,T)} &=& 0. \end{array}\right. \quad \text{in} \quad Q_T, \tag{3.6}$$

Note that this is a system of type (3.1) but with non-zero divergence. The coefficients a_{ijkm}, b_{ijk}, c_{ij}, d_{ij} arising in this system depend on the derivatives of φ of the first, second and third orders, due to the condition (2.1) they are bounded and satisfy the conditions (3.2).

Using the result of paper [2] (see section 4, estimate (4.1) in the cited paper) we can find a function $W \in W^{2,1}_{s,l}(Q_T)$ such that

$$\begin{gathered} \operatorname{div} W \;=\; g \quad \text{a.e. in} \quad Q_T, \\ W|_{\partial\Omega\times(0,T)} = 0, \qquad W|_{t=0} = 0, \\ \|W\|_{W^{2,1}_{s,l}(Q_T)} \;\leqslant\; C\Big(\|g\|_{W^{1,0}_{s,l}(Q_T)} + \|\partial_t g\|^{1/s}_{L_{s,l}(Q_T)} \|\partial_t g\|^{1/s'}_{L_l(0,T;W_s^{-1}(\Omega))}\Big). \end{gathered}$$

Then we consider the problem

$$\left\{\begin{array}{rcl} \partial_t \widetilde{w} - \mathcal{L}_\varphi \widehat{\Delta}_\varphi(\mathcal{L}_\varphi^{-1} \widetilde{w}) + \mathcal{L}_\varphi \widehat{\nabla}_\varphi q &=& \widetilde{f} \\ \operatorname{div} \widetilde{w} &=& 0 \\ \widetilde{w}|_{t=0} = 0, \qquad \widetilde{w}|_{\partial\Omega\times(0,T)} &=& 0, \\ \widetilde{f} \;:=\; \mathcal{L}_\varphi f - \Big(\partial_t W - \mathcal{L}_\varphi \widehat{\Delta}_\varphi(\mathcal{L}_\varphi^{-1} W)\Big) &\in& L_{s,l}(Q_T), \end{array}\right. \quad \text{in} \quad Q_T,$$

which has the unique solution $(\widetilde{w}, q) \in W^{2,1}_{s,l}(Q_T) \times W^{1,0}_{s,l}(Q_T)$ due to Theorem 3.1. Now we take $w := \widetilde{w} + W$ and see that $(w, q) \in W^{2,1}_{s,l}(Q_T) \times$

$W^{1,0}_{s,l}(Q_T)$ satisfy all equations in (3.6). The uniqueness of this solution follows from Theorem 3.1, and the estimate

$$\|w\|_{W^{2,1}_{s,l}(Q_T)} + \|q\|_{W^{1,0}_{s,l}(Q_T)}$$
$$\leqslant C\Big(\|f\|_{L_{s,l}(Q_T)} + \|g\|_{W^{1,0}_{s,l}(Q_T)} + \|\partial_t g\|^{1/s}_{L_{s,l}(Q_T)}\|\partial_t g\|^{1/s'}_{L_l(0,T;W_s^{-1}(\Omega))}\Big)$$

follows from the corresponding estimates of $\widetilde{w}$ and W. From this estimate taking into account $u = \mathcal{L}_\varphi^{-1} w$ and W^3_∞-smoothness of φ we obtain (2.7). Note that only here we need W^3_∞-smoothness for the function φ. Theorem 2.1 is proved. □

§4. Proof of Theorem 2.2

The estimate (2.8) follows from (2.7) by the arguments used in [2]. We reproduce this proof here just for the sake of completeness. Within this section C denotes different positive constants which depend only on n, r, R, s, l.

Take arbitrary ρ_1, ρ_2 such that

$$r \leqslant \rho_1 < \rho_2 \leqslant R - \frac{1}{10}(R-r).$$

Consider a cut-off function $\zeta \in C_0^\infty(Q_R^+)$ such that

$$0 \leqslant \zeta \leqslant 1 \ \text{ in } \ Q_R^+, \quad \zeta \equiv 1 \ \text{ in } \ Q_{\rho_1}^+, \quad \zeta \equiv 0 \ \text{ in } \ Q_R^+ \setminus Q_{\rho_2}^+,$$
$$\|\nabla^k \zeta\|_{L_\infty(Q_R^+)} \leqslant \frac{C}{(\rho_2-\rho_1)^k}, \quad k = 1, 2,$$
$$\|\partial_t \zeta\|_{L_\infty(Q_R^+)} \leqslant \frac{C}{\rho_2-\rho_1}, \quad \|\partial_t \nabla \zeta\|_{L_\infty(Q_R^+)} \leqslant \frac{C}{(\rho_2-\rho_1)^2}.$$

Let (v, p) be a solution to the system (1.1), (1.4). Fix arbitrary function $b \in L_l(-R^2, 0)$ of t-variable and denote $\bar{p} := p - b$. Let Ω be a smooth domain such that $B^+_{R-\frac{1}{10}(R-r)} \subset \Omega \subset B^+_R$. Consider functions $u := \zeta v$, $q := \zeta \bar{p}$. Then (u, q) is a solution to the initial-boundary problem of type (2.6), but in domain $\Omega \times (-R^2, 0)$ instead of $\Omega \times (0, T)$ and with "right hand sides" f, g in (2.6) equal to $\widetilde{f}$, $\widetilde{g}$, where

$$\widetilde{f} = \zeta f + v(\partial_t \zeta - \widehat{\Delta}_\varphi \zeta) - 2(\widehat{\nabla}_\varphi v)\widehat{\nabla}_\varphi \zeta + \bar{p}\widehat{\nabla}_\varphi \zeta, \quad \widetilde{g} = v \cdot \widehat{\nabla}_\varphi \zeta.$$

Applying the estimate (2.7) to the functions $(u, q, \widetilde{f}, \widetilde{g})$ and taking into account that $\zeta \equiv 1$ on $Q^+_{\rho_1}$, $\frac{1}{\rho_2-\rho_1} \geqslant C$, we obtain

$$\|v\|^s_{W^{2,1}_{s,l}(Q^+_{\rho_1})} \leqslant C\|f\|^s_{L_{s,l}(Q^+_R)} + \frac{C}{(\rho_2-\rho_1)^{2s}}\Big(\|v\|^s_{W^{1,0}_{s,l}(Q^+_R)} + \|\bar{p}\|^s_{L_{s,l}(Q^+_R)}\Big)$$
$$+ C\Big(\|\nabla(v\cdot\widehat{\nabla}_\varphi\zeta)\|^s_{L_{s,l}(Q^+_R)}$$
$$+ \|\partial_t(v\cdot\widehat{\nabla}_\varphi\zeta)\|_{L_{s,l}(Q^+_R)}\|\partial_t(v\cdot\widehat{\nabla}_\varphi\zeta)\|^{s-1}_{L_l(-R^2,0;W_s^{-1}(B^+_R))}\Big).$$

Taking into account the estimates

$$\|\nabla(v\cdot\widehat{\nabla}_\varphi\zeta)\|^s_{L_{s,l}(Q^+_R)} \leqslant \frac{C}{(\rho_2-\rho_1)^{2s}}\|v\|^s_{W^{1,0}_{s,l}(Q^+_R)},$$
$$\|\partial_t(v\cdot\widehat{\nabla}_\varphi\zeta)\|_{L_{s,l}(Q^+_R)} \leqslant \frac{C}{(\rho_2-\rho_1)^2}\Big(\|\partial_t v\|_{L_{s,l}(Q^+_{\rho_2})} + \|v\|_{L_{s,l}(Q^+_R)}\Big),$$
$$\|\partial_t(v\cdot\widehat{\nabla}_\varphi\zeta)\|^{s-1}_{L_l(-R^2,0;W_s^{-1}(B^+_R))}$$
$$\leqslant \frac{C}{(\rho_2-\rho_1)^{2s-2}}\Big(\|\partial_t v\|^{s-1}_{L_l(-R^2,0;W_s^{-1}(B^+_R))} + \|v\|^{s-1}_{L_{s,l}(Q^+_R)}\Big),$$

we find that

$$\|v\|^s_{W^{2,1}_{s,l}(Q^+_{\rho_1})} \leqslant C\|f\|^s_{L_{s,l}(Q^+_R)}$$
$$+ \frac{C}{(\rho_2-\rho_1)^{2s}}\Big(\|v\|^s_{W^{1,0}_{s,l}(Q^+_R)} + \|\bar{p}\|^s_{L_{s,l}(Q^+_R)} + \|\partial_t v\|^s_{L_l(-R^2,0;W_s^{-1}(B^+_R))}\Big)$$
$$+ \frac{C}{(\rho_2-\rho_1)^{2s}}\|\partial_t v\|_{L_{s,l}(Q^+_{\rho_2})}\Big(\|\partial_t v\|^{s-1}_{L_l(-R^2,0;W_s^{-1}(B^+_R))} + \|v\|^{s-1}_{L_{s,l}(Q^+_R)}\Big). \tag{4.1}$$

Estimating the last term in the right-hand side of (4.1) via the Young inequality $ab \leqslant \varepsilon a^s + C_\varepsilon b^{s'}$ we obtain

$$\frac{C}{(\rho_2-\rho_1)^{2s}}\|\partial_t v\|_{L_{s,l}(Q^+_{\rho_2})}\Big(\|\partial_t v\|^{s-1}_{L_l(-R^2,0;W_s^{-1}(B^+_R))} + \|v\|^{s-1}_{L_{s,l}(Q^+)}\Big)$$
$$\leqslant \varepsilon\|\partial_t v\|^s_{L_{s,l}(Q^+_{\rho_2})} + \frac{C_\varepsilon}{(\rho_2-\rho_1)^{2ss'}}\Big(\|\partial_t v\|^s_{L_l(-R^2,0;W_s^{-1}(B^+_R))} + \|v\|^s_{L_{s,l}(Q^+_R)}\Big),$$

where the constant $\varepsilon > 0$ can be chosen arbitrary small. Therefore,

$$\|v\|^s_{W^{2,1}_{s,l}(Q^+_{\rho_1})} \leqslant C\|f\|^s_{L_{s,l}(Q^+_R)} + \varepsilon\|\partial_t v\|^s_{L_{s,l}(Q^+_{\rho_2})}$$
$$+\frac{C_\varepsilon}{(\rho_2-\rho_1)^{2ss'}}\Big(\|v\|^s_{W^{1,0}_{s,l}(Q^+_R)} + \|\bar{p}\|^s_{L_{s,l}(Q^+_R)} + \|\partial_t v\|^s_{L_l(-R^2,0;W_s^{-1}(B^+_R))}\Big),$$

and by virtue of (1.9) we arrive at the estimate

$$\begin{aligned}\|v\|^s_{W^{2,1}_{s,l}(Q^+_{\rho_1})} &\leqslant \varepsilon \|\partial_t v\|^s_{L_{s,l}(Q^+_{\rho_2})} \\ &+ \frac{C_\varepsilon}{(\rho_2-\rho_1)^{2ss'}}\Big(\|f\|^s_{L_{s,l}(Q^+_R)} + \|v\|^s_{W^{1,0}_{s,l}(Q^+_R)} + \|\bar{p}\|^s_{L_{s,l}(Q^+_R)}\Big).\end{aligned} \tag{4.2}$$

Let us introduce the monotone function $\Psi(\rho) := \|v\|^s_{W^{2,1}_{s,l}(Q^+_\rho)}$, and the constant

$$A := C_\varepsilon \left(\|f\|^s_{L_{s,l}(Q^+_R)} + \|v\|^s_{W^{1,0}_{s,l}(Q^+_R)} + \|\bar{p}\|^s_{L_{s,l}(Q^+_R)}\right).$$

The inequality (4.2) implies that

$$\Psi(\rho_1) \leqslant \varepsilon \Psi(\rho_2) + \tfrac{A}{(\rho_2-\rho_1)^\alpha}, \quad \forall\ \rho_1,\ \rho_2: \quad R_1 \leqslant \rho_1 < \rho_2 \leqslant R_0, \tag{4.3}$$

for $\alpha = 2ss'$ and for $R_1 = r$, $R_0 = R - \frac{1}{10}(R-r)$. Now we shall take an advantage of the following lemma (which can be easily proved by iterations if one takes $\rho_k := R_0 - 2^{-k}(R_0 - R_1)$):

Lemma 4.1. *Assume Ψ is a nondecreasing bounded function which satisfies the inequality* (4.3) *for some $\alpha > 0$, $A > 0$, and $\varepsilon \in (0, 2^{-\alpha})$. Then there exists a constant B depending only on ε and α such that*

$$\Psi(R_1) \leqslant \frac{B\,A}{(R_0 - R_1)^\alpha}.$$

Fixing $\varepsilon = 2^{-4ss'}$ in (4.2) and applying Lemma 4.1 to our function Ψ, we obtain the estimate

$$\|v\|_{W^{2,1}_{s,l}(Q^+_r)} \leqslant C_*\Big(\|f\|^s_{L_{s,l}(Q^+_R)} + \|v\|^s_{W^{1,0}_{s,l}(Q^+_R)} + \|\bar{p}\|^s_{L_{s,l}(Q^+_R)}\Big).$$

Then from (1.1) we obtain that $\widehat{\nabla}_\varphi p \in L_{s,l}(Q^+_r)$. Taking into account (1.2) and $\|\varphi\|_{W^3_\infty(S_R)} \leqslant \mu_2$ we get

$$\nabla p \in L_{s,l}(Q^+_r), \quad \|\nabla p\|_{L_{s,l}(Q^+_r)} \ \leqslant\ c\Big(\|v\|_{W^{2,1}_{s,l}(Q^+_r)} + \|f\|_{L_{s,l}(Q^+_r)}\Big).$$

Theorem 2.2 is proved. □

§5. Proof of Theorem 2.3

For the of convenience, in subsequent presentation we fix $R = 1$ and $r = \frac{1}{2}$. The extension of our proof to the case of general $0 < r < R$ is straightforward.

Let $\rho_m \to +0$ be an arbitrary sequence. Extend all functions v, p, f from Q^+ to the set $B^+ \times \mathbb{R}$ by zero. For any extended function v denote by v^m the mollification of the function v with respect to t variable:

$$v^m(x,t) := (\omega_{\rho_m} * v)(x,t) \equiv \int\limits_{\mathbb{R}} \omega_{\rho_m}(t-\tau) v(x,\tau)\, d\tau,$$

where $\omega_\rho(t) = \frac{1}{\rho}\omega(t/\rho)$, and $\omega \in C_0^\infty(-1,1)$ is a smooth kernel normalized by the identity $\int\limits_0^1 \omega(t)dt = 1$.

As $v \in W^{1,0}_{s,l}(Q^+)$, $p \in L_{s,l}(Q^+)$, $f \in L_{s,l}(Q^+)$ we have

$$\begin{gathered} v^m \to v \quad \text{in } W^{1,0}_{s,l}(Q^+), \quad p^m \to p \quad \text{in } L_{s,l}(Q^+), \\ f^m \to f \quad \text{in } L_{s,l}(Q^+). \end{gathered} \tag{5.1}$$

Let us fix arbitrary $\delta \in (0, \frac{1}{12})$. Then for any $\rho_m < \delta$ and any $\eta \in C^\infty(\bar{Q}^+)$

$$\partial_t(\omega_{\rho_m} * \eta)(x,t) = (\omega_{\rho_m} * \partial_t\eta)(x,t), \quad \forall\, x \in B^+,\ t \in (-1+\delta, -\delta).$$

Let us set $\eta = \omega_{\rho_m} * \widetilde{\eta}$ in (1.10), where $\widetilde{\eta} \in C^\infty(\bar{Q}^+)$ is an arbitrary function vanishing on $\partial B^+ \times (-1,0)$ and on $B^+ \times (-1, -1+\delta)$ and $B^+ \times (-\delta, 0)$. Using the property of convolution

$$\int\limits_{Q^+} g \cdot (\omega_{\rho_m} * h)\ dxdt \ = \ \int\limits_{Q^+} (\omega_{\rho_m} * g) \cdot h\ dxdt,$$

$$\forall\ \rho_m < \delta,\ g \in L_1(Q^+),\ h \in C^\infty(\bar{Q}^+) :\ \operatorname{supp} h \subset \bar{B}^+ \times [-1+\delta, -\delta],$$

and taking into account the fact that convolution with respect to t commutes with the differential operators Δ_φ, ∇_φ, we obtain the identity

$$-\int\limits_{Q^+} v^m \cdot (\partial_t\widetilde{\eta} + \widehat{\Delta}_\varphi\widetilde{\eta})\ dxdt = \int\limits_{Q^+} (f^m \cdot \widetilde{\eta} + p^m \widehat{\nabla}_\varphi \cdot \widetilde{\eta})\ dxdt \tag{5.2}$$

which holds for all $\widetilde{\eta} \in C^\infty(\bar{Q}^+)$ such that $\widetilde{\eta}|_{x_n=0} = 0$ and $\widetilde{\eta}$ vanishes on $B^+ \times (-1, -1+\delta)$, $B^+ \times (-\delta, 0)$, and near the set $\partial' B^+ \times (-1, 0)$, where $\partial' B^+ := \{x \in \mathbb{R}^n : |x| = R, x_n > 0\}$.

Let $\zeta \in C^\infty(\bar{Q}^+)$ be a cut-of function vanishing in $Q^+ \setminus Q^+_{5/6}$ and such that $\zeta \equiv 1$ in $Q^+_{2/3}$. Denote $u^m := \zeta v^m$, $q^m := \zeta p^m$. Then from (5.2) we obtain that (u^m, q^m) satisfy the integral identity

$$-\int\limits_{B^+\times(-1,-\delta)} u^m \cdot (\partial_t \eta + \widehat{\Delta}_\varphi \eta)\, dxdt = \int\limits_{B^+\times(-1,-\delta)} (f_0^m \cdot \eta + q^m \widehat{\nabla}_\varphi \cdot \eta)\, dxdt$$

for any $\eta \in C^\infty(\bar{B}^+ \times [-1,-\delta])$ such that $\eta|_{\partial B^+\times(-1,-\delta)} = 0$ and $\eta|_{B^+\times\{t=-\delta\}} = 0$. Here by f_0^m we denote the expression

$$f_0^m = f^m\zeta - v^m\partial_t\zeta + v^m\widehat{\Delta}_\varphi\zeta - 2\widehat{\nabla}_\varphi v^m \widehat{\nabla}_\varphi\zeta - p^m\widehat{\nabla}_\varphi\zeta. \tag{5.3}$$

Moreover, u^m also satisfies the identity

$$\widehat{\nabla}_\varphi \cdot u^m = g^m \quad \text{a.e. in} \quad Q^+,$$

where we denote

$$g^m = v^m \cdot \widehat{\nabla}_\varphi \zeta.$$

Assume that $\Omega \subset \mathbb{R}^3$ is a smooth domain such that $B^+_{5/6} \subset \Omega \subset B^+$ and denote $\widetilde{Q}^+ := \Omega \times (-1,0)$. As v^m is smooth with respect to t variable for each fixed $m \in \mathbb{N}$ the functions f^m, g^m possess the properties

$$f_0^m \in L_{s,l}(\widetilde{Q}^+),\ g^m \in W^{1,0}_{s,l}(\widetilde{Q}^+),\ \partial_t g^m \in L_{s,l}(\widetilde{Q}^+),\ \int\limits_\Omega g^m(x,t)\, dx = 0.$$

From Theorem 2.1, we obtain that for any $m \in \mathbb{N}$ there exists a strong solution $\widetilde{u}^m \in W^{2,1}_{s,l}(\widetilde{Q}^+)$, $\widetilde{q}^m \in W^{1,0}_{s,l}(\widetilde{Q}^+)$ to the problem

$$\begin{cases} \partial_t \widetilde{u}^m - \widehat{\Delta}_\varphi \widetilde{u}^m + \widehat{\nabla}_\varphi \widetilde{q}^m = f_0^m \\ \widehat{\nabla}_\varphi \cdot \widetilde{u}^m = g^m \\ \widetilde{u}^m|_{\partial'\widetilde{Q}^+} = 0. \end{cases} \quad \text{in } \widetilde{Q}^+, \tag{5.4}$$

Note that as $\zeta \equiv 1$ in $Q^+_{2/3}$, we have the identity $g^m \equiv 0$ in $Q^+_{2/3}$. So, functions $(\widetilde{u}^m, \widetilde{q}^m)$ satisfy all assumptions of Theorem 2.2 in $Q^+_{2/3}$ and hence by Theorem 2.2 with $r = \frac{1}{2}$, $R = \frac{2}{3}$ we obtain the estimate

$$\begin{aligned} &\|\widetilde{u}^m\|_{W^{2,1}_{s,l}(Q^+_{1/2})} + \|\nabla\widetilde{q}^m\|_{L_{s,l}(Q^+_{1/2})} \\ &\leqslant C\ \left(\|f_0^m\|_{L_{s,l}(Q^+_{2/3})} + \|\widetilde{u}^m\|_{W^{1,0}_{s,l}(Q^+_{2/3})} + \|\widetilde{q}^m - b\|_{L_{s,l}(Q^+_{2/3})}\right), \end{aligned} \tag{5.5}$$

where the constant C does not depend neither on m nor on δ and $b \in L_l(-\frac{4}{9}, 0)$ is arbitrary.

As every strong solution of the Perturbed Stokes system is a generalized one, from (1.10) we obtain that $(\widetilde{u}^m, \widetilde{q}^m)$ satisfy the integral identity

$$-\int\limits_{\widetilde{Q}^+} \widetilde{u}^m \cdot (\partial_t \eta + \widehat{\Delta}_\varphi \eta)\ dxdt = \int\limits_{\widetilde{Q}^+} (f_0^m \cdot \eta + \widetilde{q}^m \widehat{\nabla}_\varphi \cdot \eta)\ dxdt$$

for all $\eta \in C^\infty(\overline{\widetilde{Q}^+})$ such that $\eta|_{\partial\Omega\times(-1,0)} = 0$ and $\eta|_{\Omega\times\{t=0\}} = 0$. Hence the differences $w^m := u^m - \widetilde{u}^m$, $\pi^m := q^m - \widetilde{q}^m$ are a generalized solution to the Perturbed Stokes system (1.1) in $\Omega\times(-1,-\delta)$ satisfying the integral identity

$$\begin{gathered} -\int\limits_{\Omega\times(-1,-\delta)} w^m \cdot (\partial_t \eta + \widehat{\Delta}_\varphi \eta)\ dxdt\ =\ \int\limits_{\Omega\times(-1,-\delta)} \pi^m \widehat{\nabla}_\varphi \cdot \eta\ dxdt, \\ \widehat{\nabla}_\varphi \cdot w^m = 0 \quad \text{a.e. in} \quad \Omega\times(-1,-\delta) \end{gathered} \tag{5.6}$$

for any $\eta \in W^{2,1}_{s',l'}(\Omega\times(-1,-\delta))$ such that $\eta|_{\partial\Omega\times(-1,-\delta)} = 0$ and $\eta|_{\Omega\times\{t=-\delta\}} = 0$. Denote $\varkappa = \min\{s,l\} > 1$. As u^m, $\widetilde{u}^m \in L_{s,l}(\widetilde{Q}^+)$ and q^m, $\widetilde{q}^m \in L_{s,l}(\widetilde{Q}^+)$ we have $w^m = u^m - \widetilde{u}^m \in L_\varkappa(\widetilde{Q}^+)$ and $\pi^m = q^m - \widetilde{q}^m \in L_\varkappa(\widetilde{Q}^+)$. Hence $|w^m|^{\varkappa-2}w^m \in L_{\varkappa'}(\widetilde{Q}^+)$, and using Theorem 2.1 we can find functions $\eta \in W^{2,1}_{\varkappa'}(\Omega\times(-1,-\delta))$ and $\kappa \in W^{1,0}_{\varkappa'}(\Omega\times(-1,-\delta))$ such that

$$\begin{cases} \partial_t \eta + \widehat{\Delta}_\varphi \eta + \widehat{\nabla}_\varphi \kappa = |w^m|^{\varkappa-2} w^m, \\ \widehat{\nabla}_\varphi \cdot \eta = 0, \\ \eta|_{\partial\Omega\times(-1,\delta)} = 0, \qquad \eta|_{t=-\delta} = 0. \end{cases} \quad \text{in} \quad \Omega\times(-1,-\delta),$$

Substituting this η as a test function into the identity (5.6) we obtain $w^m = 0$ in $\Omega\times(-1,-\delta)$. Hence $u^m = \widetilde{u}^m \in W^{2,1}_{s,l}(\Omega\times(-1,-\delta))$. Hence, from (5.6) we obtain

$$\int\limits_{\Omega\times(-1,-\delta)} \pi^m \widehat{\nabla}_\varphi \cdot \eta\ dxdt = 0, \quad \forall\ \eta \in L_{l'}((-1,-\delta); \overset{\circ}{W}{}^1_{s'}(\Omega)). \tag{5.7}$$

Correcting, if necessary, function $\widetilde{q}^m$ by a constant, we can assume that $\int\limits_\Omega \pi^m\ dx = 0$ for a.e. $t \in (-1,-\delta)$. As $\pi_m \in L_\varkappa(\Omega)$ for a.e. $t \in (-1,-\delta)$, we have $|\pi_m|^{\varkappa-2}\pi^m \in L_{\varkappa'}(\Omega)$ for a.e. $t \in (-1,-\delta)$. Taking into account the identity $\widehat{\nabla}_\varphi \cdot \eta = \operatorname{div}(\mathcal{L}_\varphi \eta)$ where $\mathcal{L}_\varphi$ is smooth invertible matrix and

using results of [1] for a.e. t we can find $\eta(\cdot,t)\in \overset{\circ}{W}{}^1_{\varkappa'}(\Omega)$ such that

$$\begin{cases} \operatorname{div}(\mathcal{L}_\varphi \eta) = |\pi_m|^{\varkappa-2}\pi^m - (|\pi_m|^{\varkappa-2}\pi^m)_\Omega, \quad \text{a.e. } t\in(-1,-\delta), \\ \|\eta\|_{W^1_{\varkappa'}(\Omega)} \leqslant C\|\pi^m\|^{\varkappa-1}_{L_\varkappa(\Omega)}. \end{cases}$$

From the last estimate, we see that

$$\eta \in L_{\varkappa'}((-1,-\delta);\overset{\circ}{W}{}^1_{\varkappa'}(\Omega)) \subset L_{l'}((-1,-\delta);\overset{\circ}{W}{}^1_{s'}(\Omega)).$$

Substituting this η into the identity (5.7), we obtain $\pi^m = 0$. This implies $q^m = \widetilde{q}^m + const$ and we obtain the inclusion $q^m \in W^{1,0}_{s,l}(\Omega\times(-1,-\delta))$. Moreover, from (5.5) we obtain

$$\begin{aligned} &\|u^m\|_{W^{2,1}_{s,l}(B^+_{1/2}\times(-\frac14,-\delta))} + \|\nabla q^m\|_{L_{s,l}(B^+_{1/2}\times(-\frac14,-\delta))} \\ &\leqslant C\ \left(\|f^m_0\|_{L_{s,l}(Q^+_{2/3})} + \|u^m\|_{W^{1,0}_{s,l}(Q^+_{2/3})} + \|q^m - b\|_{L_{s,l}(Q^+_{2/3})}\right), \end{aligned}$$

where C is independent on m and δ. Using identities $u^m = \zeta v^m$, $q^m = \zeta p^m$, $\zeta\equiv 1$ on $Q_{2/3}$ and the expression (5.3) for f^0_m we arrive at the estimate

$$\begin{aligned} &\|v^m\|_{W^{2,1}_{s,l}(B^+_{1/2}\times(-\frac14,-\delta))} + \|\nabla p^m\|_{L_{s,l}(B^+_{1/2}\times(-\frac14,-\delta))} \\ &\leqslant C\ \left(\|f^m\|_{L_{s,l}(Q^+_{2/3})} + \|v^m\|_{W^{1,0}_{s,l}(Q^+_{2/3})} + \|p^m\|_{L_{s,l}(Q^+_{2/3})}\right). \end{aligned}$$

Making use of (5.1) we obtain

$$v\in W^{2,1}_{s,l}\left(B^+_{1/2}\times(-\tfrac14,-\delta)\right), \quad p\in W^{1,0}_{s,l}\left(B^+_{1/2}\times(-\tfrac14,-\delta)\right),$$

and the estimate

$$\begin{aligned} &\|v\|_{W^{2,1}_{s,l}(B^+_{1/2}\times(-\frac14,-\delta))} + \|\nabla p\|_{L_{s,l}(B^+_{1/2}\times(-\frac14,-\delta))} \\ &\leqslant C\ \left(\|f\|_{L_{s,l}(Q^+_{2/3})} + \|v\|_{W^{1,0}_{s,l}(Q^+_{2/3})} + \|p\|_{L_{s,l}(Q^+_{2/3})}\right) \end{aligned}$$

holds for any $\delta\in(0,\frac{1}{12})$ with C independent on δ. The last inequality provides the required properties of (v,p). Theorem 2.3 is proved. □

§6. Proof of Theorem 2.4

As usually, for the presentation convenience we fix $R=1$ and $r=\frac12$. For any $k=0,1,\dots$ denote $s_k = \frac{ns}{n-ks}$ if $n>ks$ and $\frac{ns}{n-ks} < m$ and $s_k = m$ otherwise. Denote also $N = \min\{k\in\mathbb{N} : s_k = m\}$ and $\rho_k = \frac12 + \frac{1}{2^{k+1}}$.

Using Theorem 2.2 and Theorem 2.3 we see that if

$$(v,p)\in W^{1,0}_{s_k,l}(Q^+_{\rho_k})\times L_{s_k,l}(Q^+_{\rho_k})$$

is a generalized solution of the problem (1.1), (1.4) in $Q^+_{\rho_k}$, then

$$(v,p)\in W^{2,1}_{s_k,l}(Q^+_{\rho_{k+1}})\times W^{1,0}_{s_k,l}(Q^+_{\rho_{k+1}})$$

and the following estimate holds:

$$\begin{aligned}&\|v\|_{W^{2,1}_{s_k,l}(Q^+_{\rho_{k+1}})}+\|\nabla p\|_{L_{s_k,l}(Q^+_{\rho_{k+1}})}\\ &\leqslant C\,\Big(\|f\|_{L_{m,l}(Q^+)}+\|v\|_{W^{1,0}_{s_k,l}(Q^+_{\rho_k})}+\|p-b\|_{L_{s_k,l}(Q^+_{\rho_k})}\Big),\end{aligned}\tag{6.1}$$

where $b\in L_l(-1,0)$ is an arbitrary function of t-variable. Moreover, due to the imbedding $W^1_{s_k}(B^+_{\rho_{k+1}})\hookrightarrow L_{s_{k+1}}(B^+_{\rho_{k+1}})$ we obtain the estimate

$$\begin{aligned}&\|v\|_{W^{1,0}_{s_{k+1},l}(Q^+_{\rho_{k+1}})}+\|p\|_{L_{s_{k+1},l}(Q^+_{\rho_{k+1}})}\\ &\leqslant C\,\Big(\|v\|_{W^{2,1}_{s_k,l}(Q^+_{\rho_{k+1}})}+\|p\|_{W^{1,0}_{s_k,l}(Q^+_{\rho_{k+1}})}\Big).\end{aligned}\tag{6.2}$$

Iterating (6.1) and (6.2) from $k=0$ to $k=N$ we finally obtain the estimate

$$\begin{aligned}&\|v\|_{W^{2,1}_{s_N,l}(Q^+_{1/2})}+\|\nabla p\|_{L_{s_N,l}(Q^+_{1/2})}\ \leqslant\\ &\leqslant\ C^N\,\Big(\|f\|_{L_{m,l}(Q^+)}+\|v\|_{W^{1,0}_{s_0,l}(Q^+)}+\|p-b\|_{L_{s_0,l}(Q^+)}\Big).\end{aligned}$$

This estimate is equivalent to (2.9). Theorem 2.4 is proved. □

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Поступило 17 декабря 2012 г.