Записки научных
семинаров ПОМИ
Том 404, 2012 г.

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# ON INTERACTION OF SYMPLECTIC AND ORTHOGONAL HECKE-SHIMURA RINGS OF ONE-CLASS QUADRATIC FORMS

Abstract. Transformation formulas of theta-series with harmonic polynomials of one-class quadratic forms under Hecke operators are interpreted as a result of interaction of standard representation of symplectic Hecke–Shimura ring on theta-series with natural representation of orthogonal Hecke–Shimura ring on the same theta-series considered as invariants of quadratic forms. Properties of the interaction maps and their relations with action of Hecke operators are considered.

## Introduction

A number of interesting number-theoretic questions arise from study of interaction of various representations of Hecke–Shimura rings of arithmetical discrete subgroups of Lie groups on spaces of automorphic forms. Important examples of so to say "vertical" interaction are given by lifts of automorphic structures to similar groups of higher orders (see, e.g., [4]). Not less, if not more, interesting are cases of "horizontal" interaction arising from consideration of Hecke–Shimura rings of different Lie groups, say, symplectic and orthogonal (see, e.g., [6]).

In general, an automorphic structure on a Lie group consists of a Hecke–Shimura ring of an arithmetical discrete subgroup of the group and a linear representation of the ring on a space of automorphic forms given by Hecke operators. An interaction of one automorphic structure to automorphic structure on another group is formed by an interaction mapping of Hecke–Shimura rings for the discrete subgroups compatible with a mapping of spaces of automorphic forms, where the Hecke–Shimura rings act by Hecke operators.

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*Key words and phrases*: Hecke–Shimura rings, Hecke operators, interaction mappings, interaction sums, modular forms, theta-series of integral quadratic forms.

The work was partially supported by Russian Fund of Fundamental Researches (RFFI), Grant 11-01-00239.

Interaction mapping of Hecke–Shimura rings for certain subgroups of integral symplectic groups and groups of units of integral quadratic forms in even number of variables was constructed in our paper [9]. Here we consider a different case of interaction for Hecke operators acting on holomorphic theta-series of one-class positive definite quadratic forms with spherical polynomials.

**Notation.** We fix the letters $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{C}$, as usual, for the ring of rational integers, the field of rational numbers, and the field of complex numbers, respectively.

If $\mathbb{A}$ is a set, $\mathbb{A}_n^m$ denotes the set of all $m \times n$-matrices with elements in $\mathbb{A}$. If $\mathbb{A}$ is a ring with the identity element, $1_n$ and $0_n$ is identity and zero element of the ring $\mathbb{A}_n^n$ respectively.

The transpose of a matrix M is denoted by ${}^tM$. For two matrices $Q$ and $N$ of appropriate size we write

$$Q[N] = {}^tNQN.$$

For a complex square matrix $A$ we use the notation

$$\mathbf{e}\{A\} = \exp(\pi\sqrt{-1}\sigma(A)),$$

where $\sigma(A)$ is the sum of diagonal entries of $A$.

## §1. Harmonic theta-series, symplectic Hecke–Shimura rings and Hecke operators

For a real positive definite quadratic form $\mathbf{q}$ in $m$ variables with matrix $Q$ let $P = P(X)$ be a harmonic polynomial with respect to $\mathbf{q}$ of genus $n \geqslant 1$ and weight $k$ in entries of variable $m \times n$–matrix $X = (x_{ij})$, i.e. a polynomial of the form

$$P(X) = P_0(SX),$$

where $S$ is a real $m \times m$–matrix such that $Q = 2\,{}^tSS$ and $P_0(X) = P_0((x_{ij}))$ is a polynomial solution of the equation

$$\Delta P_0 = \sum_{i,j} \frac{\partial^2 P_0}{\partial x_{ij}^2} = 0$$

satisfying the homogeneity condition

$$P(XA) = (\det A)^k P(X) \quad \text{for every matrix } A \in GL_n(\mathbb{C}),$$

(see, e.g., [5, §2] or [6, §3]). The *theta-series of genus* $n$ *of the form* $\mathbf{q}$ *with polynomial* $P$ is defined as a function in variable

$$Z \in \mathbb{H}^n = \{Z = X + \sqrt{-1}Y \in \mathbb{C}_n^n \mid {}^tZ = Z, Y > 0\},$$

(the *Siegel upper half-plane of genus* $n$), given by the series

$$\Theta^n(Z, P, Q) = \sum_{N \in \mathbb{Z}_n^m} P(N)\mathbf{e}\{Q[N]Z\}. \tag{1.1}$$

The series converges absolutely and uniformly on any subset of the upper half-plane of the form $\{Z = X + \sqrt{-1}Y \in \mathbb{H}^n \mid Y \geqslant \varepsilon 1_n\}$ with $\varepsilon > 0$ and so defines an analytic function in $Z \in \mathbb{H}^n$.

Suppose that $\mathbf{q}$ is an integral quadratic form. Then the matrix $Q$ of $\mathbf{q}$ is an *even matrix*, i.e., an integral symmetric matrix with even diagonal entries. Let $q$ be the *level* of $\mathbf{q}$, i.e., the least positive integer such that the matrix $qQ^{-1}$ is even. On integral quadratic forms see, e.g., [8].

According to [2, Theorems 4.1–4.3], the theta-series (1.1) of genus $n \geqslant 1$ of an integral quadratic $\mathbf{q}$ in even number $m$ of variables with a harmonic polynomial $P$ of weight $k$ satisfies the functional equation

$$\Theta\left(M\langle Z\rangle, P, Q\right) = j(M, Z)\Theta(Z, P, Q), \tag{1.2}$$

for each matrix $M$ of the group

$$\Gamma_0^n(q) = \left\{M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp_n(\mathbb{Z}) | \quad C \equiv 0 \pmod{q}\right\},$$

where, for $M = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$,

$$M\langle Z\rangle = (AZ + B)(CZ + D)^{-1}$$

and

$$j(M, Z) = j_{Q,k}(M, Z) = \chi_Q(\det D)(\det(CZ + D))^{m/2+k}, \tag{1.3}$$

and where $\chi_Q = \chi_{\mathbf{q}}$ is the *character of the quadratic form* $\mathbf{q}$.

Now, let us introduce the multiplicative semigroups

$$\Sigma_0^n(q) =$$
$$= \left\{M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathbb{Z}_{2n}^{2n} \;\middle|\; {}^tM \begin{pmatrix} 0_n & 1_n \\ -1_n & 0_n \end{pmatrix} M = \mu(M) \begin{pmatrix} 0_n & 1_n \\ -1_n & 0_n \end{pmatrix}, \right.$$
$$\left. C \equiv 0_n \pmod{q}, \mu(M) > 0, \gcd(\mu(M), q) = 1\right\}$$

for $q \geqslant 1$. The group $\Gamma_0^n(q)$ can be considered as a subgroup of $\Sigma_0^n(q)$ consisting of matrices $M \in \Sigma_0^n(q)$ with the *multiplier* $\mu(M) = 1$. Let

$$\mathcal{H}_0^n(q) = \mathcal{H}(\Gamma_0^n(q),\, \Sigma_0^n(q))$$

be the Hecke–Shimura ring (over $\mathbb{C}$) of the semigroup $\Sigma_0^n(q)$ relative to the subgroup $\Gamma_0^n(q)$. Here we only note that this ring consists of all those finite formal linear combinations $T$ with complex coefficients of symbols $(\Gamma_0^n(q)M)$, corresponding in one-to-one way to different left cosets $\Gamma_0^n(q)M \subset \Sigma_0^n(q)$, which are invariant with respect to all right multiplication by elements of $\Gamma_0^n(q)$:

$$T = \sum_\alpha c_\alpha(\Gamma_0^n(q)\, M_\alpha) \in \mathcal{H}_0^n(q), \iff TM = \sum_\alpha c_\alpha(\Gamma_0^n(q)\, M_\alpha M) = T \tag{1.4}$$

for all $M \in \Gamma_0^n(q)$. The semigroup $\Sigma_0^n(q)$ operates on the space $\mathcal{A}^n$ of all analytic functions on $\mathcal{H}^n$ by *Petersson operators*

$$\Sigma_0^n(q) \ni M : F = F(Z) \mapsto F|M = F|_{\mathbf{j}}M = j_{Q,k}(M,\, Z)^{-1}F(M\langle Z\rangle), \tag{1.5}$$

where $j_{Q,k}(M,\, Z)$ is the automorphic factor (1.3). Petersson operators map the space $\mathcal{A}^n$ into itself and satisfy the rule $F|M|M' = F|MM'$. It allows one to define the standard representation $T \mapsto |T = |_{\mathbf{j}}T$ of the Hecke–Shimura ring $\mathcal{H}_0^n(q)$ on the subspace

$$\mathfrak{M}_{\mathbf{j}}^n(q) = \{F \in \mathcal{A}^n \mid F|_{\mathbf{j}}M = F \ \text{ for all } \ M \in \Gamma_0^n(q)\} \tag{1.6}$$

of all $\Gamma_0^n(q)-$invariant functions of $\mathcal{A}^n$ by *Hecke operators*: the Hecke operator $|T = |_{\mathbf{j}}T$ corresponding to an element of $\mathcal{H}_0^n(q)$ of the form (1.4) is defined by

$$F|T = \sum_\alpha c_\alpha F|M_\alpha \qquad \big(F = F(Z) \in \mathfrak{M}_{\mathbf{j}}^n(q)\big), \tag{1.7}$$

where $|M_\alpha$ are the Petersson operators (1.5). The Hecke operators are independent of a choice of representatives $M_\alpha \in \Gamma_0^n(q)M_\alpha$ and map the space $\mathfrak{M}_{\mathbf{j}}^n(q)$ into itself. The map $T \mapsto |T$ is a linear representation of the ring $\mathcal{H}_0^n(q)$ on the space $\mathfrak{M}_{\mathbf{j}}^n(q)$.

Theta-series of different genuses $n$ of a fixed quadratic form are related by *Siegel operators* $\Phi^{n,r} : \mathcal{A}^n \mapsto \mathcal{A}^r$, where $0 \leqslant r \leqslant n$, whereas the action of Hecke operators on the spaces are related by *Zharkovskaya homomorphisms* $\Psi_Q^{n,r} : \mathcal{H}_0^n(q) \mapsto \mathcal{H}_0^r(q)$ of Hecke–Shimura rings. For definition and properties of the mappings see [7, §4]. Here we shall only note that the

Zharkovskaya homomorphism $\Psi_Q^{n,r}$ is not always epimorphic, but it is epimorphic if $n, r \geqslant m/2$ [3, Proposition 3.3].

The functional equations (1.2) show that theta-series $\Theta^n(Z, P, Q)$, viewed as functions of $Z$, belong to the space $\mathfrak{M}_{\mathbf{j}}^n(q)$. Explicit formulas for the action of Hecke operators on theta-series show that images of the theta-series under Hecke operators can be often written as finite linear combinations with constant coefficients of similar theta-series. According to results of papers [1, Theorem 1] and [7, Propositions 5.1, 5.2(2)], for each homogeneous element $T \in \mathcal{H}_0^n(q)$ of a multiplier $\mu$ (i.e., a linear combination of left cosets $(\Gamma_0^n(q)\, M_\alpha)$ with a fixed multiplier $\mu(\Gamma_0^n(q)\, M_\alpha) = \mu(M_\alpha) = \mu$), which if $n < m$ belongs to the image of the ring $\mathcal{H}_0^m(q)$ under the Zharkovskaya map $\Psi_Q^{m,n} : \mathcal{H}_0^m(q) \mapsto \mathcal{H}_0^n(q)$, the image of the theta-series (1.1) under the Hecke operator $|T$ can be written as a linear combination with constant coefficients of similar theta-series in the form

$$\Theta^n(Z, P, Q)|T = \sum_{D \in A(Q,\mu)/\Lambda} I(D, Q, \Psi_Q^{n,m}T)\Theta^n(Z, P|\mu^{-1}D, \mu^{-1}Q[D]), \tag{1.8}$$

where

$$A(Q,\mu) = \left\{ D \in \mathbb{Z}_m^m \;\middle|\; \mu^{-1}Q[D] \text{ is even and } \det \mu^{-1}Q[D] = \det Q \right\}, \tag{1.9}$$

is the set of all *automorphes of* $Q$ *with multiplier* $\mu$, $\Lambda = GL_m(\mathbb{Z})$, and the right hand part is zero if the set $A(Q,\mu)$ is empty. The element $\Psi_Q^{n,m}T \in \mathcal{H}_0^m(q)$ is image of $T$ under the Zharkovskaya map if $n \geqslant m$ and an inverse image of $T$ if $n < m$, and $P|\mu^{-1}D = P(\mu^{-1}DX)$. In addition, coefficients $I(D, Q, \Psi_Q^{n,m}T)$ on the right are *interaction sums* defined for elements

$$\mathbf{T} = \sum_\alpha c_\alpha \left(\Gamma_0^m(q) N_\alpha\right) \subset \mathcal{H}_0^m(q) \tag{1.10}$$

written with "triangular" representatives $N_\alpha = \left(\begin{smallmatrix} A_\alpha & B_\alpha \\ 0_m & D_\alpha \end{smallmatrix}\right)$ by

$$I(D, Q, \mathbf{T}) = \sum_{\alpha,\, D\,{}^tD_\alpha \equiv 0 \pmod{\mu}} c_\alpha \chi_Q(|\det D_\alpha|)|\det D_\alpha|^{-m/2}\mathbf{e}\{\mu^{-2}Q[D]\cdot {}^tD_\alpha B_\alpha\}. \tag{1.11}$$

Note that the interaction sums satisfy the relations

$$I(MDN,\,Q,\,\mathbf{T}) = I(D,\,Q[M],\,\mathbf{T}) \qquad (\forall\ \mathbf{T} \in \mathcal{H}_0^m(q),\ M,\,N \in \Lambda^m). \tag{1.12}$$

## §2. Orthogonal Hecke–Shimura rings

Let $Q$ be the matrix of an integral positive definite quadratic form $\mathbf{q}$ in $m$ variables. Denote by $\mathcal{P}_k^n(Q)$ the set of all harmonic polynomials of genus $n$ and weight $k$ with respect to $Q$ (see [5, §2]). According to [5, Proposition 2.3], the space $\mathcal{P}_k^n(Q)$ is spanned over $\mathbb{C}$ by the polynomials

$$P(X) = \det({}^t\Omega QX)^k,$$

where $\Omega$ is a matrix of $\mathbb{C}_n^m$ satisfying ${}^t\Omega Q\Omega = 0$ if $k > 1$. The general linear group $GL_m(\mathbb{R})$ operates on functions $P : \mathbb{R}_n^m \mapsto \mathbb{C}$ by linear transformations of variables

$$U \mapsto |U\ :\ P(X) \mapsto (P|U)(X) = P(UX) \quad (U \in GL_m(\mathbb{R})), \tag{2.1}$$

and the operators satisfy $|U|V = |UV$ if $U,\,V \in GL_m(\mathbb{R})$. Each of the operators $|U$ with $U \in GL_m(\mathbb{R})$ maps the space $\mathcal{P}_k^n(Q)$ bijectively onto the space $\mathcal{P}_k^n(Q[U])$ [5, Lemma 2.1]. In particular, the operators $|U$ maps the space $\mathcal{P}_k^n(Q)$ onto itself if

$$U \in E(Q) = \{U \in GL_m(\mathbb{Z}) \mid Q[U] = Q\} \tag{2.2}$$

(*the group of units of* $Q$). Let now $\mathcal{I}_k^n(Q)$ be the subspace of all polynomials in $\mathcal{P}_k^n(Q)$, which are invariant with respect to all operators $|U$ of the form (2.1) with $U \in E(Q)$,

$$\mathcal{I}_k^n(Q) = \left\{P \in \mathcal{P}_k^n(Q)\ \middle|\ P(UX) = P(X) \quad \text{for all } U \in E(Q)\right\}. \tag{2.3}$$

On the other hand, let $\mathcal{H}(Q)$ be the (*regular*) *Hecke–Shimura ring of the quadratic form with matrix* $Q$ i.e. the *ring of* (*regular*) *classes of automorphes* of this quadratic form. The ring $\mathcal{H}(Q)$ consists of all finite formal linear combinations with integral coefficients of symbols $(E(Q)DE(Q)) = \tau(D)$ corresponding in one-to-one way to double cosets of the semigroup

$$\mathbf{A}(Q) = \bigcup_{\mu \geqslant 1,\,\gcd(\mu,q)=1} \mathbf{A}(Q,\,\mu), \tag{2.4}$$

where

$$\mathbf{A}(Q,\,\mu) = \{D \in \mathbb{Z}_m^m \mid {}^tDQD = \mu Q\}, \tag{2.5}$$

of regular automorphes of $\mathbf{q}$ modulo the group of units $E(Q)$. By writing symbols $(E(Q)DE(Q))$ corresponding to double coset as sums of left cosets modulo $E(Q)$ it contains, one can present each element of $\mathcal{H}(Q)$ as a formal sum of left cosets modulo the group $E(Q)$,

$$\tau = \sum_{\alpha} a_\alpha (E(Q)D_\alpha) \in \mathcal{H}(Q),$$

invariant with respect to all right multiplications by elements of $E(Q)$:

$$\tau U = \sum_{\alpha} a_\alpha (E(Q)D_\alpha U) = \sum_{\alpha} a_\alpha (E(Q)D_\alpha) = \tau \quad (\forall \quad U \in E(Q).$$

Then the *Hecke operator* $\circ\tau$ *on* $\mathcal{I}_k^n(Q)$ corresponding to the element $\tau$ is defined by

$$\circ\tau : P \mapsto P \circ \tau = \sum_{\alpha} a_\alpha P|D_\alpha \qquad (\tau \in \mathcal{H}(Q),\ P \in \mathcal{I}_k^n(Q)), \tag{2.6}$$

where $|D_\alpha$ are the operators (2.1). It follows from definitions of the space $\mathcal{I}_k^n(Q)$ and the ring $\mathcal{H}(Q)$ that each Hecke operator is independent of the choice of representatives in the corresponding left cosets and maps this space into itself. Moreover, it follows from (2.6) and the definition of multiplication in $\mathcal{H}(Q)$ that a product of elements in $\mathcal{H}(Q)$ goes to the product of corresponding operators. Thus, the mapping $\tau \mapsto \circ\tau$ is a linear representation of the ring $\mathcal{H}(Q)$ on the space $\mathcal{I}_k^n(Q)$.

## §3. Interaction mappings

In this section we shall assume that all different classes of integrally equivalent even matrices of an even order $m$, with divisor $d$, level $q$, and determinant $\Delta$ coincide with the class of a fixed matrix $Q$,

$$\{Q\} = \{Q[U] \mid \quad U \in \Lambda = GL_m(\mathbb{Z})\}. \tag{3.1}$$

In this case shall say that $Q$ is an *one-class matrix*. Given such a matrix, let $E(Q)$ be the group (2.2) of units of $Q$ and $\mathbf{A}(Q)$ – the semigroup (2.4) of (regular) automorphes of $Q$. It can be verified that the groups $E(Q)$ and semigroups $\mathbf{A}(Q)$ satisfy the following three condition: $\mathbf{A}(Q)\mathbf{A}(Q) \subset \mathbf{A}(Q)$, $E(Q) \subset \mathbf{A}(Q)$, and each double coset $E(Q)DE(Q)$ with $D \in \mathbf{A}(Q)$ is a finite union of left cosets modulo $E(Q)$. Finally, let $\mathcal{H}(Q)$ be the (regular) Hecke–Shimura ring of the quadratic form with matrix $Q$ defined in §2.

On the other hand, let $T \in \mathcal{H}_0^n(q)$ be an homogeneous element of a multiplier $\mu(T) = \mu$. If the set $A(Q, \mu)$ of the form (1.9) is not empty, then, for each $D \in A(Q, \mu)$, the matrix $\mu^{-1}Q[D]$ is integrally equivalent to the matrix $Q$. By choosing appropriate representative in the coset $D \cdot \Lambda$ of the group $\Lambda = GL_m(\mathbb{Z})$ one can assume that $\mu^{-1}Q[D] = Q$, i.e., $Q[D] = \mu Q$, and the coset $D \cdot \Lambda$ for such $D$ reduces to the coset $D \cdot E(Q)$ of the group $E(Q)$ of units of $Q$. It follows that one can take

$$A(Q, \mu)/\Lambda = \mathbf{A}(Q, \mu)/E(Q), \tag{3.1}$$

where $\mathbf{A}(Q, \mu)$ is the set (2.5). Then the relation (1.8) takes the form

$$\Theta^n(Z, P, Q)|T = \begin{cases} \sum_{D \in \mathbf{A}(Q,\mu)/E(Q)} I(D, Q, \Psi^{n,m}T)\Theta^n(Z, \mu^{-1}P[D], Q), \\ 0, \end{cases}$$

depending on whether the set $\mathbf{A}(Q, \mu)$ is not empty or empty, where $\Psi^{n,m} = \Psi_Q^{n,m}$. Since $\mu$ is prime to the level $q$, it follows that the condition $D \in \mathbf{A}(Q, \mu)/E(Q)$ is equivalent to the condition $\mu D^{-1} \in E(Q) \setminus \mathbf{A}(Q, \mu)$. Therefore, by replacing $D \mapsto \mu D^{-1}$, the relations can be rewritten in the form

$$\Theta^n(Z, P, Q)|T = \begin{cases} \sum_{D \in E(Q)\setminus\mathbf{A}(Q,\mu)} I(\mu D^{-1}, Q, \Psi^{n,m}T)\Theta^n(Z, \mu P[D^{-1}], Q), \\ 0, \end{cases} \tag{3.2}$$

depending on whether the set $\mathbf{A}(Q, \mu)$ is not empty or empty.

For $n \geqslant 1$ let us set

$$\tau^n(T) = \begin{cases} \sum_{D \in E(Q)\setminus\mathbf{A}(Q,\mu)} I(\mu D^{-1}, Q, \Psi^{n,m}T)\,(E(Q)D) & \text{if } \mathbf{A}(Q, \mu) \neq \varnothing, \\ 0 & \text{if } \mathbf{A}(Q, \mu) = \varnothing. \end{cases} \tag{3.3}$$

It follows from (1.12) that for each $U \in E(Q)$ sums (3.3) satisfy relations

$$\begin{aligned}\tau^n(T)U &= \sum_{D'=DU\in E(Q)\backslash \mathbf{A}(Q,\mu)} I(\mu U(D')^{-1},\, Q,\, \Psi^{n,m}T)(E(Q)D') \\ &= \sum_{D\in E(Q)\backslash \mathbf{A}(Q,\mu)} I(\mu D^{-1},\, Q[U],\, \Psi^{n,m}T)(E(Q)D) = \tau^n(T).\end{aligned}$$

Thus, $\tau^n(T) \in \mathcal{H}(Q)$ (see §2). Extending the mapping by linearity to arbitrary $T \in \mathcal{H}_0^n(q)$, we obtain a linear mapping of Hecke–Shimura rings

$$\mathcal{H}_0^n(q) \ni T \mapsto \tau^n(T) \in \mathcal{H}(Q). \tag{3.4}$$

**Theorem 3.1.** *Let $Q$ be an even positive definite one-class matrix of even order $m$. Then for each $n \geqslant 1$ the action of each Hecke operator $|T$ with $T \in \mathcal{H}_0^n(q)$ on the theta-series (1.1) with $Q$-harmonic polynomial $P$ can be written in the terms of action of the operator $\circ\tau^n(T)$ defined in §2 on polynomial $P$ in the form*

$$\Theta^n(Z,\, P,\, Q)|T = \Theta^n(Z,\, P\circ\tau^n(T),\, Q). \tag{3.5}$$

*In particular, if $P$ is an eigenfunction of the operator $\circ\tau^n(T)$, then the theta-series $\Theta^n(Z,\, P,\, Q)$ is an eigenfunction of the operator $|T$ with the same eigenvalue.*

**Proof.** Suppose first that $T$ is an homogeneous element of multiplier $\mu(T) = \mu$, and that the set $A(Q,\, \mu)$ of the form (1.9) is not empty. Then, by definition, the action of operators $\circ\tau^n(T)$ on the polynomial $P$ can be written in the form

$$(P\circ\tau^n(T))(X) = \sum_{D\in E(Q)\backslash \mathbf{A}(Q,\mu)} I(\mu D^{-1},\, Q,\, \Psi^{n,m}T)P(DX).$$

Hence the upper formula (3.2) can be rewritten in the form

$$(\Theta^n|T)(Z,\, P,\, Q) = \Theta^n(Z,\, P\circ\tau^n(T),\, Q).$$

This formula together with the upper formula (1.8) is true for all homogeneous elements $T \in \mathcal{H}_0^n(q)$ of a multiplier $\mu$ and such that the set $A(Q,\, \mu)$ of the form (1.9) is not empty. If the set $A(Q,\, \mu)$ is empty, then $\Theta^n|T = 0$, and the formula remains true with convention $\tau^n(T) = 0$. By linearity, similar formulas remain true for all $T \in \mathcal{H}_0^n(q)$, which proves the theorem. □

Note that when $1 \leqslant n < m$, inverse image $\Psi^{n,m}T \in \mathcal{H}_0^m(q)$ is not unique, which cause an indeterminacy of the definition of the mapping (3.4), but in view of the theorem it does not affect the action of operators $\tau^n(T)$ on theta-function. We call the mapping $T \mapsto \tau^n(T)$ the *interaction mapping of the Heske–Shimura rings.* In paper [An05] examples of explicit relations binding zeta-functions of eigenfunctions of Hecke operators tied by the interaction mappings were considered.

**Theorem 3.2.** *Let $Q$ be an even positive definite one-class matrix of even order $m$. Then the mapping* (3.4) *for every $n \geqslant m$ is a linear ring-homomorphism of the Hecke–Shimura rings.*

**Proof.** The mapping (3.4) is linear by definition. We consider first the case $n = m$. By linearity, it is sufficient to prove in this case that

$$\tau^m(T\,T') = \tau^m(T)\tau^m(T') \tag{3.6}$$

for every homogeneous elements $T,\, T' \in \mathcal{H}_0^m(q)$. If $\mu(T) = \mu$ and $\mu(T') = \mu'$, then $\mu(TT') = \mu\mu'$ and by (3.3) we can write relations

$$\tau^m(TT') = \sum_{D'' \in E(Q)\backslash \mathbf{A}(Q,\, \mu\mu')} I(\mu\mu'(D'')^{-1},\, Q,\, TT')\,(E(Q)D'')$$

if $\mathbf{A}(Q,\, \mu\mu') \neq \varnothing$ and $\tau^m(TT') = 0$ if $\mathbf{A}(Q,\, \mu\mu') = \varnothing$. Similarly,

$$\tau^m(T) \sum_{D \in E(Q)\backslash \mathbf{A}(Q,\, \mu)} I(\mu\, D^{-1},\, Q,\, T)\,(E(Q)D)$$

if $\mathbf{A}(Q,\, \mu) \neq \varnothing$ and $\tau^m(T) = 0$ if $\mathbf{A}(Q,\, \mu) = \varnothing$ and, analogously,

$$\tau^m(T') = \sum_{D' \in E(Q)\backslash \mathbf{A}(Q,\, \mu')} I(\mu'(D')^{-1},\, Q,\, T')\,(E(Q)D')$$

if $\mathbf{A}(Q,\, \mu') \neq \varnothing$ and $\tau^m(T') = 0$, otherwise. By definition of multiplication in Hecke–Shimura rings we can write

$$\tau^m(T)\tau^m(T') =$$
$$= \sum_{D \in E(Q)\backslash \mathbf{A}(Q,\, \mu)} \; \sum_{D' \in E(Q)\backslash \mathbf{A}(Q,\, \mu')} I(\mu D^{-1},\, Q,\, T) I(\mu'(D')^{-1},\, Q,\, T')(E(Q)DD'),$$

if $\mathbf{A}(Q,\,\mu)\neq\varnothing$ and $\mathbf{A}(Q,\,\mu')\neq\varnothing$, otherwise, $\tau^m(T)\tau^m(T')=0$. Therefore, in order to prove (3.6) it is sufficient to show that

$$\begin{aligned}&I(\mu\mu'(D'')^{-1},\,Q,\,TT')\\&=\sum_{\substack{(D,D')\in(E(Q)\backslash\mathbf{A}(Q,\,\mu),E(Q)\backslash\mathbf{A}(Q,\,\mu')),\\ DD'\in E(Q)D''}} I(\mu D^{-1},\,Q,\,T)I(\mu'(D')^{-1},\,Q,\,T')\end{aligned}\tag{3.7}$$

for each $D''\in\mathbf{A}(Q,\,\mu\mu')$, unless the left or the right sides are both zero.

On the other hand, by [An96, Proposition 3.8, formula (3.18)], for every $N\in\mathbf{A}(Q_j,\,\mu\mu')$ we can write the relation

$$\begin{aligned}&I(N,\,Q,\,TT')\\&\quad=\sum_{\substack{(D,D')\in(A(Q,\,\mu)/\Lambda,\,A(\mu^{-1}Q[D],\,\mu')/\Lambda),\\ DD'\in N\Lambda}} I(D,\,Q,\,T)I(D',\,\mu^{-1}Q[D],\,T')\end{aligned}$$

with $\Lambda=GL_m(\mathbb{Z})$. By (3.1), the condition $D\in\mathbf{A}(Q,\mu)/\Lambda$ can be replaced by inclusion $D\in\mathbf{A}(Q,\,\mu)/E(Q)$, in particular, $\mu^{-1}Q[D]=Q$. Then, similarly, the condition $D'\in A(\mu^{-1}Q[D],\,\mu')/\Lambda$ can be replaced by inclusion $D'\in\mathbf{A}(Q,\,\mu')/E(Q)$. Thus the condition $DD'\in N\Lambda$ can be replaced by the condition $DD'\in NE(Q)$, and we came to the relation

$$I(N,\,Q,\,TT')=\sum_{\substack{(D,D')\in(\mathbf{A}(Q,\,\mu)/E(Q),\mathbf{A}(Q,\,\mu')/E(Q)),\\ DD'\in NE(Q)}} I(D,\,Q,\,T)I(D',\,Q,\,T'),$$

Since $\mu$ and $\mu'$ are prime to the level $q$ of $Q$, it follows that the conditions $D\in\mathbf{A}(Q,\,\mu)/E(Q)$ and $D'\in\mathbf{A}(Q,\,\mu')/E(Q)$ are equivalent to the conditions $\mu D^{-1}\in E(Q)\setminus\mathbf{A}(Q,\,\mu)$ and $\mu'(D')^{-1}\in E(Q)\setminus\mathbf{A}(Q,\,\mu')$, respectively. Similarly, the condition $DD'\in NE(Q)$ is equivalent to the condition $\mu\mu'(DD')^{-1}\in E(Q)(\mu\mu'N^{-1})$. Therefore, by setting $D_1=\mu D^{-1}$, $D_2=\mu'(D')^{-1}$ and $N_1=\mu\mu'N^{-1}$, i.e. $D=\mu D_1^{-1}$, $D'=\mu'D_2^{-1}$ and $N=$

$\mu\mu' N_1^{-1}$ we came to the relation

$$\begin{aligned} & I(\mu\mu' N_1^{-1}, Q, TT') \\ & = \sum_{\substack{(D_1, D_2) \in (E(Q)\backslash \mathbf{A}(Q,\mu)/, E(Q)\backslash \mathbf{A}(Q,\mu')),\\ D_2 D_1 \in E(Q) N_1}} I(\mu D_1^{-1}, Q, T) I(\mu' D_2^{-1}, Q, T') \\ & = \sum_{\substack{(D_2, D_1) \in (E(Q)\backslash \mathbf{A}(Q,\mu'), E(Q)\backslash \mathbf{A}(Q,\mu)),\\ D_2 D_1 \in E(Q) N_1}} I(\mu' D_2^{-1}, Q, T') I(\mu D_1^{-1}, Q, T). \end{aligned}$$

Since the ring $\mathcal{H}^m(q)$ is commutative, we have $TT' = T'T$, and so the last relation for elements $T'$, $T$ is actually, up to notation, the relation (3.7) for these elements.

The case when $n \geqslant m$ follows from the case $n = m$, since by definition the mapping $T \mapsto \tau^n(T) = \tau^m(\Psi^{n,m}(T))$ is the composition of the Zharkovskaya homomorphism $\Psi^{n,m} = \Psi_Q^{n,m}$ and the homomorphism $\tau^m$. □

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Поступило 10 июня 2012 г.