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CONVEX HULLS OF REGULARLY VARYING PROCESSES

Abstract. We consider the asymptotic behaviour of the compact convex subset $\widetilde{W}_n$ of $\mathbb{R}^d$ defined as the closed convex hull of the ranges of independent and identically distributed (i.i.d.) random processes $(X_i)_{1\leqslant i\leqslant n}$. Under a condition of regular variations on the law of X_i's, we prove the weak convergence of the rescaled convex hulls $\widetilde{W}_n$ as $n\to\infty$ and analyse the structure and properties of the limit shape. We illustrate our results on several examples of regularly varying processes and show that, in contrast with Gaussian setting, in many cases the limit shape is a random polytope of $\mathbb{R}^d$.

§1. Introduction

Let T be a separable metric space. Let $X_i=\{X_i(t),t\in T\}$, $i=1,2,\ldots$, be i.i.d. copies of a random process X with values in $\mathbb{R}^d$. Assume that X is separable and has a.s. bounded sample paths. We consider the closed convex hulls

$$\widetilde{W}_n=\overline{\mathrm{conv}}\{X_i(t);\ t\in T,\ 1\leqslant i\leqslant n\} \tag{1}$$

and study its asymptotic behavior as $n\to\infty$.

Our work is motivated from one side by the recent works of Comptet, Majumdar, and Randon-Furling [10, 13] dealing in an ecological context with the estimation of the home range of a heard of animals with population size n. Another stimulating point is the relation with classical extreme value theory. Indeed, if $T=\{t_0\}$ is a singleton and $d=1$, then X_i, $i=1,2,\ldots$, is simply a sequence of i.i.d. real random variables and

$$\widetilde{W}_n=[\min_{i\leqslant n}X_i,\ \max_{i\leqslant n}X_i].$$

In the Gaussian case, the question was studied in details by Davydov in [4]. It was shown that for any bounded centered Gaussian process X,

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$$\frac{1}{\sqrt{2\log(n)}}\widetilde{W}_n \to \widetilde{W} \quad \text{almost surely as } n \to \infty,$$

where $\widetilde{W}$ is a *nonrandom* compact convex set and the convergence is meant in the sense of the Hausdorff distance. Hence, the growth rate of $\widetilde{W}_n$ is always equal to $\sqrt{2\log(n)}$ and there exists a nonrandom limit shape $\widetilde{W}$. The limit set $\widetilde{W}$ is given by

$$\widetilde{W}_n = \overline{\text{conv}}(K_t,\ t \in T),$$

where K_t is the ellipsoid of concentration associated to the covariance operator of $X(t)$ (see [4] for a precise definition). In particular, $\widetilde{W}$ has, in general, smooth boundary.

The aim of the present work is to consider what happens for non-Gaussian processes.

Limit theorems for unions of random sets in Euclidean or Banach space have been studied in great details by Molchanov [12] (for general results on random closed sets in stochastic geometry we refer also to this monograph and the references therein). In particular, he got necessary and sufficient conditions for convergence of convex hulls. However, these conditions are formulated in the terms of capacity or containment functionals and it is very difficult to use them in our concrete situation. Therefore we apply more direct method based on point processes. We show (and it is in a full agreement with Molchanov's results) that the natural class of processes for which the normalized sequence $(\widetilde{W}_n)_{n\geqslant 1}$ has some limit is the class of regularly varying processes (for more details on the *regular variations* property, see Hult and Lindskog [8], Davis and Mikosch [3]).

We generalize an initial problem and consider at first the convex closed hulls

$$W_n = \overline{\text{conv}}(X_1, \ldots, X_n) \tag{2}$$

of an i.i.d. sample of elements of $\mathbb{K}$ in abstract cones $\mathbb{K}$.

If $X \in \text{RV}_{\alpha,\sigma}(\mathbb{K})$, i.e. if X is regularly varying with index α and spectral measure σ (see Definition 2.2 below), then the rescaled empirical point process

$$\beta_n = \sum_{k=1}^{n} \delta_{X_k/b_n} \tag{3}$$

converges as $n \to \infty$ to a Poisson point process $\Pi_{\alpha,\sigma}$ on $\mathbb{K}$, with $(b_n)_{n\geqslant 1}$ a suitable renormalization sequence. It is then possible to deduce the convergence in distribution of the convex hulls

$$b_n^{-1} W_n \Longrightarrow C_{\alpha,\sigma}, \tag{4}$$

where $C_{\alpha,\sigma} = \overline{\text{conv}}(\Pi_{\alpha,\sigma})$. When $\mathbb{K} = \mathbb{C}(T, \mathbb{R}^d)$ is the Banach space of continuous functions $f : T \to \mathbb{R}^d$, we can go back to our original problem: under the assumption that the processes are regularly varying, we deduce from (4) the weak convergence of the convex hulls of the ranges of the processes

$$b_n^{-1} \widetilde{W}_n \Longrightarrow \widetilde{C}_{\alpha,\sigma}. \tag{5}$$

Another interesting case is the case $\mathbb{K} = \mathbb{D}([0,1], \mathbb{R}^d)$ where we consider regularly varying càd-làg processes from $[0,1]$ into $\mathbb{R}^d$. The main difficulty here is that the addition is not continuous on the Skorokhod space so that the convergence (4) may not be satisfied. Nevertheless, we can go through these technical issues and prove directly the convergence (5) in this case.

Our main conclusions about the convex hulls of non-Gaussian random processes are the following: the assumption of regular variations plays a natural role; the logarithmic scale of normalization natural in the Gaussian case is replaced by a power one; the limit shape is no longer deterministic and almost sure convergence has to be replaced by weak convergence; in many cases, the limit shape is a polytope and in particular its boundary is not smooth.

The structure of the paper is as follows: in the next section, we present general results on convex cones, convex hulls and regular variations, and then state a limit theorem for the convex hull W_n of an i.i.d. sample of size n in a convex cone $\mathbb{K}$ under a general assumption of regular variations. The structure and properties of the limit shape are analysed: we give a representation of the limit shape in terms of a LePage series and prove a stability property. This is in good correspondance with the results by Molchanov [12]. In Sec. 3, we focus on our original problem and provide a limit theorem for the convex hull $\widetilde{W}_n$ of the range of n i.i.d. regularly varying processes. The last section is devoted to examples: we present several classes of processes satisfying the regular variations properties. In many interesting cases, the limit shape $\widetilde{C}_{\alpha,\sigma}$ is a random polytope in $\mathbb{R}^d$, i.e. the convex hull of a finite number of random points. In particular, this is the case for strictly α-stable Lévy processes when the spectral measure is scattered enough on the unit sphere.

§2. Convex hulls of large samples in a convex cone

2.1. Preliminaries on convex cones. We look at our original problem of convex hulls associated to random processes in the general framework of abstract convex cones. We refer to Davydov, Molchanov and Zuyev [6] for general results on convex cones and the associated stable distributions. For the reader convenience, we recall here some definitions.

A *convex cone* $\mathbb{K}$ is a topological space with a binary operation $(x, y) \to x + y$ and an operation $(a, x) \to ax$ of multiplication by positive scalars a such that:

- i) $(\mathbb{K}, +)$ is a topological abelian semigroup: the operation $+$ is associative, commutative and continuous;
- ii) the group $(0,\infty)$ acts on $\mathbb{K}$ by continuous automorphisms: for all $a > 0$, the multiplication operator $D_a : \mathbb{K} \to \mathbb{K}$ defined by $D_a(x) = ax$ is a continuous automorphism, D_1 is the identical map and the relation $D_a D_b = D_{ab}$ holds for all $a, b > 0$;
- iii) the topological space $\mathbb{K}$ is Polish: there is a metric d on $\mathbb{K}$ that makes the space complete and separable.

Furthermore, we will always suppose that

- iv) the cone $\mathbb{K}$ is pointed, i.e. there is an element 0 called the origin such that for all $x \in \mathbb{K}$, $ax \to 0$ as $a \to 0$;
- v) the metric d on $\mathbb{K}$ satisfies the following properties: for all $x, y, h \in \mathbb{K}$ and $a > 0$

$$d(x + h, y + h) \leqslant d(x + y), \tag{6}$$

$$d(ax, ay) = ad(x, y). \tag{7}$$

Note that it is not assumed that the origin 0 is the neutral element of the operation $+$, i.e., we may have $x+0 \neq x$ for some $x \in \mathbb{K}$. Axiom v) suggests the following convention: the multiplication by $a = 0$ is defined by $0x = 0$ for all $x \in \mathbb{K}$. The *norm* of an element $x \in \mathbb{K}$ is defined by $\|x\| = d(0, x)$.

Note that the term "norm" is somehow abusive, since the function $d(0,x)$ is not supposed to be sublinear, i.e. $d(0,x+y)$ is not necessarily smaller than $d(0,x)+d(0,y)$. Clearly, the origin 0 is the unique point $x \in \mathbb{K}$ such that $\|x\| = 0$. Let $\mathbb{K}_0 = \mathbb{K} \setminus \{0\}$ and $\mathbb{S} = \{x \in \mathbb{K}_0;\ \|x\| = 1\}$ be the unit sphere. With the induced metric, $\mathbb{S}$ is also a Polish space. The polar decomposition $p : \mathbb{K}_0 \to (0,\infty) \times \mathbb{S}$ given by $x \to (\|x\|, x/\|x\|)$ is a homeomorphism.

A large class of examples of convex cones is given by the class of Banach spaces and their subcones. For example, the linear space of continuous functions $\mathbb{C}(T)$ on some compact parameter set T endowed with the usual uniform norm will be considered in the sequel when dealing with the convex hull of continous random processes. Other examples play an important role in extreme value theory:

— the space $[0,\infty)$ with the usual metric $d(x,y) = |x-y|$, the usual multiplication by positive scalars and the addition defined by $x + y = \max(x,y)$;
— the space $\mathbb{K} = \mathbb{C}^+(T)$ of nonnegative functions on a compact set T, with the distance associated to the uniform norm, the usual multiplication by positive scalars and the addition given by the pointwise maximum $x + y = \max(x,y)$.

Further examples are discussed in [6].

We recall some basic definitions and properties of convex sets and convex hulls in convex cones. They are standard in the context of Banach spaces and can be extended without major changes to abstract convex cones. A subset $C \subset \mathbb{K}$ is said to be *convex* if for all $x, y \in C$ and $\lambda \in [0,1]$, $\lambda x + (1-\lambda)y \in C$. The whole cone $\mathbb{K}$ is convex and any intersection of convex sets is convex. The *convex hull* of a subset $A \subset \mathbb{K}$, denoted by $\mathrm{conv}(A)$, is defined as the smallest convex C containing A, and is equal to the intersection of all convex sets containing A. Equivalently, a constructive definition is

$$\mathrm{conv}(A) = \left\{ x \in \mathbb{K};\ \exists n \geqslant 1,\ \exists (x_i)_{1 \leqslant i \leqslant n} \in A^n,\ \exists \lambda \in \Sigma_n,\ \sum_{i=1}^{n} \lambda_i x_i = x \right\},$$

where Σ_n denotes the simplex $\Sigma_n = \big\{\lambda = (\lambda_i)_{1 \leqslant i \leqslant n} \in [0,1]^n;\ \sum_{i=1}^{n} \lambda_i = 1\big\}$. The *closed convex hull* of a subset A, denoted by $\overline{\mathrm{conv}}(A)$ is the smallest closed convex set containing A. Using the fact that the closure $\overline{C}$ of a convex set C is convex, it is easy to see that the closed convex hull of A is equal to the closure of the convex hull of A:

$$\overline{\operatorname{conv}}(A)=\overline{\operatorname{conv}(A)}.$$

For $\varepsilon>0$, the ε-neighborhood of $A\subset\mathbb{K}$ is the set $\mathcal{V}_\varepsilon(A)$ defined by

$$\mathcal{V}_\varepsilon(A)=\{x\in K;\ \exists a\in A,\ d(x,a)<\varepsilon\}.$$

The axiom v) ensures that the ε-neighborhood of a convex set is convex and also that the closed convex hull of a relatively compact set is compact. Unlike the usual results on Banach space, singletons and balls may not to be convex sets in a general abstract convex cone.

Denote by $\mathcal{K}=\mathcal{K}(\mathbb{K})$ the set of nonempty compact convex subsets of $\mathbb{K}$ with the Hausdorff distance ρ. Recall that the Hausdorff distance between closed bounded subsets A and B of $\mathbb{K}$ is defined by

$$\rho(A,B)=\inf\{\varepsilon>0;\ A\subset\mathcal{V}_\varepsilon(B)\text{ and }B\subset\mathcal{V}_\varepsilon(A)\}.$$

It is easy to see that $(\mathcal{K},\rho)$ is a Polish space. Interestingly, there is a natural structure of convex cone on $(\mathcal{K},\rho)$: the multiplication is defined in the usual way

$$aC=\{ac;\ c\in C\},\quad C\in\mathcal{K},\ a>0,$$

and the addition, denoted $\oplus$, is defined as the convex hull of the union

$$C_1\oplus C_2=\overline{\operatorname{conv}}(C_1\cup C_2),\quad C_1,C_2\in\mathcal{K}.$$

It is easily checked that these operations satisfy the axioms i)–v); the origin of the cone $\mathcal{K}$ is equal to the singleton $\{0\}$. The following simple lemma turns out to be useful in order to estimate the Hausdorff distance between convex hulls. The proof is elementary and left to the reader.

Lemma 2.1. *Let A and B be nonempty bounded subsets of $\mathbb{K}$. Suppose that there exists $r>0$ such that*

$$\forall a\in A,\ \exists b\in B,\ d(a,b)\leqslant r\quad\text{and}\quad\forall b\in B,\ \exists a\in A,\ d(a,b)\leqslant r.\tag{8}$$

Then,

$$\rho(\overline{\operatorname{conv}}(A),\overline{\operatorname{conv}}(B))\leqslant r.$$

2.2. Regular variations. The notion of regularly varying random variable on a convex cone will play a key role to derive the asymptotic behavior of the convex hull of i.i.d. sample. We recall here some definitions and basic properties needed in the sequel, more details can be found in Hult and Lindskog [8] or Davis and Mikosch [3].

Definition 2.2. *A $\mathbb{K}_0$-valued random variable X is said to be regularly varying with exponent $\alpha > 0$, if there exist a probability measure σ on $\mathbb{S}$ and a positive sequence $b_n \to +\infty$ such that*

$$\lim_{n\to\infty} n\mathbf{P}\left[\frac{X}{\|X\|} \in A \ ; \ \|X\| > rb_n\right] = \sigma(A)r^{-\alpha} \tag{9}$$

for all $r > 0$ and all $A \in \mathcal{B}(\mathbb{S})$ such that $\sigma(\partial A) = 0$. We denote this property by $X \in \mathrm{RV}_{\alpha,\sigma}(\mathbb{K})$.

The exponent α and the spectral measure σ are uniquely determined, the sequence (b_n) is unique up to asymptotic equivalence and a possible choice is given by

$$b_n = \inf\{x > 0;\ \mathbf{P}(\|X\| > x) \leqslant n^{-1}\}. \tag{10}$$

It is known that $b_n = n^{1/\alpha}L(n)$ for some function L slowly varying at infinity.

The definition of regularly varying random variable can then be reformulated using the concept of vague convergence of measures. Following Davydov *et al.* [6], the usual definition of vague convergence is suitably amended in order to take into account possible explosion of the measures near the origin 0 and the fact that the state space $\mathbb{K}$ is not locally compact. See also Daley and Vere-Jones [2] and Matthes *et al.* [11] for a discussion on measures and vague convergence in general Polish spaces. For $r > 0$, let

$$B_r = \{x \in \mathbb{K}, \|x\| < r\} \quad \text{and} \quad B^r = \{x \in \mathbb{K}, \|x\| > r\}.$$

Denote by $\mathcal{M}(\mathbb{K})$ the set of measures μ on $\mathbb{K}_0$ such that $\mu(B^r) < \infty$ for all $r > 0$. Consider the family $\mathcal{C}$ of continuous bounded functions $f : \mathbb{K}_0 \to \mathbb{R}$ with support included in B^r for some $r > 0$. A sequence of measures $\{m_n, n \geqslant 1\}$ is said to converge vaguely to m in $\mathcal{M}(\mathbb{K})$ as $n \to \infty$, denoted $m_n \xrightarrow{v} m$ if for every $f \in \mathcal{C}$

$$\int f(x) m_n(dx) \longrightarrow \int f(x) m(dx) \quad \text{as } n \to \infty.$$

The space $\mathcal{M}(\mathbb{K})$ with the vague topology is Polish. As stated by Davis and Mikosch [3], $X \in \mathrm{RV}_{\alpha,\sigma}(\mathbb{K})$ if and only if

$$n\mathbf{P}\left[b_n^{-1}X \in \cdot\,\right] \xrightarrow{v} m_{\alpha,\sigma}(\,\cdot\,), \tag{11}$$

where $m_{\alpha,\sigma}$ is the measure on $\mathbb{K}_0$ characterized by the relation

$$m_{\alpha,\sigma}\Big(\big\{x \in \mathbb{K}_0; x/\|x\| \in A, \|x\| \geqslant r\big\}\Big) = r^{-\alpha}\sigma(A), \quad A \in \mathcal{B}(\mathbb{S}), \quad r > 0.$$

2.3. Convergence of the empirical point processes. Consider $\mathcal{M}_p(\mathbb{K}) \subset \mathcal{M}(\mathbb{K})$ the subspace of point measures: $m \in \mathcal{M}_p(\mathbb{K})$ if and only if $m = \sum_{i \in I} \delta_{x_i}$ with $\{x_i, i \in I\}$ an at most countable collection of points in $\mathbb{K}_0$ such that for all $r > 0$, B^r contains only a finite number of the x_i's. The subset $\mathcal{M}_p(\mathbb{K})$ is closed in $\mathcal{M}(\mathbb{K})$ with respect to the vague topology. Hence $\mathcal{M}_p(\mathbb{K})$ with the induced topology is a Polish space. Let X be a $\mathbb{K}_0$-valued random variable and $\{X_k, k \geqslant 1\}$ an i.i.d. sample with the same distribution. Consider the rescaled empirical point process $\beta_n = \sum_{k=1}^{n} \delta_{X_k/b_n}$ with b_n given by (10).

Theorem 2.3. *The following conditions are equivalent:*

(1) $X \in \mathrm{RV}_{\alpha,\sigma}(\mathbb{K})$.
(2) *The rescaled empirical point processes β_n weakly converge in $\mathcal{M}_p(\mathbb{K})$ as $n \to \infty$ to a Poisson point process $\Pi_{\alpha,\sigma}$ with intensity measure $m_{\alpha,\sigma}$.*

This result can be found in Resnick [14], in the case when the state space $\mathbb{K}$ is locally compact; the extension to complete and separable abstract cones is discussed in Davydov *et al.* [6], Theorem 4.3.

2.4. Convergence of the convex hulls. Let $\{X_k, k \geqslant 1\}$ be an i.i.d. sequence of $\mathbb{K}_0$-valued random variables. The convex hull W_n of the n-th order sample is defined by equality (2). Our main result in this section is a weak limit for the convex hull W_n as $n \to +\infty$. The weak limit is taken in the sense of weak convergence in the Polish space $\mathcal{K}$.

Theorem 2.4. *Suppose that $X \in \mathrm{RV}_{\alpha,\sigma}(\mathbb{K})$. Then, as $n \to \infty$,*

$$b_n^{-1} W_n \Longrightarrow C_{\alpha,\sigma},$$

where $C_{\alpha,\sigma} \stackrel{d}{=} \overline{\mathrm{conv}}(\Pi_{\alpha,\sigma})$ with $\Pi_{\alpha,\sigma}$ a Poisson point process on $\mathbb{K}$ with intensity measure $m_{\alpha,\sigma}$. The symbol $\stackrel{d}{=}$ stands for equality in distribution.

The proof of Theorem 2.4 relies on Theorem 2.3 and on the following lemma:

Lemma 2.5. *The map $\Phi : \mathcal{M}_p(\mathbb{K}) \to \mathcal{K}$ defined by*

$$\Phi\left(\sum_{i \in I} \delta_{x_i}\right) = \overline{\mathrm{conv}}(\{x_i, i \in I\} \cup \{0\}) \tag{12}$$

is continuous on $M_p(\mathbb{K})$ with respect to the vague topology.

Remark 2.6. By definition of $\mathcal{M}_p(\mathbb{K})$, the set $\{x_i, i \in I\} \cup \{0\}$ is closed and compact since 0 is the only possible accumulation point of the set $\{x_i, i \in I\}$. The closed convex hull $\overline{\text{conv}}(\{x_i, i \in I\} \cup \{0\})$ is therefore compact and the mapping C is well defined with values in $\mathcal{K}$. The possible accumulation of points at the origin 0 explains why we have to add the origin to the convex hulls. Consider indeed the following example: let $x \in \mathbb{K}_0$ and for $n \geqslant 1$, $m_n = \delta_x + \delta_{n^{-1}x}$; then it holds that $m_n \xrightarrow{v} \delta_x$ in $\mathcal{M}_p(\mathbb{K})$ and that $\overline{\text{conv}}(x, n^{-1}x) \xrightarrow{\rho} \overline{\text{conv}}(x, 0)$ in $\mathcal{K}$.

Proof of Theorem 2.4. According to Theorem 2.3, the scaled empirical point process $\beta_n = \sum_{i=1}^{n} \delta_{b_n^{-1}X_i}$ converges weakly in $\mathcal{M}_p(\mathbb{K})$ to the Poisson point process $\Pi_{\alpha,\sigma}$ as $n \to \infty$. The map Φ being continuous, the continuous mapping Theorem (see Billingsley [1]) implies that the convex hulls $\Phi(\beta_n)$ weakly converges in $\mathcal{K}$ to the convex hull $\Phi(\Pi_{\alpha,\sigma})$.

The origin 0 is almost surely an accumulation point of the set $\Pi_{\alpha,\sigma}$ because the intensity $m_{\alpha,\sigma}$ explodes at the origin ($m_{\alpha,\sigma}(B_r) = \infty$ for all $r > 0$). This implies that

$$\Phi(\Pi_{\alpha,\sigma}) = \overline{\text{conv}}(\Pi_{\alpha,\sigma} \cup \{0\}) = \overline{\text{conv}}(\Pi_{\alpha,\sigma}) \quad \text{a.s.}$$

On the other hand,

$$\Phi(\beta_n) = \overline{\text{conv}}(\{b_n^{-1}X_i, 1 \leqslant i \leqslant n\} \cup \{0\})$$

and

$$b_n^{-1}W_n = \overline{\text{conv}}(\{b_n^{-1}X_i,\ 1 \leqslant i \leqslant n\}).$$

An application of Lemma 2.1 yields

$$\rho(\Phi(\beta_n), b_n^{-1}W_n) \leqslant b_n^{-1} \min_{1 \leqslant i \leqslant n} \|X_i\|,$$

and this distance converges almost surely to 0 as $n \to \infty$. As a consequence, the weak convergence $b_n^{-1}W_n \Longrightarrow \overline{\text{conv}}(\Pi_{\alpha,\sigma})$ holds in $\mathcal{K}$ and Theorem 2.4 is proved. □

Proof of Lemma 2.5. Consider a sequence $(m_n)_{n \geqslant 0}$ that converges vaguely to m in $\mathcal{M}_p(\mathbb{K})$. Let $\varepsilon > 0$. Denote by m_n^ε (respectively, m^ε) the restriction of m_n (respectively, m) to B^ε. These are finite point processes on B^ε and $m_n^\varepsilon \xrightarrow{v} m^\varepsilon$ as $n \to \infty$. It is easy to see that we can write

$$m_n^\varepsilon = \sum_{i=1}^{k_n} \delta_{x_i^n} \quad \text{and} \quad m^\varepsilon = \sum_{i=1}^{k} \delta_{x_i}$$

with, as $n \to \infty$,

$$k_n \to k \quad \text{and} \quad x_i^n \to x_i, \quad 1 \leqslant i \leqslant k.$$

Note that k_n is equal to k for n large enough. Let n_0 be large enough so that for all $n \geqslant n_0$,

$$k_n = k \quad \text{and} \quad d(x_i^n, x_i) \leqslant \varepsilon, \quad 1 \leqslant i \leqslant k.$$

For fixed $n \geqslant n_0$, denote by A' the set of points in m_n, and by B' the set of points in m. We prove easily that the sets $A = A' \cup \{0\}$ and $B = B' \cup \{0\}$ satisfy the assumption of Lemma 2.1: for example, a given point a in A' is either in B_r, and then $d(a,0) \leqslant \varepsilon$, or in B^r, and then $a = x_i^n$ for some $1 \leqslant i \leqslant k$ and $d(a, x_i) \leqslant \varepsilon$. Using Lemma 2.1, we get that for all $n \geqslant n_0$, $\rho(C(m_n), C(m)) \leqslant \varepsilon$. This proves that $C(m_n) \to C(m)$ in $\mathcal{K}$ as $n \to \infty$ and that the mapping C is continuous. □

2.5. Stability and structure of the limit shape. Let $C_{\alpha,\sigma}$ be the limit shape $C_{\alpha,\sigma} = \overline{\operatorname{conv}}(\Pi_{\alpha,\sigma})$ appearing in Theorem 2.4. This is a random variable taking values in the Polish space $\mathcal{K}$. Recall that $\mathcal{K}$ has a natural structure of cone with operations

$$C_1 \oplus C_2 = \overline{\operatorname{conv}}(C_1 \cup C_2), \quad aC_1 = \{ac;\ c \in C_1\}, \quad C_1, C_2 \in \mathcal{K},\ a > 0.$$

The following proposition states two interesting properties of the limit shape: its stability with respect to the operation $\oplus$ and a representation in terms of LePage series. For a general discussion on LePage series in abstract convex cone, see Davydov *et al.* [6] Theorem 3.6. The results are a consequence of the study of convex-stable sets in Molchanov [12, Chap. 4, Sec. 4.2] and the proof is omitted.

Proposition 2.7. (1) *If $C^1_{\alpha,\sigma}$ and $C^2_{\alpha,\sigma}$ are two independent copies of $C_{\alpha,\sigma}$, then*

$$a_1^{1/\alpha} C^1_{\alpha,\sigma} \oplus a_2^{1/\alpha} C^1_{\alpha,\sigma} \overset{d}{=} (a_1 + a_2)^{1/\alpha} C_{\alpha,\sigma}, \quad a_1, a_2 > 0.$$

(2) *Let $(\Gamma_k)_{k \geqslant 1}$ be the nondecreasing enumeration of the points of a Poisson point process on $[0, +\infty)$ with Lebesgue intensity measure and, independently, let $(\varepsilon'_k)_{k \geqslant 1}$ be an i.i.d. sequence of $\mathbb{S}$-valued random elements with distribution σ. Define $\varepsilon_k = \overline{\operatorname{conv}}\{\varepsilon'_k\}$. Then, $C_{\alpha,\sigma}$ admits the representation*

$$C_{\alpha,\sigma} \overset{d}{=} \oplus_{k \geqslant 1} \Gamma_k^{-1/\alpha} \varepsilon_k.$$

§3. Convex hulls of independent processes

3.1. Framework. We consider the case when the random variables $\{X_k,\ k \geqslant 1\}$ are independent processes, i.e., the convex cone $\mathbb{K}$ is a space of functions. Two main cases occur: we can consider either the class of continuous processes or the class of cád-lág processes; the corresponding spaces are $\mathbb{K} = \mathbb{C}(T, \mathbb{R}^d)$ and $\mathbb{K} = \mathbb{D}([0,1], \mathbb{R}^d)$, respectively.

The space $\mathbb{C}(T, \mathbb{R}^d)$ endowed with the uniform norm

$$\|x\| = \sup_{t \in T} \max_{1 \leqslant i \leqslant d} |x_i(t)|$$

is a Banach space and hence a convex cone satisfying the basic axioms i)–v). We can therefore apply the general theory of convex hull developed in the previous section in this particuliar case $\mathbb{K} = \mathbb{C}(T, \mathbb{R}^d)$. For example, Theorem 2.4 provides the behavior of the convex hulls for a large sample of regularly varying continuous processes.

The space $\mathbb{D}([0,1], \mathbb{R}^d)$ endowed with the Skorokhod metric d is a Polish space, see Billingsley [1]. Note that the norm of $x \in \mathbb{D}([0,1], \mathbb{R}^d)$ is equal to

$$\|x\| = d(0, x) = \sup_{t \in [0,1]} \max_{1 \leqslant i \leqslant d} |x_i(t)|.$$

This case suffers from the noncontinuity of the addition $+$. Some assumptions in the definition of convex cones are violated: axiom i) holds except that the addition $+$ is not continuous and in axiom v), the property $d(x+h, y+h) \leqslant d(x,y)$ is not fulfilled. Recall this simple example: let $x = 1_{[1/2,1]}$, $y = -x$ and for $n \geqslant 2$, $x_n = 1_{[1/2+1/n,1]}$; then $x_n \to x$ but $x_n + y \not\to x + y$. As a consequence, Theorem 2.4 cannot be applied for the sample convex hull – see Example 3.2 below.

As explained in the Introduction, we are interested rather in the convex hull of the ranges of the process than in the sample convex hull. Denote by $\mathcal{K}_d$ the space of nonempty compact convex subsets of $\mathbb{R}^d$ endowed with the Hausdorff distance ρ_d. Let $\{X_k,\ k \geqslant 1\}$ be independent and identically distributed random elements in $\mathbb{K} = \mathbb{C}(T, \mathbb{R}^d)$ or in $\mathbb{D}([0,1], \mathbb{R}^d)$ and consider the convex hull of their range $\widetilde{W}_n$ defined by (1), with $T = [0,1]$ when $\mathbb{K} = \mathbb{D}([0,1], \mathbb{R}^d)$. In the sequel, we will provide results for the limit behavior of the $\mathcal{K}_d$-valued random variable $\widetilde{W}_n$. We will consider only the case $\mathbb{K} = \mathbb{D}([0,1], \mathbb{R}^d)$, the results being easily adapted to the case $\mathbb{K} = \mathbb{C}(T, \mathbb{R}^d)$. For notational convenience, we will use the shorter notation $\mathbb{D} = \mathbb{D}([0,1], \mathbb{R}^d)$.

3.2. Regular variations for the convex hull of the range of a random process. Let $\pi : \mathbb{D} \to \mathcal{K}_d$ be the application defined by

$$\pi(x) = \overline{\mathrm{conv}}\{x(t),\ t \in [0,1]\}, \quad x \in \mathbb{D}.$$

Our proof uses the representation

$$\widetilde{W}_n = \oplus_{k=1}^n \pi(X_k). \tag{13}$$

The following lemma plays an important role in the proof.

Lemma 3.1. *Suppose that $X \in \mathrm{RV}_{\alpha,\sigma}(\mathbb{D})$. Then the random element $\pi(X)$ of the cone $\mathcal{K}_d$ satisfies the condition $\mathrm{RV}_{\alpha,\widetilde{\sigma}}(\mathcal{K}_d)$, where $\widetilde{\sigma} = \sigma\pi^{-1}$.*

Proof of Lemma 3.1. Let b_n be such that (9) holds in the definition $X \in \mathrm{RV}_{\alpha,\sigma}(\mathbb{D})$. We use the notation ρ_d for the Hausdorff distance in $\mathcal{K}_d$ and $|.|$ for the norm in $\mathcal{K}_d$:

$$|C| = \rho_d(\{0\}, C), \quad C \in \mathcal{K}_d.$$

Let $\mathbb{S}_{\mathcal{K}_d} = \{C \in \mathcal{K}_d;\ |C| = 1\}$ and consider $A \in \mathcal{B}(\mathbb{S}_{\mathcal{K}_d})$ such that $\widetilde{\sigma}(\partial A) = 0$. We need to state that for all $r > 0$,

$$\lim_{n\to\infty} n\mathbf{P}\left[\frac{\pi(X)}{|\pi(X)|} \in A\ ;\ |\pi(X)| > rb_n\right] = \widetilde{\sigma}(A)r^{-\alpha}. \tag{14}$$

Remark that for all $x \in \mathbb{D}$ and $a > 0$,

$$|\pi(x)| = \|x\| \quad \text{and} \quad \pi(ax) = a\pi(x)$$

so that the condition $|\pi(X)| > rb_n$ is equivalent to $\|X\| > rb_n$ and the condition $\frac{\pi(X)}{|\pi(X)|} \in A$ is equivalent to $\frac{X}{\|X\|} \in \pi^{-1}(A)$. Furthermore, it is easily checked that the map π is Lipschitz:

$$\rho_d(\pi(x), \pi(y)) \leqslant d(x, y), \quad x, y \in \mathbb{D}.$$

From the continuity of π, it follows that $\partial(\pi^{-1}(A)) \subset \pi^{-1}(\partial A)$ and we get

$$\sigma(\partial(\pi^{-1}(A))) \leqslant \sigma(\pi^{-1}(\partial A)) = \widetilde{\sigma}(\partial A) = 0.$$

Now we see that (14) is equivalent to

$$\lim_{n\to\infty} n\mathbf{P}\left[\frac{X}{\|X\|} \in \pi^{-1}A\ ;\ \|X\| > rb_n\right] = \sigma(\pi^{-1}(A))r^{-\alpha}$$

which follows from $X \in \mathrm{RV}_{\alpha,\sigma}(\mathbb{D})$. $\square$

3.3. Convergence of the convex hulls $\widetilde{W}_n$. As noted above, the cone $\mathbb{D}$ violates the assumption that the addition + is continuous. As a consequence, Theorem 2.4 does not hold true, as the following example shows.

Example 3.2. Consider a positive random variable η such that $\mathbf{P}(\eta > r) \sim r^{-1}$ as $r \to \infty$. Let $\phi : (0, +\infty) \to (0, 1/2)$ be a decreasing function such that $\phi(r) \to 0$ as $r \to \infty$. Consider X the $\mathbb{D}$-valued random variable defined by

$$X = (\eta 1_{(1/2-\phi(\eta),1]}(t))_{0 \leqslant t \leqslant 1}.$$

It is easy to check that the random variable X is regularly varying in $\mathbb{K} = \mathbb{D}$ with exponent $\alpha = 1$ and spectral measure $\sigma = \delta_{x_0}$, where $x_0 = (1_{(1/2,1]}(t))_{0 \leqslant t \leqslant 1}$. The corresponding normalization sequence is simply $b_n \equiv n$. On the other hand, given i.i.d. copies $X_n, n \geqslant 1$, of X, one can show that $W_n = \overline{\mathrm{conv}}(X_k, 1 \leqslant k \leqslant n)$ does not satisfy $b_n^{-1} W_n \Longrightarrow f(\Pi_{\alpha,\sigma})$ as in Theorem 2.4.

Even though no limit theorem holds for the convex hull W_n of the sample of processes in $\mathbb{D}$, we can prove a limit theorem for the convex hull $\widetilde{W}_n$ of their ranges in $\mathbb{R}^d$. Recall that the multiplication operation in $\mathcal{M}_p(\mathcal{K}_d)$ is defined by

$$a\left(\sum_{i \in I} \delta_{C_i}\right) = \sum_{i \in I} \delta_{aC_i}.$$

Theorem 3.3. *Suppose that $X \in \mathrm{RV}_{\alpha,\sigma}(\mathbb{D})$. Then the following weak convergence holds in the space $\mathcal{K}_d$*

$$b_n^{-1} \widetilde{W}_n \Longrightarrow \widetilde{C}_{\alpha,\sigma} \quad \text{as } n \to \infty$$

where $\widetilde{C}_{\alpha,\sigma}$ is given by the LePage series

$$\widetilde{C}_{\alpha,\sigma} = \oplus_{k \geqslant 1} \Gamma_k^{-1/\alpha} \widetilde{\varepsilon}_k$$

with the sequence $(\Gamma_k)_{k \geqslant 1}$ as in Proposition 2.7 and $(\widetilde{\varepsilon}_k)_{k \geqslant 1}$ an independent i.i.d. sequence in $\mathbb{S}_{\mathcal{K}_d}$ with distribution $\widetilde{\sigma} = \sigma \pi^{-1}$.

An alternative representation of the limit is as follows:

$$\widetilde{C}_{\alpha,\sigma} \stackrel{d}{=} \overline{\mathrm{conv}}\Big(\big\{x(t);\ x \in \Pi_{\alpha,\sigma},\ t \in [0,1]\big\}\Big),$$

where $\Pi_{\alpha,\sigma}$ is a Poisson point process on $\mathbb{D}$ with intensity $m_{\alpha,\sigma}$. The proof of Theorem 3.3 makes use of the following lemma.

Lemma 3.4. *The mapping* $\widetilde{\Phi} : \mathcal{M}_p(\mathcal{K}_d) \to \mathcal{K}_d$ *defined by*

$$\widetilde{\Phi}\left(\sum_{i\in I}\delta_{C_i}\right) = \overline{\text{conv}}\left(\bigcup_{i\in I} C_i \bigcup \{0\}\right)$$

satisfies the following properties:

$$\widetilde{\Phi}(am) = a\widetilde{\Phi}(m), \tag{15}$$

$$\widetilde{\Phi}(m_1 + m_2) = \widetilde{\Phi}(m_1) \oplus \widetilde{\Phi}(m_2), \tag{16}$$

$$\widetilde{\Phi} \text{ is continuous.} \tag{17}$$

Proof of Lemma 3.4. Equality (15) is straightforward. To prove (16), consider $m_1 = \sum_{i\in I_1} \delta_{C_{1,i}}$ and $m_2 = \sum_{i\in I_2} \delta_{C_{2,i}}$. It is obvious that $\widetilde{\Phi}(m_1) \cup \widetilde{\Phi}(m_2) \subset \widetilde{\Phi}(m_1 + m_2)$ so that $\widetilde{\Phi}(m_1) \oplus \widetilde{\Phi}(m_2) \subset g(m_1 + m_2)$. Conversely, the set

$$C = \bigcup_{i\in I_1} C_{1,i} \bigcup_{i\in I_2} C_{2,i} \bigcup \{0\}$$

is such that $\widetilde{\Phi}(m_1 + m_2) = \overline{\text{conv}}(C)$, and we have $C \subset \widetilde{\Phi}(m_1) \oplus \widetilde{\Phi}(m_2)$. As a consequence, $\widetilde{\Phi}(m_1 + m_2) \subset \widetilde{\Phi}(m_1) \oplus \widetilde{\Phi}(m_2)$ and relation (16) holds true.

Finally, we consider the continuity of $\widetilde{\Phi}$. The proof is similar to the proof of Lemma 2.5. Consider a converging sequence $(m_n)_{n\geqslant 0}$ that converges vaguely to m in $\mathcal{M}_p(\mathcal{K}_d)$. Let $\varepsilon > 0$ and denote by m_n^ε (resp. m^ε) the restriction of m_n (resp. m) to B^ε. These are finite point processes on B^ε and $m_n^\varepsilon \xrightarrow{v} m^\varepsilon$ as $n \to \infty$. As a consequence, we can write

$$m_n^\varepsilon = \sum_{i=1}^{k_n} \delta_{C_i^n} \quad \text{and} \quad m^\varepsilon = \sum_{i=1}^{k} \delta_{C_i}$$

with, as $n \to \infty$,

$$k_n \to k \quad \text{and} \quad C_i^n \to C_i, \quad 1 \leqslant i \leqslant k.$$

Note that k_n is equal to k for n large enough. Let n_0 be large enough so that for all $n \geqslant n_0$,

$$k_n = k \quad \text{and} \quad d(x_i^n, x_i) \leqslant \varepsilon, \quad 1 \leqslant i \leqslant k.$$

For fixed $n \geqslant n_0$, denote by A' the set of points in m_n, and by B' the set of points in m. We prove easily that the sets $A = A' \cup \{0\}$ and $B = B' \cup \{0\}$ satisfy the assumption of Lemma 2.1: for example, a given point a in A'

is either in B_ε, and then $d(a,0) \leqslant \varepsilon$, or in B^ε, and then $a = x_i^n$ for some $1 \leqslant i \leqslant k$ and $d(a,x_i) \leqslant \varepsilon$. Using Lemma 2.1, we get that for all $n \geqslant n_0$, $\rho(\widetilde{\Phi}(m_n), \widetilde{\Phi}(m)) \leqslant \varepsilon$. This proves that $\widetilde{\Phi}(m_n) \to \widetilde{\Phi}(m)$ in $\mathcal{K}_d$ as $n \to \infty$ and that the mapping $\widetilde{\Phi}$ is continuous. □

Proof of Theorem 3.3. In view of (13), the definition and properties of $\widetilde{\Phi}$ imply

$$b_n^{-1}\widetilde{W}_n = \widetilde{\Phi}\left(\sum_{k=1}^n \delta_{\pi(X_k)/b_n}\right).$$

Using the fact that $\pi(X) \in \mathrm{RV}_{\alpha,\widetilde{\sigma}}(\mathcal{K}_d)$, Theorem 2.3 implies the convergence of the empirical point processes in $\mathcal{M}_p(\mathcal{K}_d)$:

$$\sum_{k=1}^n \delta_{\pi(X_k)/b_n} \Longrightarrow \Pi_{\alpha,\widetilde{\sigma}} \quad \text{as } n \to \infty$$

with $\Pi_{\alpha,\widetilde{\sigma}}$ a Poisson point process with intensity $m_{\alpha,\widetilde{\sigma}}$. The continuity of $\widetilde{\Phi}$ and the continuous mapping Theorem imply the weak convergence in $\mathcal{K}_d$:

$$b_n^{-1}\widetilde{W}_n \Longrightarrow \widetilde{\Phi}(\Pi_{\alpha,\widetilde{\sigma}}) \quad \text{as } n \to \infty.$$

Finally, it is well known that $\Pi_{\alpha,\widetilde{\sigma}}$ admits the LePage series representation

$$\Pi_{\alpha,\widetilde{\sigma}} \stackrel{d}{=} \sum_{k \geqslant 1} \delta_{\Gamma_k^{-1/\alpha}\widetilde{\varepsilon}_k}$$

with the sequence $(\Gamma_k)_{k\geqslant 1}$ and $(\widetilde{\varepsilon}_k)_{k\geqslant 1}$ as in Theorem 3.3. Using the morphism properties and continuity of $\widetilde{\Phi}$ proved in Lemma 3.4, we deduce that

$$\widetilde{\Phi}(\Pi_{\alpha,\widetilde{\sigma}}) \stackrel{d}{=} \oplus_{k\geqslant 1}\Gamma_k^{-1/\alpha}\widetilde{\varepsilon}_k.$$ □

3.4. Properties of the limit shape $\widetilde{C}_{\alpha,\sigma}$. The limit shape $\widetilde{C}_{\alpha,\sigma}$ is a random element of the convex cone $\mathcal{K}_d$ that enjoys the stability property.

Proposition 3.5. *Let $\widetilde{C}^1_{\alpha,\sigma}$ and $\widetilde{C}^2_{\alpha,\sigma}$ be two independent copies of $\widetilde{C}_{\alpha,\sigma}$. For all $a_1, a_2 > 0$ it is true that*

$$a_1^{1/\alpha}\widetilde{C}^1_{\alpha,\sigma} \oplus a_2^{1/\alpha}\widetilde{C}^1_{\alpha,\sigma} \stackrel{d}{=} (a_1+a_2)^{1/\alpha}\widetilde{C}_{\alpha,\sigma}.$$

The proof of Proposition 3.5 is very similar to the proof of Proposition 2.7 and is omitted.

Interestingly, in many examples, the limit shape $\widetilde{C}_{\alpha,\sigma}$ is a random polytope of $\mathbb{R}^d$, i.e., the convex hull of a finite numbers of points in $\mathbb{R}^d$. This can happen when the spectral measure σ is concentrated on $\mathcal{F}$, the subset of functions $x \in \mathbb{S}_{\mathbb{D}}$ such that

$$x(t) = f(t)s, \quad t \in [0,1],$$

for some $s \in \mathbb{S}_{\mathbb{R}^d}$ and some cád-lág function $f : [0,1] \to [0,1]$ such that $f(0)=0$ and $\sup_{t\in[0,1]} f(t)=1$. Note that the element $s \in \mathbb{S}_{\mathbb{R}^d}$ is uniquely defined and let $\theta : \mathcal{F} \to \mathbb{S}_{\mathbb{R}^d}$ be the measurable function defined by $\theta(x) = s$.

Proposition 3.6. *Suppose that the spectral measure σ is concentrated on $\mathcal{F}$ and let $\overline{\sigma} = \sigma\theta^{-1}$. Let $\{\overline{\varepsilon}_k,\ k \geqslant 1\}$ be i.i.d. random elements on the unit sphere $\mathbb{S}_{\mathbb{R}^d}$ with distribution $\overline{\sigma}$. Then,*

$$\widetilde{C}_{\alpha,\sigma} \overset{d}{=} \oplus_{k\geqslant 1}\{\Gamma_k^{-1/\alpha}\overline{\varepsilon}_k\}.$$

If furthermore the interior of the closed convex hull of the support of $\overline{\sigma}$ contains 0, then $\widetilde{C}_{\alpha,\sigma}$ is a random polytope with 0 as an interior point.

Proof of Proposition 3.6. When $x \in \mathcal{F}$,

$$\pi(x) = \overline{\mathrm{conv}}\{x(t); 0 \leqslant t \leqslant 1\} = \mathrm{conv}\{0, \theta(\mathrm{x})\}.$$

Let ε_k an i.i.d. sequence with distribution σ. The LePage series in Theorem 3.3 can then be rewritten as

$$\widetilde{C}_{\alpha,\sigma} \overset{d}{=} \oplus_{k\geqslant 1}\Gamma_k^{-1/\alpha}\pi(\varepsilon_k) = \oplus_{k\geqslant 1}\Gamma_k^{-1/\alpha}\mathrm{conv}\{0, \theta(\varepsilon_\mathrm{k})\}.$$

Let $\overline{\varepsilon}_k = \theta(\varepsilon_k) \in \mathbb{S}_{\mathbb{R}^d}$. The sequence $\overline{\varepsilon}_k$ is i.i.d. in $\mathbb{S}_{\mathbb{R}^d}$ with distribution $\overline{\sigma}$. Furthermore, $\Gamma_k^{-1/\alpha}\overline{\varepsilon}_k \to 0$ almost surely, so that we can remove the point 0 in the summands and

$$\widetilde{C}_{\alpha,\sigma} \overset{d}{=} \oplus_{k\geqslant 1}\{\Gamma_k^{-1/\alpha}\overline{\varepsilon}_k)\}.$$

Let $C_k = \oplus_{1\leqslant j\leqslant k}\Gamma_k^{-1/\alpha}\{\overline{\varepsilon}_k\}$ be the partial sum. Denote by $\mathrm{Int}(C)$ the interior of $C \subset \mathbb{R}^d$. We first prove that $0 \in \mathrm{Int}(C_k)$ if and only if $0 \in \mathrm{Int}(\oplus_{1\leqslant j\leqslant k}\{\overline{\varepsilon}_j\})$. To see this, note that $0 \notin \mathrm{Int}(C_k)$ if and only if there exist some nonzero linear form $L : \mathbb{R}^d \to \mathbb{R}$ such that $L(\Gamma_j^{-1/\alpha}\overline{\varepsilon}_j) \geqslant 0$ for all $1 \leqslant j \leqslant k$; since the Γ_j's are nonnegative, this is equivalent to $L(\overline{\varepsilon}_j) \geqslant 0$ for all $1 \leqslant j \leqslant k$; hence to the fact that $0 \notin \mathrm{Int}(\oplus_{1\leqslant j\leqslant k}\{\overline{\varepsilon}_j\})$.

Denote by $\mathrm{Supp}\ \overline{\sigma}$ the support of the measure $\overline{\sigma}$. Under the assumption that $0 \in \mathrm{Int}(\overline{\mathrm{conv}}(\mathrm{Supp}\ \overline{\sigma}))$, we prove that $0 \in \mathrm{Int}(C_{k_0})$ for some random integer k_0. To see this, note that the assumption implies that there exists some $\theta_1, \ldots, \theta_l \in \mathrm{Supp}\ \overline{\sigma}$ such that $0 \in \mathrm{Int}(\mathrm{conv}(\theta_1, \ldots, \theta_l))$. There exists also $\delta > 0$ such that for all $\theta_1', \ldots, \theta_l'$ satisfying $\max\{|\theta_j - \theta_j'|; 1 \leqslant j \leqslant l\} \leqslant \delta$, it holds $0 \in \mathrm{Int}(\mathrm{conv}(\theta'_1, \dots, \theta'_l))$. By the definition of the support of $\overline{\sigma}$, there is some integers k_0 such that the set $\{\overline{\varepsilon}_j; 1 \leqslant j \leqslant k_0\}$ intersects all balls of radius δ and centered at θ_j for some $j \in \{1, \dots, l\}$. Then, we have $0 \in \mathrm{Int}(\oplus_{1 \leqslant i \leqslant k_0}\{\overline{\varepsilon}_k\})$, and also $0 \in \mathrm{Int}(C_{k_0})$.

Then, C_{k_0} contains some ball centered at the origin and with radius $\delta > 0$. Since $\Gamma_k^{-1/\alpha}\overline{\varepsilon}_k \to 0$ almost surely, there is some (random) integer $k_1 \geqslant k_0$ such that $\Gamma_k^{-1/\alpha} < \delta$ for all $k \geqslant k_1$; this implies $C_k = C_{k_1}$ for all $k \geqslant k_1$. Hence $\widetilde{C}_{\alpha,\sigma} = C_{k_1}$ is the convex hull of a finite number of points, i.e., a polytope in $\mathbb{R}^d$, and 0 is an interior point. □

§4. Examples

In this section, we recall some examples of regularly varying random processes due to Davis and Mikosch [3] and Hult and Lindskog [7, 9] and illustrate our results in these cases.

4.1. Simple multiplicative process. Consider the $\mathbb{D}$-valued random process

$$X(t) = \eta Y(t), \quad t \in [0,1]$$

where Y is a càd-làg process with values in $\mathbb{R}^d$ and η is a non-negative regularly varying random variable with index $\alpha > 0$, and independent of Y. Assume that one of the following condition is satisfied:

i) $\mathbf{E}[\|Y\|_\infty^{\alpha+\delta}] < \infty$ for some $\delta > 0$;

ii) $\mathbf{E}[\|Y\|_\infty^{\alpha}] < \infty$ and $\mathbf{P}(\eta > x) \sim_{x\to\infty} Cx^{-\alpha}$ for some $C > 0$.

Then, according to Davis and Mikosch [3], Sec. 4.1, the process X is regularly varying on $\mathbb{D}$ with index α and spectral measure given by

$$\sigma(A) = \frac{\mathbf{E}\left[\|Y\|^\alpha 1_{\{\widetilde{Y} \in A\}}\right]}{\mathbf{E}[\|Y\|^\alpha]}, \quad A \in \mathcal{B}(\mathbb{S}_{\mathbb{D}}),$$

where $\widetilde{Y} = Y/\|Y\| \in \mathbb{S}_{\mathbb{D}}$. Theorem 3.3 can be applied. However, in this context, a slightly different formulation is more natural. Let

$$b_n = \inf\{x > 0;\ \mathbf{P}(\eta > x) \leqslant n^{-1}\}$$

and $\widetilde{\varepsilon}_k$ an i.i.d. sequence in $\mathcal{K}_d$ such that

$$\widetilde{\varepsilon}_k \stackrel{d}{=} \overline{\mathrm{conv}}\{Y(t);\ 0 \leqslant t \leqslant 1\}, \quad k \geqslant 1.$$

Then, the renormalized convex hulls $b_n^{-1}\widetilde{W}_n$ converge weakly to the limit shape defined by the LePage series $\oplus_{k\geqslant 1}\Gamma_k^{-1/\alpha}\widetilde{\varepsilon}_k$.

4.2. Regularly varying Lévy processes. Consider a random process X with independent and stationary increments. We suppose that its paths belong to $\mathbb{D}$ a.s. and that for all $t \in [0,1]$ and $u \in \mathbb{R}^d$,

$$\mathbf{E}[e^{i\langle u, X(t)\rangle}] = \exp\left(t \int_{\mathbb{R}^d\setminus\{0\}} (e^{i\langle u,y\rangle} - 1 - i\langle u,y\rangle 1_{\{|y|\leqslant 1\}})\nu(dy)\right),$$

where ν is a Lévy measure on $\mathbb{R}^d \setminus \{0\}$ satisfying

$$\int_{\mathbb{R}^d\setminus\{0\}} (1 \wedge |y|^2)\nu(dy) < \infty.$$

According to Hult and Lindskog [9], Lemma 2.1, X is regularly varying with index α on $\mathbb{D}([0,1],\mathbb{R}^d)$ if and only if ν is regularly varying with index α on $\mathbb{R}^d$; then if μ is the spectral measure on $\mathbb{S}_d$ associated to ν, the spectral measure σ on $\mathbb{S}_{\mathbb{D}}$ associated to X is given by

$$\sigma(B) = \int_0^1 \int_{\mathbb{S}_d} 1_B(y 1_{[t,1]})\mu(dy)\, dt, \quad B \in \mathbb{S}_{\mathbb{D}}.$$

In other words, σ is the distribution of the random function $t \mapsto Y 1_{[T,1]}(t)$, where Y and T are independent random variables, Y with distribution μ on $\mathbb{S}_d$ and T uniform on $[0,1]$. Note that the spectral measure σ has the particular form discussed in Proposition 3.6: it is concentrated on the set of functions $\mathcal{F}$ and in this case $\overline{\sigma} = \mu$. Hence, we can apply Theorems 3.3 and 3.6, so that the renormalized convex hull $b_n^{-1}\widetilde{W_n}$ converges to $\oplus_{k\geqslant 1}\{\Gamma_k^{-1/\alpha}\overline{\varepsilon}_k\}$, where the $\overline{\varepsilon}_k$'s and Γ_k's are as in Proposition 3.6. If furthermore the convex cone generated by the support of the spectral measure μ is equal to $\mathbb{R}^d$, then the limit shape is a random polytope with the origin 0 in its interior.

4.3. Regularly varying Ornstein–Uhlenbeck processes. Consider the cád-lág process X defined by

$$X(t) = \int_0^t e^{-\lambda(t-y)} L(dy), \quad t \in [0,1],$$

where $\lambda > 0$ and L is a regularly varying Lévy process with Lévy measure $\nu \in \mathrm{RV}_{\alpha,\mu}(\mathbb{R}^d)$ (see Sec. 4.2). According to Hult and Lindskog [9], X is regularly varying on $\mathbb{D}([0,1],\mathbb{R}^d)$ with exponent α and its spectral measure σ

is equal to the distribution of the random function $t \mapsto Ye^{-\lambda(t-T)}1_{[T,1]}(t)$, where Y and T are independent random variables with distribution μ and $\mathcal{U}_{[0,1]}$ respectively. Here again, the spectral measure σ is concentrated on the function space $\mathcal{F}$ and $\overline{\sigma} = \mu$. The same conclusions as in Sec. 4.2 hold.

4.4. Symmetric α-stable series. We now consider processes of the form

$$X(t) = \sum_{i=1}^{\infty} r_i \Gamma_i^{-1/\alpha} Y_i(t), \quad t \in [0,1], \tag{18}$$

where $\alpha \in (0,2)$ and the sequences (r_i), (Γ_i) and (Y_i) are independent. The sequence (Γ_i) is as that of usual a nondecreasing enumeration of the points of a Poisson point process with Lebesgue intensity on $\mathbb{R}^+$, the Y_i's are i.i.d. $\mathbb{D}$-valued random elements, and the r_i's are i.i.d. Rademacher random variables, i.e.,

$$\mathbf{P}(r_i = +1) = \mathbf{P}(r_i = -1) = 1/2.$$

We will always assume that the series (18) converges a.s. in $\mathbb{D}$ and discuss several conditions ensuring the convergence of this series, see Rosinsky [15]. In these cases, X is regularly varying with index α and our results apply.

Symmetric α-stable series in $\mathbb{C} = \mathbb{C}([0,1], \mathbb{R}^d)$.
It is known (see, e.g., Corollary 5.3 in Ledoux and Talagrand [16]) that every symmetric α-stable random process X in $\mathbb{C}$ admits a representation in law in the form of a LePage series (18) with $\mathbf{E}[\|Y\|^\alpha] < \infty$. In this case the series converges a.s. in $\mathbb{C}$ and $X \in \mathrm{RV}_{\alpha,\sigma}(\mathbb{C})$ with the spectral measure given by

$$\sigma(A) = \frac{\mathbf{E}\left[\|Y_1\|^\alpha 1_{\{r_1 \widetilde{Y}_1 \in A\}}\right]}{\mathbf{E}[\|Y_1\|^\alpha]}, \quad A \in \mathcal{B}(\mathbb{S}_{\mathbb{C}}),$$

where $\widetilde{Y}_1 = Y_1/\|Y_1\| \in \mathbb{S}_{\mathbb{C}}$. The renormalizing sequence satisfies $b_n \sim \mathbf{E}[\|Y_1\|^\alpha]^{1/\alpha} n^{1/\alpha}$ so that we can apply Theorem 3.3. In these settings however, it is more natural to consider the renormalization $n^{-1/\alpha}\widetilde{W}_n$ and the limit shape $\oplus_{k\geqslant 1}\Gamma_k^{-1/\alpha}\widetilde{\varepsilon}_k$, with $\widetilde{\varepsilon}_k$ and i.i.d. sequence in $\mathcal{K}_d$ with the same distribution as thar of $\overline{\mathrm{conv}}\{r_1 Y_1(t); 0 \leqslant t \leqslant 1\}$.

Symmetric α-stable series in $\mathbb{D} = \mathbb{D}([0,1], \mathbb{R}^d)$.
In the case when $0 < \alpha < 1$ and the Y_i's are $\mathbb{D}$-valued and $0 < \alpha < 1$, the series (18) converges a.s. uniformly in $\mathbb{D}$ if and only if $\mathbf{E}[\|Y_1\|^\alpha] < \infty$ (see

Davis and Mikosch [3]). The case $\alpha \in [1,2)$ is more tricky and only some examples can be found in the literature: for example if $Y_i(t) = 1_{[U_i,1]}(t)$ with U_i an i.i.d. sequence uniformly distributed on $[0,1]$, the series (18) converges a.s. uniformly on $[0,1]$ and the limit process is a symmetric α-stable Lévy process (see [3,15]). A more general criterion for convergence of symmetric α-stable series is provided by Davydov and Dombry [5].

Proposition 4.1. *Let $\alpha \in [1,2)$ and consider $\{Y_i;\ i \geqslant 1\}$ an i.i.d sequence of processes in $\mathbb{D}([0,1],\mathbb{R}^d)$ such that $\mathbf{E}\,[\|Y_1\|^\alpha] < \infty$. Suppose furthermore that there exist a continuous nondecreasing function $F : [0,1] \to [0,\infty)$ and constants $\beta > 1/2$ and $\gamma > 1/2$ such that:*

$$\mathbf{E}[|Y_1(t_2) - Y_1(t_1)|^2] \leqslant |F(t_2) - F(t_1)|^\beta, \quad 0 \leqslant t_1 \leqslant t_2 \leqslant 1,$$

$$\mathbf{E}[|Y_1(t_2) - Y_1(t)|^2 |Y_1(t) - Y_1(t_1)|^2] \leqslant |F(t_2) - F(t_1)|^{2\gamma}, \quad 0 \leqslant t_1 \leqslant t \leqslant t_2 \leqslant 1.$$

Then the random series (18) converges a.s. in $\mathbb{D}([0,1],\mathbb{R}^d)$.

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