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A. S. Mikhaylov, V. S. Mikhaylov

EQUATIONS OF THE BOUNDARY CONTROL METHOD FOR THE INVERSE SOURCE PROBLEM

Abstract. We consider the dynamical inverse source problem for the abstract nonselfadjoint operator in the Hilbert space and derive the equation of the Boundary Control method for this problem. It is shown that the solution of this equation crucially depends on the property of certain exponential family. We provide the applications of this equation to inverse source problem and to the problem of the extension of the inverse data.

§1. Introduction

Recently S. A. Avdonin et. al. in [4] derived a new type of equations and apply them to solve the spectral estimation problem (see also [2, 3] for the discrete case). The authors used quite a sophisticated approach that includes using the auxiliary dynamical system and the ideas of the Boundary Control (BC) method (see [13]). Later on we used the same equations to treat the one-dimensional inverse source problem and the dynamical inverse problem by one measurement for the Schrödinger equation in [8, 7]. In the present paper we give a direct derivation of these equations in the abstract setting as a equations of the BC method for the inverse source problem.

Let H be a Hilbert space, A be an operator in H (not necessarily self-adjoint), Y be another Hilbert space, $O : H \supset D(O) \mapsto Y$ be an observation operator (see [16]). Let us consider the dynamical system in H:

$$u_t - Au = 0, \quad t > 0, \tag{1.1}$$

$$u(0) = a. \tag{1.2}$$

We denote by u^a the solution of the system (1.1), (1.2), and by $y(t) := (Ou^a)(t)$ the observation (output of the system). The operator that realize the correspondence $a \mapsto (Ou^a)(t)$ is called *observation* operator $\mathbb{O}^T : H \mapsto$

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$L_2(0,T;Y)$. We fix some $T>0$ and assume that $y(t)\in L_2(0,T;Y)$. In the present paper we will be dealing with the following questions:

1. What information on the operator A could be recovered from the observation $y(t)$?
2. Provided the operator A is known, is it possible to recover the source a from the observation $y(t)$?

Concerning the first problem me mention the multidimensional inverse problems for the Schrödinger, heat and wave equations by one measurement. Some of the results (on the Schrödinger equation) are given in [9, 10, 14]. Using techniques based on Carleman estimates, the authors established global uniqueness and stability results for different geometrical conditions on the domain in arbitrary small time under certain restrictions on the initial source. To answer the first question in the abstract setting, we derive the (suitable) version of the BC method equations. In this method the possibility of recovering the spectral data from the dynamical one is well-known for the dynamical system with boundary control [11, 12]. We elaborate this ideas to the case of the dual system. It happens that the possibility of extraction of the spectral data from the observation $y(t)$ crucially depends on the controllability property of the adjoint system and on the minimality property of the certain exponential family.

The second question also received a lot of attention. It is worth to mention the works [1, 15] and [17], where the authors are interested in the reconstruction of the initial source from the boundary measurements. The method they used are based on the exact observability (controllability) of the underlying dynamical systems. Meanwhile in the approach we propose in the paper, we use less restrictive property – the spectral controllability.

In the second section we derive the equations of the BC-method. The last section is devoted to the applications to inverse source problem.

§2. Equations of the BC method

Let us denote by A^* the operator adjoint to A and denote $B:=O^*$, $B:Y\mapsto H$. Along with the system (1.1), (1.2) we consider the following dynamical control system:

$$v_t+A^*v=Bf,\quad t<T, \tag{2.1}$$

$$v(T)=0, \tag{2.2}$$

and denote its solution by v^f. The reason we consider the system (2.1), (2.2) backward in time is that it is adjoint to (1.1), (1.2) (see [5, 16]). For every $0 \leqslant s < T$ we introduce the *control* operator by $W^s f := v^f(s)$, it is not hard to see that $-W^0$ is adjoint to $\mathbb{O}^T$. Indeed, let us evaluate

$$\int_0^T \left(v_t^f(t), u^a(t)\right)_H \, dt = \int_0^T \left((Bf, u^a)_H - \left(A^* v^f, u^a\right)_H\right) \, dt$$

$$= \int_0^T \left((f, Ou^a)_Y - \left(v^f, Au^a\right)_H\right) \, dt \qquad (2.3)$$

on the other hand,

$$\int_0^T \left(v_t^f(t), u^a(t)\right)_H \, dt = -\int_0^T \left((v^f, u_t^a\right)_H \, dt + \left(v^f, u^a\right)_H\Big)\Big|_{t=0}^{t=T}$$

$$= -\int_0^T \left((v^f, Au\right)_H \, dt - \left(v^f(0), a\right)_H . \qquad (2.4)$$

Equating (2.3) and (2.4), bearing in mind that $v^f(0) = W^0 f$, we arrive at

$$\int_0^T (f, Ou^a)_Y = -\left(W^0 f, a\right)_H .$$

Due to the arbitrariness of f and a, the last equality is equivalent to $\left(\mathbb{O}^T\right)^* = -W^0$.

We assume that the operator A satisfies the following assumptions:

Assumption 1. a) *The spectrum of A is simple: i.e. it consists of (infinite number) eigenvalues with algebraic multiplicity one. We denote them by $\{\lambda_k\}_{k=1}^{\infty}$, the adjoint operator A^* has the spectrum $\{\overline{\lambda_k}\}_{k=1}^{\infty}$.*

b) *The eigenfunctions of A forms a Riesz basis in H, we denote it by $\{\psi_k\}_{k=1}^{\infty}$, the basis of A^* we denote by $\{\varphi_k\}_{k=1}^{\infty}$, the property $(\psi_k, \varphi_l) = \delta_{kl}$ holds.*

c) *The system (2.1), (2.2) is spectrally controllable: i.e. there exists the controls $f_k \in H_0^1(0, T; Y)$ such that $W^0 f_k = \varphi_k$.*

We say that the vector a is *generic* if in its Fourier representation in the basis $\{\psi_k\}_{k=1}^{\infty}$, $a = \sum_{k=1}^{\infty} a_k \psi_k$, is such that $a_k \neq 0$ for $k = 1, \ldots, \infty$.

Now we are ready to formulate the main theorem of the paper.

Theorem 1. *Let A satisfies Assumption 1 and the source a is generic. If $Y=\mathbb{R}$, then the spectrum of A and controls f_k could be found from the generalized spectral problem:*

$$\int_0^{2T}\Big((\dot{Oa})(t)+\lambda_k(Oa)(t),f_k(t+\tau)\Big)_Y\,dt=0,\quad 0<\tau<T. \tag{2.5}$$

Proof. It would be instructive for us to work without assuming $Y=\mathbb{R}$, we use this assumption in the end of the proof. Since the condition c) is valid, we denote by $\{\widetilde{f}_k\}_{k=1}^\infty$ the set of controls which satisfy $W^0\widetilde{f}_k=\varphi_k$. By $\{f_k\}_{k=1}^\infty$ we denote the set of shifted controls: $f_k(t)=\widetilde{f}_k(t-T)$. Thus the controls f_k acts on the time interval $(T,2T)$. By dot we denote the differentiation with respect to t. Let us fix some $k\in 1,\dots,\infty$, $\tau\in(0,T)$ and consider $W^0\left(\dot{f}_k(\cdot+\tau)\right)$:

$$\begin{aligned}W^0\left(\dot{f}_k(\cdot+\tau)\right)&=v^{\dot{f}_k(\cdot+\tau)}(0)=v_t^{f_k(\cdot+\tau)}(0)\\&=\left(Bf(\cdot+\tau)\right)(0)-A^*v^{f_k(\cdot+\tau)}(0).\end{aligned} \tag{2.6}$$

Since by condition d), $f_k\in H_0^1(T,2T,Y)$, it follows that $(Bf(\cdot+\tau))(0)=0$. The second term in the right hand side in (2.6) could be evaluated from the following reasons: the function v^{f_k} solves:

$$\begin{aligned}v_t^{f_k(\cdot+\tau)}+A^*v^{f_k(\cdot+\tau)}&=0,\quad 0\leqslant t\leqslant T-\tau,\\ v^{f_k(\cdot+\tau)}(T-\tau)&=\varphi_k.\end{aligned}$$

Then it is easy to see that $v^{f_k(\cdot+\tau)}(t)=\varphi_k e^{\overline{\lambda_k}(T-\tau-t)}$ for $0\leqslant t\leqslant T$, so

$$A^*v^{f_k(\cdot+\tau)}(0)=e^{\overline{\lambda_k}(T-\tau)}A^*\varphi_k=e^{\overline{\lambda_k}(T-\tau)}\overline{\lambda_k}\varphi_k=\overline{\lambda_k}W^0f_k(\cdot+\tau). \tag{2.7}$$

Combining (2.6) and (2.7) we arrive at

$$W^0\left(\dot{f}_k(\cdot+\tau)\right)=-\overline{\lambda_k}W^0f_k(\cdot+\tau). \tag{2.8}$$

We evaluate integrating by parts and taking into account that $f_n(0) = f_n(T) = 0$:

$$\int_0^{2T} \Big((Oa)(t), \dot f_k(t+\tau)\Big)_Y \, dt = -\int_0^{2T} \Big((\dot{O}a)(t), f_k(t+\tau)\Big)_Y \, dt$$

$$+ \Big((\dot{O}a)(t+\tau), f_k(t)\Big)_Y \Big|_{t=0}^{t=2T} = -\int_0^{2T} \Big((\dot{O}a)(t), f_k(t+\tau)\Big)_Y \, dt. \qquad (2.9)$$

One the other hand, using the duality between W^0 and $\mathbb{O}^T$ and (2.8), we have:

$$\int_0^{2T} \Big((Oa)(t), \dot f_k(t+\tau)\Big)_Y \, dt = -\Big(a, W^0 \dot f_k(\cdot+\tau)\Big)_H = \big(a, \overline{\lambda_k} W^0 f_k(\cdot+\tau)\big)_H$$

$$= \big(\lambda_k a, W^0 f_k(\cdot+\tau)\big)_H = \int_0^{2T} (\lambda_k (Oa)(t), f_k(t+\tau))_Y \, dt. \qquad (2.10)$$

Combining (2.9) and (2.10) we see that the pair λ_k, f_k satisfies

$$\int_0^{2T} \Big((\dot{O}a)(t) + \lambda_k (Oa)(t), f_k(t+\tau)\Big)_Y \, dt = 0, \quad 0 < \tau < T. \qquad (2.11)$$

On the contrary, if the pair λ, f satisfies (2.11), then it follows from the proof that

$$\Big(a, W^0 \dot f(t+\tau)\Big)_H = \big(a, \overline{\lambda} W^0 f(t+\tau)\big)_H,$$

which is equivalent to

$$\Big(a, A^* v^{f(t+\tau)}(0)\Big)_H = \Big(a, \overline{\lambda} v^{f(t+\tau)}(0)\Big)_H, \quad \tau \in (0,T). \qquad (2.12)$$

The latter is possible if

$$A^* v^{f(t+\tau)}(T-\tau) = \overline{\lambda} v^{f(t+\tau)}(T-\tau),$$

i.e., $\overline{\lambda} = \overline{\lambda}_n$ is a point of the specter of A^* and f is a control that drives the system (2.1), (2.2) to φ_n for some n. Now we assume that it is not true – so there exist $\{\lambda, f\}$ solution to (2.11) such that $v^{f(T+\tau)}(T-\tau) = \sum_1^\infty c_k \varphi_k$.

Then we develop the condition (2.12) for our pair λ, f using the fact that $v^{f(\cdot+\tau)}$ satisfies

$$v_t^{f(\cdot+\tau)} + A^* v^{f(\cdot+\tau)} = 0, \quad 0 \leqslant t \leqslant T-\tau,$$
$$v^{f(\cdot+\tau)}(T-\tau) = \textstyle\sum_1^\infty c_k\varphi_k,$$

then the expansion

$$v^{f(t+\tau)}(t) = \sum_{k=1}^{\infty} c_k\varphi_k e^{\overline{\lambda_k}(T-\tau-t)}$$

holds. We plug it into (2.12) and get

$$\left(a, \sum_{k=1\infty} c_k\varphi_k\overline{\lambda}_k e^{\lambda_k(T-\tau)}\right)_H = \left(a, \sum_{k=1\infty} c_k\varphi_k\overline{\lambda} e^{\lambda_k(T-\tau)}\right)_H.$$

Which is equivalent to the equality

$$\sum_{k=0}^{\infty} a_k c_k(\overline{\lambda}_k - \overline{\lambda})e^{\overline{\lambda}_k(T-\tau)} = 0.$$

If $Y = \mathbb{R}$ then it is known (see [5]) that the spectral controllability of (2.1), (2.2) in time T is equivalent to the minimality of the family $\{e^{\lambda_k(T-\tau)}\}_{k=1}^\infty$ in $L_2(0,T)$. The minimality implies that for some $i \in \mathbb{N}$, $c_i \neq 0$, and $c_k = 0$ for $k \in \mathbb{N}\backslash\{i\}$, and $\lambda = \lambda_i$, i.e., λ is a point of the spectrum and f is a corresponding control. □

Equation (2.5) coincides with one derived by Avdonin et. al. in [4]. We notice that as it follows from the proof, without the assumption $Y = R$ we can only show that every pair λ_k, f_k with λ_k being an eigenvalue, f_k being a corresponding control, satisfies equation (2.5). And to prove the converse: that every solution λ, f to (2.5) is in fact eigenvalue and control which drives system to eigenfunction, we made use of the minimality of the family $\{e^{\lambda_k(T-\tau)}\}_{k=1}^\infty$ in $L_2(0,T)$. This minimality condition holds for example when $Y = \mathbb{R}$, but not only in this situation. The characterization of the minimality of the exponential family $\{e^{i\lambda_k t)}\}_{k=1}^\infty$ in $L_2(0,T)$ in terms of the spectrum $\{\lambda_k\}_{k=1}^\infty$ is given in terms of the existence of the entire function with special properties. This subject is discussed in [5], chapter 1.4. So we make another

Assumption 2. *The spectrum $\{\lambda_k\}_{k=1}^\infty$ of A is such that the family $\{e^{\lambda_k t}\}_{k=1}^\infty$ is minimal in $L_2(0,T)$.*

Thus in the case we know the additional information on the spectrum of A, we can formulate

Remark 1. *Let A satisfies Assumptions* 1, 2 *and the source a is generic, then the spectrum of A and controls f_k. could be found from the generalized spectral problem* (2.5).

Let us notice that using equation (2.5) one can determine eigenvalues λ_n and non-normalized controls f_n. The latter means that the function f_n has the property:

$$W^0 f_n = b_n \varphi_n,$$

with some constant b_n.

§3. Applications

Inverse source problem. Let us consider the dynamical system (1.2), (1.2). Assume that the operator A satisfy assumptions 1, 2, and we are given with the observation on the interval $(0, 2T)$. Making use of equation (2.5) we find the sequence $\{\lambda_k, f_k\}_{k=1}^{\infty}$ of eigenvalues and non-normalized controls. Since we know the operator A, we can normalize the controls f_k by using the bi-orthogonality with the family $\{O\psi_k e^{\lambda_k t}\}_{k=1}^{\infty}$:

$$1 = \left(W^0 f_k, \psi_k\right)_H = (f_k, -O\psi_k)_{L_2(0,T;Y)} = -(f_k, O\psi_k e^{\lambda_k t})_{L_2(0,T;Y)}.$$

(we keep the same notation for normalized controls). Then we recover the Fourier coefficients of the source and the source by

$$a_k = (Oa(t), f_k(t)), \quad a = \sum_{k=1}^{\infty} a_k \psi_k.$$

The same problem has been considered in [1, 15] and [17], but the authors utilized methods which were based on stronger controllability result – they utilized the exact controllability property. Thus our approach could be useful for wider class of the dynamical systems which are spectrally controllable but not exactly controllable. The classical example of such a system is a wave equation on graph with circles, see [5].

One-dimensional observation. Let us now consider the special case when $Y = \mathbb{R}$. The latter means that the control in (2.1) and the observation $y(t) = (Oa)(t)$ are functions and $L_2(0, T; Y) = L_2(0, T)$. Let a has the representation $a = \sum_k a_k\psi_k$. We remind that the observation is given by

$$y(t) = \sum_{k=0}^{\infty} a_k O\psi_k e^{\lambda_k t}. \tag{3.1}$$

We assume that we already used equations (2.5) to find $\{\lambda_k, \widetilde{f}_k\}$, where $\widetilde{f}_k$ – non-normalized controls. Denote by f_k, $k = 1,\dots\infty$ the normalized controls: $W^0 f_k = \psi_k$. We can find the products $\overline{O\psi_k} f_k(t)$ by using the bi-orthogonality, indeed:

$$-1 = \left(O\psi_k e^{\lambda_k t}, f_k(t)\right)_{L_2(0,T)} = \left(e^{\lambda_k t}, \overline{O\psi_k} f_k(t)\right)_{L_2(0,T)}. \tag{3.2}$$

After that we can determine the product $O\psi_k a_k$ by

$$(Oa, \overline{O\psi_k} f_k)_{L_2(0,T)} = (a, -\overline{O\psi_k} W^0 f_k)_H = (a, \overline{O\psi_k}\varphi_k)_H = O\psi_k a_k. \tag{3.3}$$

Thus we have recovered the factors $a_k O\psi_k$, $k = 1,\dots\infty$ in (3.1). This in particular means that we can extend the observation $y(t)$ from $(0,T)$ to $(0,\infty)$ by formula (3.1). The possibility of the extension of the inverse data is important in connection with the solving inverse problems (see [12]). Other applications the reader can find in [8, 7], see also [4].

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St. Petersburg Department of V. A. Steklov
Institute of Mathematics
of the Russian Academy of Sciences,
27, Fontanka, 191023 St. Petersburg; St.Petersburg State University,
Universitetsky prospekt, 28,
Peterhof, 198504 St. Petersburg, Russia
E-mail: mikhaylov@pdmi.ras.ru

St.Petersburg Department of V. A. Steklov
Institute of Mathematics
of the Russian Academy of Sciences,
27, Fontanka, 191023 St.Petersburg, Russia
E-mail: vsmikhaylov@pdmi.ras.ru