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ON EQUIDISSECTION OF BALANCED POLYGONS

Abstract. In this paper. we show that a lattice balanced polygon of odd area cannot be cut into an odd number of triangles of equal areas. The first result of this type was obtained by Paul Monsky in 1970. He proved that a square cannot be cut into an odd number of triangles of equal areas. In 2000, Sherman Stein conjectured that the same holds for any balanced polygon.

We also show connections between the equidissection problem and tropical geometry.

§1. The equidissection problem

Theorem (P. Monsky, 1970). *A square cannot be cut*[1] *into an odd number of triangles*[2] *of equal areas.*

The only known proof of this theorem was published by Monsky [3] in 1970. The proof is based on two key ideas: Sperner's lemma and the coloring of the plane in three colors based on a 2-adic valuation.

After that, several generalizations of Monsky's results appeared. The first generalization was conjectured by Stein and proved by Monsky [4] in 1990. It claims that a centrally symmetric polygon cannot be cut into an odd number of triangles of equal areas. Though its proof is based on the same idea of 3-coloring, it is technically more challenging than the proof in the case of the square and uses a nontrivial homological technique.

In 1994, Bekker and Netsvetaev [1] proved a similar result in higher dimensions.

To state another generalization, we need a definition. Let us call a finite union of squares of area 1 with integer coordinates of vertices a *polyomino*. First, Stein [8] proved in 1999 that a polyomino of odd area cannot be cut into an odd number of triangles of equal areas, and in 2002 Praton [5]

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[1]By the phrase "*a polygon B is cut into triangles*" we mean that B can be presented as the union of a finite number of triangles so that the interiors of the triangles have an empty intersection with each other. Figure 1 illustrates this.

[2]Throughout this article, "triangle" is taken to include the degenerate case.

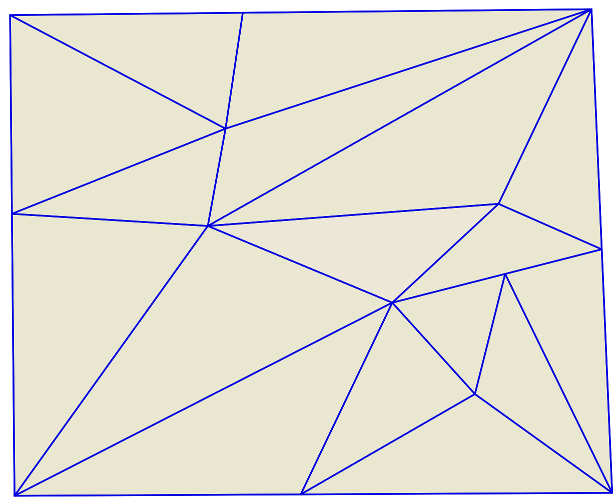

Fig. 1. A square is cut into triangles.

proved the same for an even-area polyomino. In 2000, Stein [6] suggested a conjectural generalization of Theorem 1, see also [7].

Let B be a plane polygon with clockwise oriented boundary. It is called *balanced* if its edges can be divided into pairs so that the edges in each pair are parallel, equal in length, and have opposite orientations (the edges are oriented, their orientation comes from the orientation of the boundary).

Now we are ready to formulate Stein's conjecture.

Conjecture 1 (S. Stein, 2000)**.** *A balanced polygon cannot be cut into an odd number of triangles of equal areas.*

In this note, we present a proof of a special case of Conjecture 1. Namely, we prove the following theorem.

Theorem (Nonequidissectibility of a balanced lattice polygon)**.** *Consider a balanced polygon B of integer odd area and assume that the coordinates of all vertices are integers. Then B cannot be cut into an odd number of triangles of equal areas.*

For an example of a balanced lattice polygon of area 15, see Fig. 2.

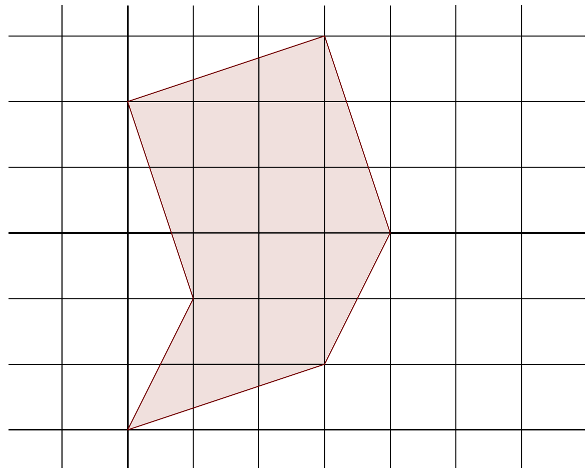

Fig. 2. A balanced lattice polygon of area 15.

The proof of the nonexistence of an equidissection of a balanced lattice polygon consists of several steps.

In Sec. 2, we review the coloring of the plane in three colors introduced by Monsky.

In Sec. 3, we introduce the notion of the *degree* of a broken line. It is an integer that depends both on a coloring and a broken line. We prove that if a polygon can be cut into triangles with nonnegative 2-adic valuations of areas, then its degree is 0.

In Sec. 4, the previous results are applied to the case of a lattice polygon.

The proof of the nonexistence of an equidissection of a balanced lattice polygon is completed in Sec. 5.

In the appendix, we show connections between tropical geometry and 3-colorings of the projective plane.

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§2. Tropical colorings

The main tool for us will be a special type of colorings of the plane in three colors. To begin with, let us recall the notion of a discrete valuation and sketch its basic properties. A function $\nu_2 : \mathbb{R} \longrightarrow \mathbb{R} \cup \{\infty\}$ is a 2-adic valuation on the field of real numbers if for any two numbers $a, b \in \mathbb{R}$ the following five properties hold.

Property 1: $\nu_2(ab) = \nu_2(a) + \nu_2(b)$, $\nu_2(\frac{a}{b}) = \nu_2(a) - \nu_2(b)$.
Property 2: $\nu_2(a+b) \geqslant \min\{\nu_2(a), \nu_2(b)\}$.
Property 3: if $\nu_2(a) < \nu_2(b)$ then $\nu_2(a+b) = \nu_2(b)$.
Property 4: $\nu_2(0) = \infty$.
Property 5: it extends the standard 2-adic valuation on the rationals: for every $q \in \mathbb{Q} \setminus \{0\}$,

$$\nu_2(q) = s \iff q = 2^s \frac{2k+1}{2l+1} \text{ for some } k, l, s \in \mathbb{Z}.$$

The existence of such a function follows from the theorem on the extension of valuations, see [2]. This function is not unique, and its construction is based on the axiom of choice.

Our goal now is to construct a family of 3-colorings of the plane (we will call these colorings "tropical") with two properties:

(P1) On any line, points of only two colors occur.

(P2) For any triangle with vertices having all three different colors, its area has a negative 2-adic valuation.

Let us color points of the plane in three colors A, B, C according to the following rule: a point Z with coordinates (x, y) is colored

in color A if $\nu_2(x) > 0$, $\nu_2(y) > 0$;

in color B if $\nu_2(y) \leqslant 0$, $\nu_2(x) > \nu_2(y)$;

in color C if $\nu_2(x) \leqslant 0$, $\nu_2(y) \geqslant \nu_2(x)$.

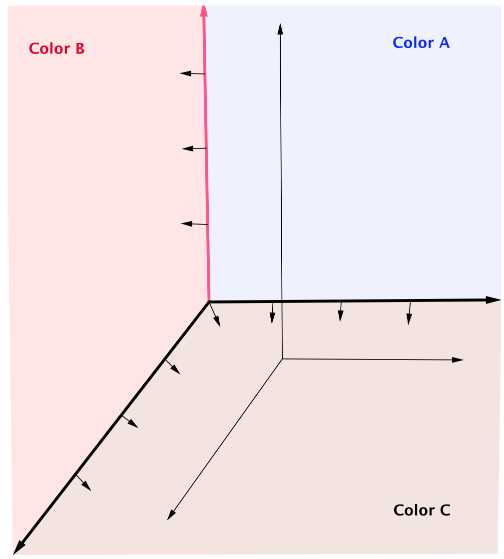

Fig. 3. The tropical coloring of the plane and the image of a line.

This rule defines a map from the plane to a three-element set:

$$\pi : \mathbb{R}^2 \longrightarrow \{A, B, C\}.$$

In Fig. 3, the method of coloring is presented in the coordinates $\nu_2(x), \nu_2(y)$.

For any area-preserving affine transformation $\mathcal{A} \in \mathbb{R}^2 \rtimes SL_2(\mathbb{R})$, we can define another coloring $\pi^{\mathcal{A}}$ by the rule

$$\pi^{\mathcal{A}}(Z) = \pi(\mathcal{A}(Z)) \quad \text{for every point} \quad Z.$$

This defines a family of 3-colorings, which we will call tropical.

Lemma 1. *For the* 3*-coloring* $\pi^{\mathcal{A}}$*, properties* **P1** *and* **P2** *hold.*

Proof. **P2** $\Longrightarrow$ **P1**. If there were three points of different colors on the same line, they would form a triangle of area 0. Since $\nu_2(0) = \infty$, this is impossible.

P2. For the coloring $\pi^{\mathcal{A}}$ we need to prove that for any triangle $\triangle$ whose image under $\mathcal{A}$ has vertices of all three different colors, the following holds true:

$$\nu_2(\mathrm{Area}(\triangle)) < 0.$$

Since $\mathcal{A}$ is area-preserving, it suffices to prove that the area of $\mathcal{A}(\triangle)$ has a negative valuation. Suppose that the triangle $\mathcal{A}(\triangle)$ has a vertex (x_1, y_1) of color A, a vertex (x_2, y_2) of color B, and a vertex (x_3, y_3) of color C. Then its area is equal to

$$\mathrm{Area}(\triangle) = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = \frac{1}{2}\Big((x_2 - x_1)(y_3 - y_1) - (y_2 - y_1)(x_3 - x_1)\Big).$$

Applying the properties of a valuation and the definition of the coloring leads to

$$\begin{aligned} &\nu_2\Big(\frac{1}{2}(y_2 - y_1)(x_3 - x_1)\Big) = -1 + \nu_2(y_2 - y_1) + \nu_2(x_3 - x_1) \\ &\quad = -1 + \nu_2(y_2) + \nu_2(x_3) < -1 + \min\{\nu_2(x_2), 0\} + \min\{\nu_2(y_3), 0\} \\ &\quad \leqslant -1 + \nu_2(x_2 - x_1) + \nu_2(y_3 - y_1) = \nu_2\Big(\frac{1}{2}(x_2 - x_1)(y_3 - y_1)\Big). \end{aligned}$$

Therefore,

$$\begin{aligned} \nu_2(\mathcal{A}(\triangle)) &= \nu_2\Big(\frac{1}{2}(y_2 - y_1)(x_3 - x_1)\Big) \\ &= -1 + \nu_2(y_2) + \nu_2(x_3) \leqslant -1 + 0 + 0 = -1. \qquad \square \end{aligned}$$

§3. The degree of a broken line

Given a tropical coloring $\pi^{\mathcal{A}}$, one can construct a degree map associated with it. It assigns an integer to any oriented broken line.

Let K_n be the complete graph with n vertices regarded as a one-dimensional simplicial complex. Suppose that we are given a one-dimensional simplicial complex K and a map Col sending vertices of K to vertices of K_n. Then this map can be extended to a continuous map from the complex K to K_n according to the following rules:

- A vertex X is sent to the point $\mathrm{Col}(X)$.

- An edge XY is sent to the edge $\mathrm{Col}(X)\,\mathrm{Col}(Y)$ by the linear map determined by its endpoints.

This map is, obviously, a continuous simplicial map from one simplicial complex to the other. We will use the same letter for both the original map (coloring) and the extended one.

For the following, let us fix a tropical coloring $\pi^{\mathcal{A}}$.

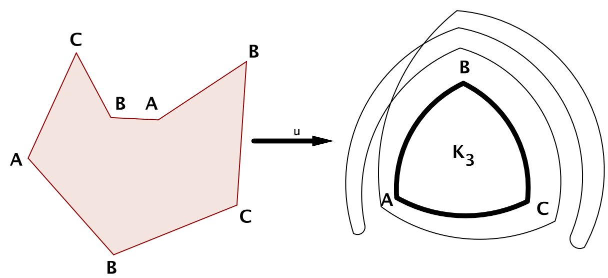

Fig. 4. The degree of a polygon is -1.

Definition 1. *Any closed broken line $L = L_1L_2\dots L_n$ has a natural structure of a simplicial complex. This complex is homeomorphic to a circle. Its vertices are 3-colored by $\pi^{\mathcal{A}}$. The extension of $\pi^{\mathcal{A}}$ gives a continuous map from the topological circle L to the topological circle K_3. We denote by $\mathrm{Deg}(L, \pi^{\mathcal{A}})$ its topological degree. See Fig. 4.*

For a polygon M, we denote its boundary by ∂M.

Lemma 2. *If a polygon M can be cut into triangles whose areas have nonnegative 2-adic valuations, then for every tropical coloring $\pi^{\mathcal{A}}$,*

$$\mathrm{Deg}(\partial M, \pi^{\mathcal{A}}) = 0.$$

Proof. Suppose that a polygon M has a triangulation $\mathcal{T}$ with each triangle having a nonnegative valuation of the area. This triangulation carries a natural structure of a one-dimensional simplicial complex, induced from the plane, with vertices colored by $\pi^{\mathcal{A}}$. The boundary ∂M is a subcomplex of $\mathcal{T}$ homeomorphic to a circle, so there is a class $[\partial M] \in H_1(\mathcal{T}, \mathbb{Z})$

corresponding to ∂M. The degree $\mathrm{Deg}(\partial M, \pi^{\mathcal{A}})$ is equal to the image of the class $[\partial M]$ under the map

$$H_1(\mathcal{T}, \mathbb{Z}) \xrightarrow{\pi^{\mathcal{A}}} H_1(K_3, \mathbb{Z}) \cong \mathbb{Z}.$$

One can orient all triangles in the cut in a coherent way. Then the triangles sharing an edge will induce the opposite orientations on this edge. Adding up the classes of the boundaries of all triangles, we obtain the class of ∂M:

$$[\partial M] = \sum_{\triangle \in \mathcal{T}} [\partial \triangle];$$

applying $\pi^{\mathcal{A}}$ yields

$$\pi^{\mathcal{A}}([\partial M]) = \sum_{\triangle \in \mathcal{T}} \pi^{\mathcal{A}}([\partial \triangle]).$$

Since any triangle $\triangle \in \mathcal{T}$ has a nonnegative valuation of the area, at least two of its vertices are of the same color, according to Lemma 1. Thus $\pi^{\mathcal{A}}([\partial \triangle]) = 0$ for any triangle in $\mathcal{T}$, and hence

$$\pi^{\mathcal{A}}([\partial M]) = \mathrm{Deg}(\partial M, \pi^{\mathcal{A}}) = 0.$$

$\square$

§4. Lattice polygons

The points with integer coordinates in the plane form a two-dimensional lattice in $\mathbb{R}^2$; we will denote it by $\mathcal{L}$. We say that a polygon M or a closed broken line L is a *lattice* if all its vertices have integer coordinates.

Let us denote by K_4 the simplicial complex with four vertices labeled by the elements of the group $\mathbb{Z}_2 \times \mathbb{Z}_2$, and edges connecting any two of its vertices. We can map $\mathcal{L}$ to K_4 by the map

$$\overline{*} : (x_1, x_2) \longrightarrow \overline{(x_1, x_2)} = (x_1 \bmod 2, x_2 \bmod 2).$$

For any lattice broken line L, we can consider its image under the map $\overline{*}$ using the construction from the previous section. This map induces a map on the simplicial homology groups:

$$\mathbb{Z} \cong H_1(L, \mathbb{Z}) \xrightarrow{\overline{*}} H_1(K_4, \mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}.$$

The image of $1 \in \mathbb{Z}$ in the group $H_1(K_4, \mathbb{Z})$ will be denoted by $\langle L \rangle$ and called the *class* of the broken line L.

Lemma 3. *If a lattice polygon M can be dissected into triangles whose areas have nonnegative 2-adic valuations, then the map $\overline{*}$ sends $H_1(\partial M, \mathbb{Z})$ to* 0.

Proof. Let us denote the points of K_4 in the following way:

$$X_1 = (0,0),\ X_2 = (0,1),\ X_3 = (1,0),\ X_4 = (1,1).$$

The three cycles

$$\begin{aligned} \sigma_1 &= X_1X_2 + X_2X_3 + X_3X_1, \\ \sigma_2 &= X_1X_3 + X_3X_4 + X_4X_1, \\ \sigma_3 &= X_3X_2 + X_4X_3 + X_2X_4 \end{aligned}$$

generate $H_1(K_4, \mathbb{Z}) \cong \mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z}$. Suppose that $\langle \partial M \rangle = \lambda_1\sigma_1 + \lambda_2\sigma_2 + \lambda_3\sigma_3 \in H_1(K_4, \mathbb{Z})$.

The 2-adic valuation of an integer is always nonnegative, and it is equal to zero if and only if the integer is odd. It is clear from the definition of a 2-adic valuation that for a point $(x, y) \in \mathcal{L}$,

$$\begin{aligned} \overline{(x,y)} = (0,0) \in \mathbb{Z}_2 \times \mathbb{Z}_2 &\implies \pi\Big((x,y)\Big) = A, \\ \overline{(x,y)} = (0,1) \in \mathbb{Z}_2 \times \mathbb{Z}_2 &\implies \pi\Big((x,y)\Big) = B, \\ \overline{(x,y)} = (1,0) \in \mathbb{Z}_2 \times \mathbb{Z}_2 &\implies \pi\Big((x,y)\Big) = C, \\ \overline{(x,y)} = (1,1) \in \mathbb{Z}_2 \times \mathbb{Z}_2 &\implies \pi\Big((x,y)\Big) = C. \end{aligned}$$

This means that the map π is well defined on $K_4 \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ and $\pi\Big((x,y)\Big) = \pi\Big(\overline{(x,y)}\Big)$. Any area-preserving affine transformation $\mathcal{A} \in \mathbb{Z}^2 \rtimes SL_2(\mathbb{Z}) \subseteq \mathbb{R}^2 \rtimes SL_2(\mathbb{R})$ acts on the four vertices of $K_4 \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ by a permutation; a simple check shows that $\pi^{\mathcal{A}}\Big((x,y)\Big) = \pi\Big(\mathcal{A}\overline{(x,y)}\Big)$.

We will apply Lemma 2 to the three colorings corresponding to the following area-preserving affine transformations:

$$\begin{aligned} E:&\quad (x,y) \longrightarrow (x,y),\\ U:&\quad (x,y) \longrightarrow (x+y,y),\\ V:&\quad (x,y) \longrightarrow (y+1,x). \end{aligned}$$

By Lemma 2, $\operatorname{Deg}(\partial M, \pi^E) = 0$. We have

$$\begin{aligned} 0 &= \pi^E(\langle \partial M \rangle) = \lambda_1 \pi^E(\sigma_1) + \lambda_2 \pi^E(\sigma_2) + \lambda_3 \pi^E(\sigma_3)\\ &= \lambda_1(AB+BC+CA) + \lambda_2(AC+CC+CA) + \lambda_3 \pi^E(CB+CC+BC)\\ &= \lambda_1(AB+BC+CA). \end{aligned}$$

Thus $\lambda_1 = 0$. Similarly, applying the same procedure to the affine transformations U and V, we see that $\lambda_2 = 0$ and $\lambda_3 = 0$. □

§5. Balanced polygons

For two vectors $v = (v_x, v_y)$ and $w = (w_x, w_y)$, their *wedge product* is defined as the oriented area of the parallelogram formed by these vectors. It can be calculated as the following determinant:

$$v \wedge w = \begin{vmatrix} v_x & w_x \\ v_y & w_y \end{vmatrix} = v_x w_y - v_y w_x.$$

Definition 2. *For any closed broken line $L = L_1 L_2 \dots L_n$, we define its generalized area as*

$$\operatorname{Area}(L) = \frac{1}{2} \sum_{i=1}^{n} \overline{OL_i} \wedge \overline{OL_{i+1}}, \qquad \textit{where} \qquad L_{n+1} := L_1.$$

For a non-selfintersecting broken line, the notion defined above gives the oriented area of the polygon bounded by the broken line.

Lemma 4. *For a lattice parallelogram P the following is true:*

- *If the area of P is even, then $\langle \partial P \rangle = 0$.*
- *If the area of P is odd, then*

$$\langle \partial P \rangle \in \{\pm(\sigma_2 + \sigma_3), \pm(\sigma_3 + \sigma_1), \pm(\sigma_1 + \sigma_2)\}.$$

Proof. A parallelogram can be cut into two equal triangles of integer area. An application of Lemma 2 gives the first assertion.

To prove the second assertion, we will show that if a parallelogram P has an odd area, then all its pairs of coordinates of vertices are different modulo 2. If the vertices of the parallelogram have coordinates $(x_1, y_1), (x_2, y_2), (x_3, y_3), (x_4, y_4)$, then its area is equal to

$$\begin{aligned}\mathrm{Area}(P) &= (x_2 - x_1)(y_3 - y_1) - (y_2 - y_1)(x_3 - x_1)\\ &= (x_4 - x_1)(y_3 - y_1) - (y_4 - y_1)(x_3 - x_1).\end{aligned}$$

It is clear from this formula that if there are two vertices with both x and y coordinates being conjugate modulo 2, then $\mathrm{Area}(P)$ is even.

Thus the vertices of the parallelogram are colored in colors A, B, C, C. Depending on the order in which these colors follow each other, we obtain one of the cycles $\pm(\sigma_2 + \sigma_3)$, $\pm(\sigma_3 + \sigma_1)$, $\pm(\sigma_1 + \sigma_2)$. □

The following lemma generalizes Lemma 4.

Lemma 5. *If B is a balanced lattice polygon, then the image of its boundary under the map $\overline{*}$ represents a class $\langle \partial B \rangle$ in the group $H_1(K_4, \mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$ that lies in the subgroup of index 2 generated by $\sigma_2 + \sigma_3$, $\sigma_3 + \sigma_1$, and $\sigma_1 + \sigma_2$:*

$$\langle \partial B \rangle = \mu_1(\sigma_2 + \sigma_3) + \mu_2(\sigma_3 + \sigma_1) + \mu_3(\sigma_1 + \sigma_2)$$

for some $\mu_1, \mu_2, \mu_3 \in \mathbb{Z}$.

Furthermore,

$$\mathrm{Area}(B) \equiv \mu_1 + \mu_2 + \mu_3 \pmod 2.$$

Proof. Parallelograms are basic examples of balanced polygons, and we have seen that Lemma 5 holds true for them. Now we are going to show that any balanced polygon is built from parallelograms in some sense. For this we need to describe an action of the group S_n on the set of broken lines.

For a broken line $L = L_1 L_2 \dots L_n$, denote by $v_i = \overline{L_i L_{i+1}}$ the side vector of L (here $L_{n+1} := L_1$). Any permutation $\sigma \in S_n$ acts on the set of broken lines according to the following rule: $\sigma(L) = M$ where M is the broken line $M_1 M_2 \dots M_n$ with $M_1 = L_1$ and

$$M_{i+1} = L_1 + v_{\sigma^{-1}(1)} + v_{\sigma^{-1}(2)} + \dots + v_{\sigma^{-1}(i)}$$

for every i. This action sends balanced broken lines to balanced ones and lattice broken lines to lattice ones.

Let τ_i denote the transposition $(i, i+1)$. It is well known that the set $\{\tau_i \mid 1 \leqslant i \leqslant n-1\}$ generates S_n. One can check that

$$\mathrm{Area}(\tau_j(L)) = \mathrm{Area}(L) - v_{i+1} \wedge v_i$$

and

$$\langle \tau_j(L) \rangle = \langle L \rangle - \langle P \rangle.$$

Here P is the parallelogram $L_i L_{i+1} L_{i+2} X$ with $X = L_i + v_{i+1}$. These properties guarantee that the broken lines L and $\tau_i(L)$ satisfy the conclusions of Lemma 5 simultaneously.

Since the lattice polygon B is balanced, the number n of its vertices is even and the sides of B can be indexed by numbers $\alpha_1, \alpha_2, \dots, \alpha_k$ and $\beta_1, \beta_2, \dots, \beta_k$ so that $n = 2k$ and the sides with indices α_i and β_i are parallel, equal in length, and have the opposite orientations inherited from the boundary of the polygon. The numbers α_i and β_i are just positive integers from 1 to $n = 2k$, so one can consider the permutation

$$\sigma = \begin{pmatrix} \alpha_1 & \beta_1 & \alpha_2 & \beta_2 & \dots & \alpha_k & \beta_k \\ 1 & 2 & 3 & 4 & \dots & 2k-1 & 2k \end{pmatrix}.$$

In the broken line $\sigma(\partial B)$, any side with an odd index is followed by a side that is parallel, equal in length, and has the opposite direction. Both the area and the class of $\sigma(\partial B)$ in $H_1(K_4, \mathbb{Z})$ are equal to 0. Since σ can be presented as a product of transpositions τ_i, the boundary ∂B satisfies the conclusions of Lemma 5. $\square$

Now we are ready to complete the proof of our main result.

Proof of the nonequidissectibility of a balanced lattice polygon. Suppose that for a balanced lattice polygon B of integer odd area there exists a cut into an odd number of triangles of equal areas. If $\mathrm{Area}(B) = S$

and the number of triangles is equal to N, then the area of each of them is $\frac{S}{N}$. Since S and N are odd numbers,

$$\nu_2(S/N) = \nu_2(S) - \nu_2(N) = 0.$$

By Lemma 3, the class of the broken line ∂B in $H_1(K_4, \mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$ is equal to 0, and, according to Lemma 5, there exist $\mu_1, \mu_2, \mu_3 \in \mathbb{Z}$ for which

$$\langle \partial B \rangle = \mu_1(\sigma_2 + \sigma_3) + \mu_2(\sigma_3 + \sigma_1) + \mu_3(\sigma_1 + \sigma_2) = 0.$$

Therefore, $\mu_1 = \mu_2 = \mu_3 = 0$ and

$$\mathrm{Area}(B) = S \equiv \mu_1 + \mu_2 + \mu_3 \equiv 0 \pmod 2.$$

This contradicts the oddness of S. □

§6. Appendix: Connections with tropical geometry

In this section, no new results are obtained, so the style is rather informal.

It is more natural to define tropical colorings on $\mathbb{RP}^2$ – the real projective plane. It is well known that a point of $\mathbb{RP}^2$ is defined by its homogeneous coordinates, a triple of real numbers $[x : y : z]$ with not all x, y, z equal to 0. For any nonzero λ, the triples $[x : y : z]$ and $[\lambda x : \lambda y : \lambda z]$ determine the same point. One can define a *momentum map*

$$m : \mathbb{RP}^2 \longrightarrow T$$

from the projective plane to the triangle T in the plane with vertices (1,0), (0,1), and (0,0) by the formula

$$m([x : y : z]) = \frac{2^{-\nu_2(x)}\Big(1,0\Big) + 2^{-\nu_2(y)}\Big(0,1\Big) + 2^{-\nu_2(z)}\Big(0,0\Big)}{2^{-\nu_2(x)} + 2^{-\nu_2(y)} + 2^{-\nu_2(z)}}.$$

One can check that the image of any line in $\mathbb{RP}^2$ under the momentum map is the union of three segments sharing a common endpoint. For each segment, the remaining endpoint lies on a side of the triangle T, and the whole segment is contained within a line passing through the vertex of T opposite to this side.

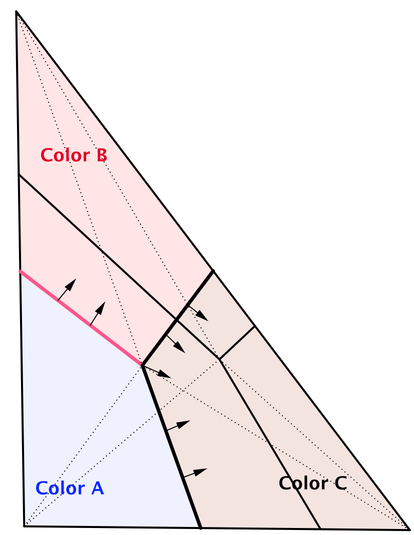

Fig. 5. The tropical coloring of the plane and the image of a line.

The image of the line $x + y + z = 0$ cuts T into three pieces, whose points we color in three colors A, B, and C. Now we can color each point in $\mathbb{RP}^2$ in the color of its image under the momentum map. This coloring coincides with the coloring π constructed at the beginning of the paper if regarded on the affine chart of $\mathbb{RP}^2$ with $z = 1$.

Property **P1** is obvious now: one can see that the image of any other line under the momentum map can intersect only two parts into which the image of the line $x + y + z = 0$ cuts the triangle T.

In the paper [9] by A. Hales and E. Straus, colorings of $\mathbb{RP}^2$ are studied in more detail. One of their results is the following theorem.

Theorem (A. Hales, E. Straus, 1982)**.** *Let C be a set of algebraic curves in $\mathbb{R}^2$ having the same Newton polygon P with n integer points inside (C is an n-dimensional linear system of algebraic curves). Then there exists* *a coloring of* $\mathbb{R}^2$ *in* $n+1$ *colors such that no curve in* C *contains all* $n+1$ *colors and no color is confined within a curve in* C.[3]

As a specific instance of this theorem, one obtains a coloring of the plane in six colors such that any conic contains at most five colors.

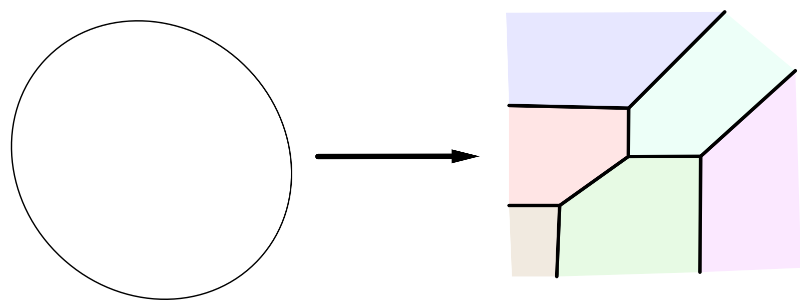

Fig. 6. The image of a conic under the momentum map and the corresponding coloring.

In [9], such colorings are constructed algebraically; here we will give a tropical explanation of this construction. By $\mathbb{R}^*$ we mean $\mathbb{R} \setminus \{0\}$. Let us consider the algebraic torus $\mathbb{R}^* \times \mathbb{R}^*$ and the "momentum map"

$$m : \mathbb{R}^* \times \mathbb{R}^* \longrightarrow \mathbb{R}^2,$$

where

$$m\Big((x,y)\Big) = (\nu_2(x), \nu_2(y)) \in \mathbb{R}^2.$$

There exists a curve with Newton polygon P whose image divides $\mathbb{R}^2$ into $n+1$ regions. We assign to them different colors. It can be proved that the image of any other curve with Newton polygon P intersects at most n regions. Now, we can color each point of $\mathbb{R}^* \times \mathbb{R}^* \in \mathbb{R}^2$ in the color of the region of $\mathbb{R}^2$ containing its image. This coloring coincides with that constructed by Hales and Straus.

[3]Actually, the result obtained in [9] is stronger: it holds for colorings of the projective plane over any field that has a nontrivial non-Archimedean valuation, and for arbitrary n-dimensional linear systems of algebraic curves without based points.

This illustration shows that the colorings constructed in [9] are natural from the perspective suggested by tropical geometry. Unfortunately, this does not lead to a simpler way of proving Theorem 6, because of both combinatorial difficulties in analyzing the way in which two tropical curves intersect and algebraic difficulties in extending colorings from $\mathbb{R}^* \times \mathbb{R}^*$ to the whole $\mathbb{R}^2$.

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