Записки научных
семинаров ПОМИ
Том 404, 2012 г.

E. G. Emel'yanov, G. V. Kuz'mina

THE VUORINEN PROBLEM ON THE MAXIMUM OF THE CONFORMAL MODULE

Abstract. A problem on the maximum of the conformal module over a family of doubly connected domains in the unit disk, being a hyperbolic analog of the Teichmüller problem, is solved. The cases when the desired maximum is expressed in elliptic functions are showed.

Problems on extremal values of conformal invariants and its analogs play the important role in the geometric function theory. The classical problem in this direction was the Teichmüller problem on the maximum of the conformal module in the family of all doubly connected domains on the z-sphere, separating given pairs of points of this sphere. Investigations of M. Schiffer, Z. Nehary, L. Ahlfors, H. Wittich and other analysts were devoted to this problem. In completed form the solution of the Teichmüller problem was obtained in [1]. An explicit expression for the desired maximum and some properties of this function were obtained in [1]. Later in the work of A. Yu. Solynin and M. Vuorinen [2] the dependence of the indicated maximum from parameters of the problem was investigated in more detail. In connection with the questions of the quasi-conformal mapping theory, recently Matti Vuorinen raised the following problem. Find the maximum of the conformal module in the family of all doubly connected domains in the disk $|z| < 1$, separating the points z_1, z_2 of the disk $|z| < 1$ from the third point z_3 of this disk and the circle $|z| = 1$. The present work is devoted to solution of this problem.

The Vuorinen problem is connected in naturally way with the Teichmüller problem and some results in the Teichmüller problem are used in investigating of the Vuorinen problem.

§1 of the present work is devoted to a brief account of some results relating to the Teichmüller problem.

Key words and phrases: conformal module, hyperbolic plane, extremal metric problem, quadratic differential, symmetrization. condenser capacity.

§1. Preliminary results: the Teicmüller problem and related results

1.1. Considering the Teichmüller problem it is sufficient to assume that distinguished pairs of points are $-1,1$ and a,∞, where $a\in\mathbb{C}, a\neq -1,1$. The family of all doubly connected domains separating indicated pairs of points is denoted by $\mathcal{D}(a)$, the maximum of the conformal module in this family of domains is denoted by $M(a)$.

Theorem 1.1 [1]. *Let $a\in\mathbb{C},\quad a\neq -1,1$. We have the equality*

$$\log M(a)=\pi\,\mathrm{Im}\,\tau(k^2),\quad k^2=\frac{2}{a+1},\tag{1}$$

where

$$\tau(k^2)=\frac{i\mathbf{K}'(k)}{\mathbf{K}(k)}.$$

Here the elliptic integrals $\mathbf{K}(k)$ and $\mathbf{K}'(k)$ are understood to be the functions that are positive for $k^2\in(0,1)$, defined for $\mathrm{Im}\,k^2\neq 0$ by analytic continuation along any part not intersecting the real axis of the k^2–plane, and defined for $\mathrm{Im}\,k^2=0$ and $k^2\notin[0,1]$ by analytic continuation along any part in the half-plane $\mathrm{Im}\,k^2\leqslant 0$. In the case $a\notin(-1,1)$ this maximum is realized only by the domain $D(a)$, obtained from the z-sphere by making cuts along the closures of the critical trajectories of the quadratic differential

$$Q(z,a)dz^2=-\frac{e^{i\beta(a)}dz^2}{(z^2-1)(z-a)},\quad \beta(a)=-\arg k^2\mathbf{K}^2(k),\tag{2}$$

in the case $a\in(-1,1)$ it is realized by the domain $D(a)$ and the domain symmetric to it with respect to the real axis of the z-plane, and only by these domains.

The closures of the critical trajectories of the differential (1) connecting the points $-1,1$ and a,∞, will be denoted respectively by γ and γ_a.

Let $\mathbf{I}=\{a: a\in\mathbb{C}, \mathrm{Re}\,a\geqslant 0, \mathrm{Im}\,a\geqslant 0\}$. By virtue of symmetric reasons it is sufficient to assume that $a\in\mathbf{I},\quad a\neq 1$.

Corollary 1.1. *Let $a>1$. Then*

$$\log M(a)=\pi\frac{\mathbf{K}'(k)}{\mathbf{K}(k)},\quad k^2=\frac{2}{a+1}.$$

The extremal domain $D(a)$ for this problem is the z-sphere with cuts along the segment $[-1,1]$ and along the half-line $z > a$. The domain $D(a)$ is the ring domain for the quadratic differential

$$Q^{(1)}(z,a)dz^2 = -\frac{dz^2}{(z^2-1)(z-a)}.$$

Corollary 1.2. *Let $a = il, \quad l \geqslant 0$. Then*

$$\log M(a) = \frac{\pi}{2}\frac{\mathbf{K}'(\mu)}{\mathbf{K}(\mu)}, \mu = \sin\omega, \qquad \text{where} \quad \omega = \operatorname{arc\,ctg} l.$$

The extremal domain $D(a)$ is the z-sphere with cuts along the arc of the circle $|z-a| = \sqrt{l^2+1}$ connecting the points $-1, 1$ and lying in the half-plane $\operatorname{Im} z \leqslant 0$ and along the half-line $z = it, t \geqslant l$. In the case $l > 0$ the domain $D(a)$ is the unique extremal domain of the problem, in the case $l = 0$ the extremal domains are $D(0)$ and $D'(0)$, where $D'(0)$ is the domain symmetric to $D(0)$ with respect to the real axes of the z-plane, and only these domains. For $l \geqslant 0$ the domain $D(a)$ is the ring domain for the quadratic differential

$$Q^{(2)}(z,a)dz^2 = -\frac{i dz^2}{(z^2-1)(z-a)},$$

the domain $D'(0)$ is the ring domain for the differential $-Q^{(2)}(z,0)dz^2$.

1.2. The present results of the theory of modules of families of curves and the method of symmetrization reduce to a more simple proof of series of facts related to the Teichmüller problem and are used in investigating of the Vuorinen problem. Note two such results.

The first result is the gradient theorem in the module theory (see, for instance, [3]).

Theorem 1.2. *Let $\mathcal{M}$ be the module of a extremal metric problem, $Q(z)dz^2$ be the associated quadratic differential of this module problem and a be a simple pole of this differential. We have the equality*

$$\arg \operatorname{grad} \mathcal{M}(a) = -\arg \widetilde{Q}(z)|_{z=a},$$

where $\widetilde{Q}(z) = (z-a)Q(z)$.

Theorem 1.2 has the following geometrical meaning: the gradient of the module $\mathcal{M}$, considered as a function in a, is directed along the critical trajectory of the associated quadratic differential outgoing from the point

a. Thus, Theorem 1.2 indicates in which direction the trajectory outgoes from the point a.

In application to the Teichmüller problem we have

Corollary 1.3. *Let $Q(z,a)dz^2$ be the differential* (1), *$M(a)$ be the maximum of the conformal module in the Teichmüller problem. Then*

$$\arg \operatorname{grad} M(a) = \arg \frac{a^2-1}{\beta(a)}. \tag{3}$$

The second mentioned fact is the result of the symmetrization theory relating to the capacity of condensers [4, 5]. Reduce the necessary definitions.

Let α be an arbitrary oriented curve or a circle on $\overline{\mathbb{C}}$ and let $\mathbb{C}_\alpha^-$ and $\mathbb{C}_\alpha^+$ be the closures of the domains on which the curve α divides $\overline{\mathbb{C}}$ (the set $\mathbb{C}_\alpha^-$ lies to the left of α). Let z^* be the point symmetric to the point $z \in \overline{\mathbb{C}}$ with respect to the curve α. Let A be a set on $\overline{\mathbb{C}}$,

$$A^* = \{z : z^* \in A\}, \quad A^- = A \cap \mathbb{C}_\alpha^-, \quad A^+ = A \cap \mathbb{C}_\alpha^+.$$

Set

$$P_\alpha^- = (A \cup A^*)^- \cup (A \cap A^*)^+, \quad P_\alpha^+ A = (A \cup A^*)^+ \cup (A \cap A^*)^-.$$

Theorem 1.3. *Let $C = (E_0, E_1)$ be a condenser on $\overline{\mathbb{C}}$,*

$$P_\alpha C = (P_\alpha^- E_0, P_\alpha^+ E_1).$$

The inequality

$$\operatorname{cap} C \geqslant \operatorname{cap} P_\alpha C. \tag{4}$$

is valid. If C is a condenser with a connected field, then the equality in (4) *takes place in the case when the condenser C either coincides with the condenser $P_\alpha C$, or symmetric to it with respect to the curve α, and only in these cases.*

1.3. It is of interest to investigate the change of $M(a)$ if the point a moves subject to some condition. Here the following lemma turn out to be useful. Its proof follows from the definition of $\beta(a)$ given by Theorem 1.1 and some properties of elliptic integrals.

Lemma 1.1. *Let $a \in \operatorname{Int} \mathbf{I}$ or $a \in (0,1)$. Then*

$$\arg(a + \sqrt{a^2-1} < \beta(a) < \arg \sqrt{a^2-1}.$$

Lemma 1.1 reduces to the proof of monotonicity of change of the value $M(a)$ under moving of the point a along the arcs of circles with the center at the point 1 and the segments of rays outgoing from this point ([1], Corollary 5.4).

Note the following result.

Corollary 1.4. *Let $\mathcal{E}$ be an ellipse with the focuses at the points $-1, 1$, and let $\mathcal{H}$ be a confocal hyperbola. If the point a moves along an arc of an ellipse $\mathcal{E}$ belonging to* $\operatorname{Int}\mathbf{I}$ *or along an arc of a hyperbola $\mathcal{H}$ belonging to the same set so that* $\operatorname{Im} a$ *increases then the value $M(a)$ in Theorem* 1.1 *strictly increases.*

This result was obtained in the work of A. Yu. Solynin [3]. Since this work is difficulty accessible then we give here the proof of this result.

We have the equalities

$$\arg\operatorname{grad} M(a) = \arg\frac{\sqrt{a^2-1}}{e^{i\beta(a)}} + \arg\sqrt{a^2-1}.$$

By virtue of the condition $\beta(a) < \arg\sqrt{a^2-1}$, established by Lemma 1.1, and the obvious inequality $\arg(e^{-i\beta(a)}\sqrt{a^2-1}) < \pi/2$ we find

$$\arg\sqrt{a^2-1} < \arg\operatorname{grad} M(a) < \pi/2 + \arg\sqrt{a^2-1}.$$

Let $\mathcal{H}$ and $\mathcal{E}$ be the hyperbola and the ellipse passing through the point a. The obtained inequalities show that the arc γ_a outgoes from the point a in interior of the angle formed by the directed tangents to $\mathcal{H}$ and $\mathcal{E}$ at the point a, that proves Corollary 1.4.

Let $L(1,a)$ and $L(-1,a)$ be the lines passing through indicated pairs of points. These lines divide the z-plane on the angles with the common vertex at the point a. Denote by $\Delta^+(a)$ and $\Delta^-(a)$ the closures of these from indicated angles which are placed in the half-planes $\operatorname{Im} z > \operatorname{Im} a$ and $\operatorname{Im} z < \operatorname{Im} a$ respectively. In the case $\operatorname{Im} a = 0$ we have $\Delta^+(a) = \{z : \operatorname{Im} z \geqslant 0\}$, $\Delta^-(a) = \{z : \operatorname{Im} z \leqslant 0\}$.

From Theorem 1.3 it follows

Corollary 1.5. *Let $a \in$* $\operatorname{Int}\mathbf{I}$ *or $a \in [0,1)$. We have the conditions*

$$\gamma_a \in \Delta^+(a), \quad \gamma \in \Delta^-(a).$$

Corollary 1.5 reduces to the definition of the homotopic class of curves separating the pairs of points $(-1,1)$ and (a,∞), corresponding to the extremal configuration of the Teichmüller problem. Notice that in the proof of the Theorem 1.1 in [1] the indicated homotopic class is determined on the basis of the properties of the elliptic modular function.

Theorems 1.1, 1.2, and 1.3 are used in §2 of the present work.

§1. The Vuorinen problem

2.1. Let us take as a model for the hyperbolic plane the disc $U_R = \{z : |z| < R\}, \quad R > 1$, with the metric defined by the line element $ds = |dz|/\sqrt{1 - R^{-2}|z|^2}$.

We denote $\mathbf{I}_R = \{z : z \in U_R, \operatorname{Re} z \geqslant 0, \operatorname{Im} z \geqslant 0\}, \quad C_R = \{z : |z| = R\}$.

Considering beforehand, if it is need, the the linear fractional transformation of the disc U_R into itself we will assume that the distinguished points z_1, z_2, z_3 in the disc U_R are $-1, 1, a$, where $a \in \mathbf{I}_R$. In this case the Vuorinen problem is formulated in the following way.

Let $1 < R < \infty$, $a \in \mathbf{I}_R$, $a \neq 1$. Let $\mathcal{D}_R(a)$ be the family of all doubly connected domains in the disc U_R, separating the points $-1, 1$ from the point a and the circle C_R. Find the maximum $M_R(a)$ of the conformal module in the family $\mathcal{D}_R(a)$ and the domains, realizing this maximum, and investigate the properties of $M_R(a)$ as a function of a.

Use following notation. The geodesic in the disc U_R, passing through the points z_1 and z_2, denote by $L_R(z_1, z_2)$. By $(z_1, z_2)_R$ denote the open arc of the geodesics $L_R(z_1, z_2)$ connecting the indicated points and by $[z_1, z_2]_R$ denote its closure.

The geodesics $L_R(-1, a)$ and $L_R(1, a)$ separate the disc U_R into angle domains with the common vertex at the point a. Denote by $\Delta_R^+(a)$ and $\Delta_R^-(a)$ the closures of the domains for which $\operatorname{Im} z > \operatorname{Im} a$ and $\operatorname{Im} z < \operatorname{Im} a$ respectively. The set $\Delta_R^-(a)$ contains the segment $[-1, 1]$. In the case $a \in [-1, 1]$ we have $\Delta_R^+(a) = \{z : z \in U_R, \operatorname{Im} z \geqslant 0\}, \Delta_R^-(a) = \{z : z \in U_R, \operatorname{Im} z \leqslant 0\}$.

We have the following theorem.

Theorem 2.1. *Let $a \in \mathbf{I}_R, \quad a \neq 1$. The maximum $M_R(a)$ of the conformal module in the family $\mathcal{D}_R(a)$ is realized by the ring domain $D_R(a)$ of* *the quadratic differential*

$$Q_R(z,a)dz^2 = -\frac{(z-b)(1-\overline{b}r^2z)dz^2}{(z^2-1)(1-r^4z^2)(z-a)(1-\overline{a}r^2z)}, \quad b \in C_R, \quad r = 1/R \tag{5}$$

belonging to the disc U_R. The structure of trajectories of the differential (5) *is symmetric respect to the circle C_R, being the closure of a critical trajectory of this differential. The points $-1, 1, a$ are poles of first order of the differential* (5) *in the disc U_R, the point b is a zero of second order of this differential, $b \in C_R, \operatorname{Im} b \geqslant 0$. The points $-1, 1$ and a, b are the limiting end points of critical trajectories of the differential* (5)*. The closures of these trajectories denote by Γ and Γ_a respectively. The curve Γ and the set $\Gamma_a \cup C_R$ are the bound continua of the domain $D_R(a)$. If $a \in \operatorname{Int} \mathbf{I}_R$ or $a \in [0,1)$, then the curve Γ belongs to the set $\Delta_R^-(a)$, the curve Γ_a belongs to the set $\Delta_R^+(a)$. If $a \in (1,R)$, then $\Gamma = [-1,1], \Gamma_a = [a,R]$. If $a = ih, 0 \leqslant h < R$, then $\Gamma_a = [a, iR]$.*

In the case $a \notin [0,1)$ the domain $D_R(a)$ is the unique extremal domain of the given problem, in the case $a \in [0,1)$ the extremal ones are the domains $D_R(a)$ and $D'_R(a)$, where $D'_R(a)$ is symmetric to $D_R(a)$ with respect to the real axis of the z-plane, and only these domains.

The zero $b = b(a)$ of the differential (5) is determined by the condition:

$$2\int_{\Gamma} Q_R^{1/2}(z,a)dz = \int_{C_R} Q_R^{1/2}(z,a)dz + 2\int_{\Gamma_a} Q_R^{1/2}(z,a)dz. \tag{6}$$

For the maximum $M_R(a)$ we have the equality

$$\log M_R(a) = \operatorname{Im} \int_{[a,1]} Q_R^{1/2}(z,a)dz \left[2\int_{\Gamma} Q_R^{1/2}(z,a)dz \right]^{-1}. \tag{7}$$

Here, positive values of the integrals in (6) *and* $\operatorname{Im} \int_{[a,1]} Q_R^{1/2}(z,a)dz$ *are considered.*

Proof. The considered problem is a module problem for a family of classes of curves. From the general theory of the method of extremal metric and the symmetry in the condition of the problem it follows that the associated quadratic differential of the problem is a differential of the form (5), the structure of trajectories of this differential possess properties mentioned in the theorem. The differential (5) has simple poles at the points $-1, 1, a$

and the points symmetric to these points with respect to the circle C_R, and a doubly zero at the point b on the circle C_R.

There is denumerable quantity of homotopic classes of curves, separating the points $-1, 1$ from the point a and the circle C_R. From Theorem 1.3 the conditions

$$\Gamma_a \in \Delta_R^+, \quad \Gamma \in \Delta_R^- \tag{8}$$

follow immediately, that determines the homotopic class of curves, corresponding to the extremal configuration of the problem. Considering the integrals of a suitable branch of $Q_R^{1/2}(z,a)$ along the boundary arcs of the domain $D_R(a)$ and along an arc of orthogonal trajectory of the differential (5) and using the Cauchy theorem, we obtain the condition (6) for finding the point b and the equality (7) for the desired maximum. From uniqueness results of the method of extremal metric it follows that in the case $a \notin [0,1)$ the extremal domain of the Vuorinen problem is unique, and in the case $a \in [0,1)$ there are two and only two extremal domains and these domains are symmetric with respect to the real axis of the z-plane. $\square$

Remark 2.1. For the maxima $M(a)$ and $M_R(a)$ of the conformal module in the Teichmüller and Vuorinen problem it is true the limit equality:

$$\lim_{R\to\infty} M_R(a) = M(a). \tag{9}$$

Proof. Let as above Γ and γ be the bound continua of the extremal domains of Theorems 2.1 and 1.1, containing the points $-1, 1$. Evidently, on the curve Γ a point of tangency with trajectories of the differential (1) exists, and on the curve γ a point of tangency with the trajectories of the differential (5)exists. Let z_0 be one of such points of tangency. From the expressions (1) and (5) we obtain the condition (set $b_R = \mathrm{R}e^{i\beta_R(a)}$)

$$-\frac{(z_0-b)(1-\overline{b}z_0/R^2)}{e^{i\beta(a)}(1-z_0^2/R^4)(1-\overline{a}z_0/R^2)}$$
$$= \frac{\mathrm{R}e^{i\beta_R(a)}(1-z_0e^{-i\beta_R(a)}/R)^2}{e^{i\beta_R(a)}(1-z_0^2/R^4)(1-\overline{a}z_0/R^2)} > 0.$$

This condition shows, that for $R \to \infty$

$$e^{i\beta_R(a)} \to e^{i\beta(a)}$$

and for $z \in U_R$

$$R^{-1}Q_R(z,a) \to Q(z,a).$$

From which and the form of metrical conditions, determining $M_R(a)$ and $M(a)$, the equality (9) follows. □

2.3. Theorem 2.1 permits to find the values $M_R(a)$ in the cases $a \in (1, R)$ and $a \in [0, iR)$ in the terms of the elliptic modular function and also to establish a connection of the Vuorinen problem with the problem on the maximum of the conformal module on the z-sphere.

Let

$$f_\beta(z) = \frac{rz}{(1 - e^{-i\beta} rz)^2}, \quad r = 1/R.$$

This function maps the disk U_R on the z-sphere with cut along the half-line $z = te^{i\beta}, t \geqslant 1/4$.

Set $f_\beta(1) = w_1, f_\beta(-1) = w_2$ and let

$$\begin{aligned} F_\beta(z) &= \frac{2}{w_1 - w_2} f_\beta(z) - \frac{w_1 + w_2}{w_1 - w_2} \\ &= \frac{1}{1 + e^{-2i\beta} r^2} \left[\frac{(1 - e^{-2i\beta} r^2)^2 z}{(1 + e^{-i\beta} rz)^2} + 2e^{-i\beta} r \right], \end{aligned}$$

$$F_\beta(1) = 1, \quad F_\beta(-1) = -1.$$

Note two corollaries of Theorem 2.1.

Corollary 2.1. *Let, in the conditions of Theorem* 2.1*,* $a \in (1, R)$*. Then*

$$\log M_R(a) = \pi \frac{K'(k)}{K(k)}, \quad k^2 = 2/(a^{(1)} + 1), \tag{10}$$

where

$$a^{(1)} = \frac{a(1 - r^2)^2 + 2r(1 + ar)^2}{(1 + ar)^2 (1 + r^2)}, \quad r = 1/R.$$

Proof. In the case under consideration the differential (5) has the form (set $a = \rho, \quad r = 1/R$):

$$Q_R^{(1)}(z, a) dz^2 = -\frac{(z - 1/r)(1 - rz) dz^2}{(z^2 - 1)(1 - r^4 z^2)(z - \rho)(1 - \rho r^2 z)}. \tag{11}$$

The domain $D_R(a)$ is the disc U_R with the cuts along the segments $[-1, 1]$ and $[\rho, R]$. By the mapping

$$w = F_0(z) = \frac{1}{1 + r^2} \left[\frac{(1 - r^2) z}{1 + rz)^2} + 2r \right], \quad r = 1/R,$$

the points $-1, 1, a, -R$ are mapped to the points $-1, 1, a^{(1)} = F_0(a), \infty$, the domain $D_R(a)$ is mapped into the domain $D^{(1)}(a^{(1)})$ which is bounded by the segment $[-1, 1]$ and the half-line $[a^{(1)}, \infty]$. As Corollary 1.1 shows, this domain is the extremal one for the Teichmüller problem for the pairs of points $-1, 1$ and $a^{(1)}, \infty$, and we have the equality (10). $\square$

Corollary 2.2. *Let $a = ih \in [0, iR)$. Then*

$$\log M_R(a) = \frac{\pi}{2}\frac{K'(\sin\omega)}{K(\sin\omega)}, \quad \operatorname{ctg} 2\omega = l, \quad 0 < \omega < \pi/2, \tag{12}$$

where

$$l = \frac{1}{1-r^2}\left[\frac{(1+r^2)^2 h}{(1+rh)^2} - 2r\right].$$

Proof. For considered values of a the differential (5) has the form (set $a = ih, 0 \leqslant h < R = 1/r$):

$$Q_R^{(2)}(z,a)dz^2 = -\frac{(z-i/r)(1+irz)dz^2}{(z^2-1)(1-r^4z^2)(z-ih)(1+ihr^2z)}. \tag{13}$$

The domain $D_R(a)$ is the disc U_R with the cuts along the segment $\Gamma_a = \{z = it, h \leqslant t \leqslant R\}$ and the closure Γ of the trajectory of the differential (13) with the ends at the points $-1, 1$. Under the mapping

$$w = F_{\pi/2} = \frac{1}{1-r^2}\left[\frac{(1+r^2)^2 z}{(1-irz)^2} - 2ir\right]$$

the points $-1, 1, ih, -iR$ are mapped to the points $1, -1, il, \infty$, the domain $D_R(a)$ is mapped into the domain $D^{(2)}(il)$. This domain is bounded by the half-line $\{w = it, l \leqslant t\}$ and the arc of the circle $|w - il| = \sqrt{l^2+1}$ connecting the points -1,1 and lying in the half-plane $\operatorname{Im} w \leqslant 0$. The bound continua of the domain $D^{(2)}(il)$ are the closures of the critical trajectories of the quadratic differential

$$\frac{idw^2}{(w^2-1)(w-il)}.$$

By virtue of Corollary 1.2, in the case $l \geqslant 0$ the domain $D^{(2)}(il)$ realizes the maximum $M(il)$ in the Teihm̈uller problem. Using the expression (7) for the maximum in this problem, taking into account the equality

$$k'^2 = 1 - 2/(1+il) = \frac{1+il}{1-il} = e^{4i\omega},$$

where $\operatorname{ctg} 2\omega = l, \quad 0 < \omega \leqslant \pi/4$, and considering the substitution

$$\mu = \frac{1+k'}{2\sqrt{k'}} = \cos\omega,$$

we come to the assertion of Corollary 2.2 in the case $l \geqslant 0$.

Let $l < 0$. Let $\widetilde{D}^{(2)}(il)$ is the "complement" domain for $D^{(2)}(il)$: the domain $\widetilde{D}^{(2)}(il)$ is bounded by the half-line $w = it$, $t \leqslant l$, and the arc of the circle $|w - il| = \sqrt{l^2+1}$, connecting the points $-1, 1$ and lying in the half-plane $\operatorname{Im} w \geqslant 0$. The boundary continua of the domain $\widetilde{D}^{(2)}(il)$ are the closures of the critical orthogonal trajectories of the differential (13). By virtue of the known relation (see, for instance, [3]), we have the equality

$$\log \operatorname{mod} D^{(2)}(il) + \log \operatorname{mod} \widetilde{D}^{(2)}(il) = 0.$$

This shows that in the case $l < 0$ the equality (12) is true with $\pi/4 < \omega < \pi/2$. □

Remark 2.2. Let $a \in \operatorname{Int} \mathbf{I}_R$ or $a \in [0, 1)$. Let $D_R(a)$ be the extremal domain of the Vuorinen problem, $b = \mathrm{R}e^{i\beta_R(a)}$ be the zero of the differential (5). Let $D^{(3)}$ be the image of $D_R(a)$ by the mapping (9), $a^{(3)} = F_\beta(a)$, where $\beta = \beta_R(a)$. The domain $D^{(3)}$ belongs to the family $\mathcal{D}(a^{(3)})$ of all doubly connected domains on the w-sphere, separating the pairs of points $-1, 1$ and $a^{(3)}, \infty$. From easy geometrical properties of the domain $D_R(a)$ stated by Theorem 2.1 it follows that $M_R(a) < M(a^{(3)})$, where $M(a^{(3)})$ is the maximum for the Teichmüller problem in the family $\mathcal{D}(a^{(3)})$.

Remark 2.3. In the cases $a \in \operatorname{Int} \mathbf{I_R}$ or $a \in [0, 1)$ the value $M_R(a)$ is determined by enough complicated condition. Theorem 2.1 can be reformulated in the terms of elliptic functions. For instance, under the transformation

$$z = sn(u, k), \quad k = r^2,$$

the disk U_R with the slits $[-R, -1]$ and $[1, R]$ pass into the quadrangle with the vertexes $\pm\mathbf{K}(k) \pm i\mathbf{K}'(k)/2$, and the differential (5) pass in the differential

$$\widetilde{Q}(u, a)du^2 = -\frac{(sn(u,k) - b)(1 - \overline{b}r^2 sn(u,k))du^2}{(sn(u,k) - a)(1 - \overline{a}r^2 sn(u,k))}.$$

By this way we obtain for finding of the maximum $M_R(a)$ of Theorem 2.1 rapidly convergent power series in u.

In the sequel, we continue to consider the configurations on the z-sphere.

2.4. Investigate the character of change of the value $M_R(a)$ under moving of the point a along suitable hyperbolic ellipses and hyperbolic hyperbolas. Give necessary definitions.

As usually, as the hyperbolic distance between the points z_1, z_2 of the disc U_R we understand the quantity

$$\rho_R(z_1, z_2) = \frac{1}{2} R \log \frac{1 + R^{-1}\rho_R^*(z_1, z_2)}{1 - R^{-1}\rho_R^*(z_1, z_2)},$$

where

$$\rho_R^*(z_1, z_2) = \left| \frac{z_1 - z_2}{1 - R^{-2}\overline{z}_1, z_2} \right|.$$

A hyperbolic ellipse (h.ellipse) $\mathcal{E}_R$ in the disc U_R with the focuses at the points $-1, 1$ is determined by the condition

$$\mathcal{E}_R = \{z : \rho_R(z, 1) + \rho_R(z, -1)\} = 2l_1,$$

a confocal hyperbolic hyperbola (h.hyperbola) $\mathcal{H}_R$ is determined by the condition

$$\mathcal{H}_R = \{z : \rho_R(z, 1) - \rho_R(z, -1)\} = 2l_2$$

(l_1, l_2 are some positive constants).

Indicated h.ellipses and h.hyperbolas are trajectories and orthogonal trajectories respectively of the quadratic differential

$$-\frac{dz^2}{(z^2 - 1)(1 - R^{-4}z^2)}.$$

Every h.ellipse $\mathcal{E}_R$ is characterized by the following property: the argument of the outer normal to $\mathcal{E}_R$ at a point $z_0 \in \mathcal{E}_R$ is equal to $1/2 \arg\{(z_0^2 - 1)(1 - R^{-4}\bar{z}_0^2)\}$. Every h.hyperbola has an analogous property.

We need the following two lemmas which supplement a information about the geometric properties of the extremal domain $D_R(a)$ of Theorem 2.1.

Lemma 2.1. *Let $a = \rho e^{it}$, where $0 < t < \pi/2$. We have the inequalities*

$$t < \beta_R(a) < \pi/2.$$

Proof. 1) Assume that $\beta_R(a) < t$. As $\beta_R(a) > t$ for $R = \infty$, then it exists some $R < \infty$, for which $\beta_R(a) = t$.

The condition of the tangency of a trajectory of the differential (5) with a hyperbolic hyperbola at the point $z \in U_R$ is the equality

$$\frac{(z-b)(1-\overline{b}R^{-2}z)}{(z-a)(1-\overline{a}R^{-2}z)} = -x, \quad x > 0. \tag{14}$$

From (14) we obtain

$$-(\overline{b}+x\overline{a})R^{-2}z^2 + z[2+x(1+\rho^2R^{-2})] - (b+xa) = 0.$$

In our case

$$b + xa = (R + x\rho)e^{it}$$

and, setting $z_1 = Rz$, we come to the equation

$$z_1^2 - z_1\frac{2+x(1+\rho^2R^{-2})}{R+x\rho}Re^{it} + e^{2it} = 0.$$

The roots of this equation are z_1^+ and z_2^-, where

$$\arg z_1^+ = \arg z_1^- = t.$$

This shows, that the trajectory of the differential (5), connecting the points a and b, is a segment. Since $0 < t < \pi/2$, then latter is impossible. Consequently, $\beta_R(a) > t$ for all R.

2) Show that $\beta_R(a) < \pi/2$. Assume that $\beta_R(a) \geqslant \pi/2$. Since $\beta_R(a) < \pi/2$ for $R = \infty$, then it exists the value $R < \infty$, for which $\beta_R(a) = \pi/2$. In this case the condition of tangency of a trajectory of the differential (5) with the imaginary axis of z-plane at the point z_0 is the inequality

$$\frac{(z_0 - iR)(1+iz_0R^{-1})}{(z_0-\rho e^{it})(1-\rho e^{-it}R^{-2}z_0)} < 0. \tag{15}$$

Setting $z_0 = ih$, rewrite (15) in the form

$$\frac{i(h-R)(1-hR^{-1})}{(ih-\rho e^{-it})(1-\rho hR^{-2}e^{-it})} < 0. \tag{16}$$

We have

$$(ih-\rho e^{it})(1-i\rho hR^{-2}e^{-it}) = i[h(1+\rho^2R^{-2}-\rho\sin t(1-h^2R^{-2})]-\rho\cos t(1-h^2R^{-2}).$$

This contradicts the inequality (16). Consequently, the inequality (16)is impossible and therefore $\beta_R(a) < \pi/2$ for $t < \pi/2$. □

Remark 2.4. The inequality (13) is unimprovable in the following sense. For $R \to 1$ the differential (5) tends to a quadratic differential having double poles at the points $-1, 1$ and $\arg b = \beta_R(a) \to t = \arg a$.

Lemma 2.2. *Let $a = \rho e^{it}, 0 < t < \pi/2$, be determined in Theorem 2.1. The set of the points of tangency of trajectories of the differential (5) with hyperbolic hyperbolas in the disc U_R is the arc $(a, b)_R$ of the geodesic in U_R, having its ends at the points a and b. For $z \in (a, b)_R$ we have the inequality*

$$\arg a < \arg z < \arg b.$$

Proof. Let z be a point of tangency of a trajectory of differential (5) with a hyperbolic hyperbola in the disc U_R. Then we have the condition (15), where x is a positive number. From the form of the condition (15) it follows, that the points z, satisfying this condition, fill the arc of the geodesic $(a, b)_R$. Setting $z_1 = R^{-1}z$, we obtain from (15) the equality

$$z_1^2 - z_1 R \frac{2 + x(1 + \rho^2/R^2)}{\overline{b} + x\overline{a}} + \frac{b + xa}{\overline{b} + x\overline{a}} = 0.$$

The roots of the last equation are the values $z_1(1) \in U_1$ and $z_1^{(2)} = 1/\overline{z}_1^{(1)}$, where

$$\arg z_1^{(1)} = \arg z_1^{(2)} = \arg(b + xa).$$

Evidently, $\arg(b + xa)$ decreases from $\arg b$ to $\arg a$ if x increases from 0 to ∞. Considering the values $z^{(1)} = Rz_1^{(1)}$ and $z^{(2)} = Rz_1^{(2)} = R^2/\overline{z}^{(1)}$, we come to the assertion of Lemma 2.3. □

From lemmas 2.1 and 2.2 it follows

Theorem 2.2. *The maximum $M_R(a)$ of Theorem 2.1 strictly increases if the point a moves along an arc of a hyperbolic ellipse $\mathcal{E}_R$ belonging to $\mathbf{I}_R$ and if the point a moves along an arc of a hyperbolic hyperbola $\mathcal{H}_R$ belonging to the same set, so that Ima increases.*

Proof. Let $a = \rho e^{it}$, $0 < t < \pi/2$. Let $\mathcal{H}_R(a)$ be the h.hyperbola in the disc U_R, passing across the point a. Evidently, $\mathcal{H}_R(a)$ is directed at the point a along the normal to the h.ellipse $\mathcal{E}_R(a)$, passing across the point a. As before, let Γ_a be the closure of the trajectory of the differential (5) connecting the points a and b. The curve Γ_a can not coincide with a arc of $\mathcal{H}_R(a)$. Assume, that the curve Γ_a outgoes from the point a to the right of the h.hyperbola $\mathcal{H}_R(a)$. Since Γ_a has its end at the point b and $\arg b > \arg a$, then the curve Γ_a has the point of intersection with $\mathcal{H}_R(a)$ and therefore on the curve $\mathcal{H}_R(a)$ it exists at least one point of tangency with some trajectory of the differential (5), lying to the left of

$\mathcal{H}_R(a)$. Since, by Lemma 2.2, all points of tangency of trajectories of the differential (5) with h.hyperbolas $\mathcal{H}_R$ lie on the arc of geodesic $(a, b)_R$, disposed to the left of the curve $\mathcal{H}_R(a)$, then we come to a contradiction. This shows, that the curve Γ_a outgoes from the point a to the left of the h.hyperbola $\mathcal{H}_R(a)$. Further, the first condition (8)shows that the curve Γ_a outgoes from the point a to the above of the h.ellipse $\mathcal{E}_R(a)$. Now, both assertions of Theorem 2.2 follow from Gradient Theorem 1.2. $\square$

Remark 2.5. Estimates for $\beta_R(a)$ reduced above can be corrected. With the help of such estimates the more complete description of properties of the function $M_R(a)$ can be obtained.

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С.-Петербургский университет
экономики и финансов,
Садовая ул., 21,
191023 С.-Петербург, Россия
E-mail: emelyanoveg@rambler.ru

Поступило 25 июня 2012 г.

С.-Петербургское отделение
Математического института
им. В. А. Стеклова РАН
Фонтанка, 27, 191023 С.-Петербург, Россия
E-mail: kuzmina@pdmi.ras.ru