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MULTIPLE-GRAIN DISSIPATIVE SANDPILES

ABSTRACT. The two-dimensional directed sandpile with dissipation is transformed into a $(1 + 1)$ -dimensional problem with discrete space and continuous “time.” The master equation for the conditional probability that K grains preserve their initial order during an avalanche is solved exactly. Explicit expressions for asymptotic forms of solutions are given for the cases of the infinite and semi-infinite lattices. A nontrivial scaling is found in both cases.

§1. INTRODUCTION

Nonequilibrium dynamic systems have been for some time of considerable interest, as they can exhibit critical behavior in close analogy with systems at thermal equilibrium. A certain class of such dynamic systems, various sandpile models [1–7], have become a standard framework when analyzing self-organized criticality [8,9], that is, when the dynamics of the system inevitably drives it to a critical state independent of the initial state. Despite the extensive work on these systems, it is only fairly recently that a more detailed understanding of problems like when exactly sandpile models exhibit self-organized criticality, or what are the possible universality classes of their critical behaviors, have begun to emerge. The contemporary theory of low-dimensional nonequilibrium physics includes different subjects intensively studied in combinatorial mathematics, in particular, the theory of Young diagrams, random walks, and random matrices [10–12].

Most of the work so far on sandpiles was thus concentrated on properties such as the average duration of avalanches and their size distribution, which both exhibit scaling in a critical state. However, there may well be interesting many-particle correlations in sandpiles which likewise exhibit scaling, similarly to that found for problems like vicious walkers [13,14]

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and nonintersecting Brownian walkers [15], for which it was also found that the scaling exponents may depend in a nontrivial way on the number of walkers as well as on the boundary conditions imposed [16–21].

In the present paper, we provide an example of such a correlation for the case of the two-dimensional (2D) directed sandpile model introduced in [1]. For the sake of generality, the sandpile model is assumed to be dissipative, that is, the number of “grains” is not conserved in the topplings of unstable sites. We consider two different boundary conditions, namely, the infinite lattice and the lattice with an absorbing boundary at the origin (the “semi-infinite” lattice). As is well known by now, the model is critical only at vanishing dissipation [6, 22]. For this sandpile model, we study critical properties of the probability that K particles preserve their initial order during an avalanche. We find nontrivial scalings for both cases of the infinite and semi-infinite lattices.

To obtain an exact analytic form for the multiple-grain correlations, we first reformulate the sandpile model as a problem in $(1 + 1)$ dimensions, where the “time” direction is continuous. The master equation for the continuous-time model can be solved exactly. The obtained exact analytic form for the multiple-grain correlations allowed us next to show that nonzero dissipation introduces an exponential cut-off in their asymptotic form, which also include a power law with scaling exponents that depend nonlinearly on K . The scaling exponents are found different for the two boundary conditions.

§2. DISCRETE-TIME SANDPILE MODEL

The 2D directed sandpile model on a lattice introduced in [1] (see also [2]) is constructed such that to each site (j, n) an integer height variable (the number of grains) $z_{(j,n)}$ is assigned. The site has a threshold height $z_{(j,n)}^c$ below which it is stable. The dynamics of the model consists of two steps. First, we choose a site (j, n) at random and add one grain to it, i.e., $z_{(j,n)} \mapsto z_{(j,n)} + 1$. For $z_{(j,n)} \geq z_{(j,n)}^c$, the site (j, n) becomes unstable and its grains are distributed among the “downhill” neighboring sites. In the following, we will use the notation in which the locations of lattice sites in the horizontal direction are labelled by j , k , or l , and in the downhill direction, by n . By n we can equivalently denote the number of steps in a cascade of toppling processes. In a toppling at a site (j, n) , grains are thus distributed to the sites $(j + 1, n + 1)$ and $(j - 1, n + 1)$. By suppressing the n labels (understanding that two adjacent columns in the lattice are

connected in a toppling and that there is no dependence on n), we can express a toppling in the form

$$z_j \mapsto z_j - \Delta_{lj}, \quad (2.1)$$

in which the elements of the toppling matrix Δ satisfy $\Delta_{jj} > 0$, and $\Delta_{lj} < 0$ for $l \neq j$. The condition $\sum_j \Delta_{lj} \geq 0$ for every l guarantees that no grains are created in the toppling process. Without loss of generality, we can put $\Delta_{jj} = z_{(j,n)}^c$. The allowed number of grains in a stable site (j, n) is now $1, 2, \dots, \Delta_{jj} - 1$. The sites (j, n) such that $\sum_j \Delta_{lj} > 0$ are called dissipative. Boundary sites are always dissipative, so that grains can leave the system through the boundaries. After an initial toppling at a site, neighboring sites can also become unstable, and sites keep on relaxing with parallel updating until all sites are stable. In this way an avalanche of topplings is generated. The existence of dissipative sites ensures that all avalanches terminate in a finite time.

Assume now that all lattice sites are initially in a stationary state (i.e., are stable): $z_{(j,n)} = z_{(j,n)}^c - 1$. If we add a grain at a randomly chosen site $(l, 0)$, then the conditional probability $G_{jl}(n)$ that an extra grain is at site (j, n) satisfies the equation

$$G_{jl}(n) = \frac{1}{2} \{G_{j+1l}(n-1) + G_{j-1l}(n-1)\}, \quad (2.2)$$

with the initial condition $G_{jl}(0) = \delta_{jl}$. Since we consider only symmetric topplings, the conditional probability also satisfies $G_{jl}(n) = G_{lj}(n)$. It is easy to verify that (2.2) coincides with the equation for the corresponding probability expressed in the conventional “light cone” coordinates, see [1, Eq. (5)].

§3. CONTINUOUS-TIME SANDPILE MODEL

We can also express (2.2) in the form

$$G_{j,l}(n+1) - G_{j,l}(n) = \frac{1}{2} \{G_{j+1,l}(n) + G_{j-1,l}(n) - 2G_{j,l}(n)\}. \quad (3.1)$$

Consider now a process in which the discrete number of steps is replaced by a continuous parameter, which will be called “time” in the following. Let $P_{jl}(t)$ be the conditional probability that a grain is at a horizontal location j at time t after an arbitrary number of steps since it was dropped at a horizontal location l at $t = 0$. Transforming (3.1) into such a continuous

time, we find that, during a short time interval dt , the probability $P_{jl}(t)$ changes so that

$$P_{j,l}(t+dt) - P_{j,l}(t) = \frac{1}{2} \{P_{j+1,l}(t) + P_{j-1,l}(t) - 2P_{j,l}(t)\} dt, \quad (3.2)$$

which leads to the master equation

$$\frac{d}{dt}P_{jl}(t) = -\frac{1}{2} \sum_k \Delta_{jk} P_{kl}(t) \quad (3.3)$$

with the toppling matrix

$$\Delta_{jk} = 2\delta_{jk} - (\delta_{j+1,k} + \delta_{j-1,k}). \quad (3.4)$$

This toppling matrix means that, as above, at each toppling two grains are removed from the site and distributed to its nearest-neighbor downhill sites. Here we consider only symmetric topplings, for which $\Delta_{jk} = \Delta_{kj}$, and thus $P_{jk}(t) = P_{kj}(t)$. The initial conditions are $P_{jk}(0) = \delta_{jk}$. For a model of N sites in the horizontal direction, the lateral boundary elements of the toppling matrix can be defined so that $\Delta_{0,1} = \Delta_{N,N+1} = 0$, and hence the boundary sites $j = 1$ and $j = N$ are always dissipative, as required.

Note that the continuous time is not a simple continuum formed by the discrete variables n , but $P_{jl}(t)$ includes processes with all possible numbers of steps. In fact, the function $P_{jl}(t)$ can be regarded as a generating function of the conditional probabilities $G_{jl}(n)$, since we find that

$$e^t P_{jl}(t) = \sum_{n=0}^{\infty} G_{jl}(n) \frac{t^n}{n!}. \quad (3.5)$$

The expected number of topplings at a site j in the avalanche resulting from a perturbation (adding a grain) at a site l is given by $\Gamma_{jl}(0) = \sum_{n=0}^{\infty} G_{jl}(n)$. The Laplace transform of the conditional probability,

$$\Gamma_{jl}(f) = \int_0^{\infty} e^{-tf} P_{jl}(t) dt, \quad (3.6)$$

is the Green function of the master equation (3.3). It is easy to verify that $\Gamma_{jl}(0)$ satisfies the condition $\sum_k \Delta_{jk} \Gamma_{kl}(0) = \delta_{jl}$ (see [2]).

The master equation (3.3) can easily be solved for the toppling matrix (3.4) with the initial condition $P_{jl}(0) = \delta_{jl}$. Consider first the case of the

infinite lattice in the horizontal direction, so that $-\infty < j, l < \infty$. In this case, we find that

$$P_{jl}^{(-\infty, \infty)}(t) = e^{-t} I_{l-j}(t), \quad (3.7)$$

where $I_j(x)$ is a modified Bessel function. Asymptotically, for large t , the conditional probability for a single grain thus behaves as

$$P_{jl}^{(-\infty, \infty)}(t) \propto t^{-1/2}. \quad (3.8)$$

This is a pure power law, so that the duration of avalanches scales with the exponent $\xi_{(-\infty, \infty)} = 1/2$. This exponent coincides with the known result for 2D directed sandpiles [1], as it should.

We can analyze the effect of boundary conditions by introducing an absorbing boundary at the origin. To this end, we first recall the solution for a finite lattice of N sites (see, e.g., [22]) for which the boundary conditions are $P_{jl}(t) = 0$ for $j, l = 0$ and $j, l = N + 1$. In this case, we have

$$P_{jl}(t) = \frac{2}{N+1} \sum_{k=1}^N e^{-tE_k} \sin \frac{\pi j k}{N+1} \sin \frac{\pi l k}{N+1}, \quad (3.9)$$

where the spectrum is of the Bloch form,

$$E_k = 1 - \cos \frac{\pi k}{N+1}. \quad (3.10)$$

In the limit $N \rightarrow \infty$, the sum in Eq. (3.9) can be replaced by an integral, with the result

$$\begin{aligned} P_{jl}^{(0, \infty)}(t) &= \frac{2}{\pi} \int_0^\pi e^{-t(1 - \cos x)} \sin(lx) \sin(jx) dx \\ &= e^{-t} [I_{l-j}(t) - I_{l+j}(t)]. \end{aligned} \quad (3.11)$$

The asymptotic behavior for large t of the conditional probability is now given by

$$P_{jl}^{(0, \infty)}(t) \propto t^{-3/2}. \quad (3.12)$$

The scaling exponent, indeed, depends on having a boundary at a finite distance: $\xi_{(0, \infty)} = \frac{3}{2} = \xi_{(-\infty, \infty)} + 1$. As expected, the same exponent was found for the scaling of avalanche sizes with the corresponding boundary conditions [23, 24].

§4. DISCRETE- AND CONTINUOUS-TIME MULTIPLE-GRAIN CORRELATIONS

Having established that our master equation method indeed reproduces the previously known results, we now turn to the more interesting problem of correlations between multiple grains during the “avalanche dynamics.”

To this end, let us address the following problem. Consider the same lattice as above with all its sites in a stationary state, $z_{(j,n)} = z_{(j,n)}^c - 1$, and add K grains at randomly chosen K horizontal locations $l_1 > l_2 > \dots > l_K$. The toppling rules are the same as above: at each toppling two grains are removed from the toppling site j , and a grain can jump to each of the two nearest-neighbor sites in the downhill direction. However, if $z_{(j,n)} - z_{(j\pm 1,n+1)} = 0$, the site (j,n) cannot topple. The probability that the additional grains will be at dissipative sites $j_1 > j_2 > \dots > j_K$ at time t (after an arbitrary number of topplings) satisfies a generalized version of the master equation (3.3), namely,

$$\begin{aligned} \frac{d}{dt} P_{j_1, \dots, j_K; l_1, \dots, l_K}(t) = & \frac{1}{2} \sum_{r=1}^K [P_{j_1, \dots, j_K; l_1, \dots, l_{r-1}, l_r+1, l_{r+1}, \dots, l_K}(t) \\ & + P_{j_1, \dots, j_K; l_1, \dots, l_{r-1}, l_r-1, l_{r+1}, \dots, l_K}(t)] - K P_{j_1, \dots, j_K; l_1, \dots, l_K}(t), \end{aligned} \quad (4.1)$$

supplemented by the condition $P_{j_1, \dots, j_K; l_1, \dots, l_K}(t) = 0$ if $j_r = j_{r+1}$, for all $r = 1, \dots, K-1$. The solution to this equation is given by

$$P_{j_1 \dots j_K, l_1 \dots l_K}(t) = \det_{1 \leq r, s \leq K} \{P_{j_r l_s}(t)\}, \quad (4.2)$$

where $P_{jl}(t)$ is the one-grain conditional probability which satisfies Eq. (3.3) with the same boundary conditions as the solution of Eq. (4.1).

As for the single grain, in the multi-grain case the continuous conditional probabilities $P_{j_1, \dots, j_K; l_1, \dots, l_K}(t)$ are generating functions of the discrete ones, $G_{j_1, \dots, j_K; l_1, \dots, l_K}(n)$, and we find that

$$e^{Kt} P_{j_1, \dots, j_K; l_1, \dots, l_K}(t) = \sum_{n=0}^{\infty} G_{j_1, \dots, j_K; l_1, \dots, l_K}(n) \frac{K^n t^n}{n!}. \quad (4.3)$$

The discrete probabilities satisfy the equation

$$\begin{aligned} G_{j_1, \dots, j_K; l_1, \dots, l_K}(n) = & \frac{1}{2K} \sum_{r=1}^K \{G_{j_1, \dots, j_{r-1}, j_r+1, j_{r+1}, \dots, j_K; l_1, \dots, l_K}(n-1) \\ & + G_{j_1, \dots, j_{r-1}, j_r-1, j_{r+1}, \dots, j_K; l_1, \dots, l_K}(n-1)\}, \end{aligned} \quad (4.4)$$

supplemented by the condition $G_{j_1, \dots, j_K; l_1, \dots, l_K}(n) = 0$ if $j_r = j_{r+1}$, for all $r = 1, \dots, K-1$.

§5. SCALING OF CONTINUOUS-TIME MULTIPLE-GRAIN CORRELATIONS

5.1. The infinite lattice. Let us now consider the asymptotic behavior as $t \rightarrow \infty$ of the above multi-grain conditional probability. We first consider the case of the infinite lattice in the horizontal direction, when the one-grain probability is given by (3.7). Using the integral representation for the modified Bessel function, we arrive at the expression

$$P_{j_1, \dots, j_K; l_1, \dots, l_K}^{(-\infty, \infty)}(t) = \frac{1}{(2\pi)^K} \int_{-\pi}^{\pi} dx_1 \cdots \int_{-\pi}^{\pi} dx_K e^{-t \sum_{m=1}^K (1 - \cos x_m)} \times \det_{1 \leq r, s \leq K} \left\{ e^{i(l_s - j_r)x_r} \right\}. \quad (5.1)$$

Making use of the symmetry of the integrand with respect to permutations of the integration variables x_1, \dots, x_K , the determinant in this expression can be transformed as

$$\begin{aligned} \det_{1 \leq r, s \leq K} \left\{ e^{i(l_s - j_r)x_r} \right\} &\longrightarrow \det_{1 \leq r, s \leq K} \left\{ e^{il_s x_r} \right\} \prod_{r=1}^K e^{-ij_s x_r} \\ &\longrightarrow \frac{1}{K!} \det_{1 \leq r, s \leq K} \left\{ e^{-ij_s x_r} \right\} \det_{1 \leq r, s \leq K} \left\{ e^{il_s x_r} \right\}. \end{aligned} \quad (5.2)$$

The two determinants above can be represented in terms of Schur functions (for a survey on Schur functions, see, e.g., [25]):

$$\begin{aligned} s_{\lambda}(x_1, x_2, \dots, x_K) &:= \frac{\det_{1 \leq s, k \leq K} (x_s^{\lambda_k + K - k})}{\det_{1 \leq s, k \leq K} (x_s^{K - k})} \\ &= \det_{1 \leq s, k \leq K} (x_s^{\lambda_k + K - k}) \prod_{1 \leq s < k \leq K} (x_s - x_k)^{-1}, \end{aligned} \quad (5.3)$$

where $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_K)$ is a partition, i.e., a nonincreasing series of nonnegative integers $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_K \geq 0$. If we consider the case

where $j_r \geq -K$ and $l_r \geq -K$, we find that

$$P_{j_1, \dots, j_K; l_1, \dots, l_K}^{(-\infty, \infty)}(t) = \frac{1}{(2\pi)^K K!} \int_{-\pi}^{\pi} dx_1 \cdots \int_{-\pi}^{\pi} dx_K e^{-t \sum_{m=1}^K (1 - \cos x_m)} \\ \times s_\lambda(e^{ix_1}, e^{ix_2}, \dots, e^{ix_K}) s_\mu(e^{-ix_1}, e^{-ix_2}, \dots, e^{-ix_K}) \\ \times \prod_{1 \leq r < s \leq K} |e^{ix_r} - e^{ix_s}|^2, \quad (5.4)$$

where $\lambda_r = j_r - K + r$ and $\mu_r = l_r - K + r$.

As $t \rightarrow \infty$ (and $j_s - l_r \ll t$ for all $r, s = 1, \dots, K$), the main contributions to the above integrals come from near the origin of the integration variables, and in the leading order we find that

$$P_{j_1, \dots, j_K; l_1, \dots, l_K}^{(-\infty, \infty)}(t) \sim \frac{s_\lambda(1, 1, \dots, 1) s_\mu(1, 1, \dots, 1)}{(2\pi)^K K!} \\ \times \int_{-\infty}^{\infty} dx_1 \cdots \int_{-\infty}^{\infty} dx_K e^{-\frac{t}{2} \sum_{m=1}^K x_m^2} \prod_{1 \leq r < s \leq K} (x_r - x_s)^2. \quad (5.5)$$

The prefactor of the integral can be computed (see, e.g., [25]) using the well-known result

$$s_\lambda(1, 1, \dots, 1) = \frac{\prod_{1 \leq r < s \leq K} (\lambda_r - r - \lambda_s + s)}{\prod_{m=1}^{K-1} m!}, \quad (5.6)$$

while the integral is the Mehta integral of the Gaussian unitary ensemble of random matrices [26], which can be explicitly evaluated:

$$\int_{-\infty}^{\infty} dx_1 \cdots \int_{-\infty}^{\infty} dx_K e^{-\frac{t}{2} \sum_{m=1}^K x_m^2} \prod_{1 \leq r < s \leq K} (x_r - x_s)^2 \\ = \frac{(2\pi)^{K/2} \prod_{m=1}^K m!}{t^{K^2/2}}. \quad (5.7)$$

We thus find that, as $t \rightarrow \infty$, in the leading order the multi-grain conditional probability is given by

$$P_{j_1, \dots, j_K; l_1, \dots, l_K}^{(-\infty, \infty)}(t) \sim A_{j_1, \dots, j_K; l_1, \dots, l_K} t^{-\gamma} \quad (5.8)$$

with the scaling exponent

$$\gamma = \frac{K^2}{2} \quad (5.9)$$

and the amplitude

$$A_{j_1, \dots, j_K; l_1, \dots, l_K} = \frac{\prod_{1 \leq s < r \leq K} (l_r - l_s)(j_r - j_s)}{(2\pi)^{\frac{K}{2}} \prod_{m=1}^{K-1} m!}. \quad (5.10)$$

5.2. The semi-infinite lattice. Let us now consider the conditional probability in the presence of an absorbing boundary at the origin. As in the one-grain case, let us start with a finite lattice of N sites in the horizontal direction. Substituting (3.9) into (4.2), we find that

$$P_{j_1, \dots, j_K; l_1, \dots, l_K}(t) = \frac{2^K}{(N+1)^K} \sum_{k_1=1}^N \cdots \sum_{k_K=1}^N e^{-t \sum_{m=1}^K E_{k_m}} \\ \times \det_{1 \leq r, s \leq K} \left\{ \sin \frac{\pi j_r k_r}{N+1} \sin \frac{\pi l_s k_r}{N+1} \right\}, \quad (5.11)$$

where E_k is given by (3.10). The multi-grain conditional probability for the semi-infinite lattice follows from this result by taking the large N limit; the resulting expression is similar to (5.1), but with a determinant that now contains sine functions instead of exponential functions:

$$P_{j_1, \dots, j_K; l_1, \dots, l_K}^{(0, \infty)}(t) = \frac{1}{\pi^K} \int_{-\pi}^{\pi} dx_1 \cdots \int_{-\pi}^{\pi} dx_K e^{-t \sum_{m=1}^K (1 - \cos x_m)} \\ \times \det_{1 \leq r, s \leq K} \{ \sin(j_r x_r) \sin(l_s x_r) \}. \quad (5.12)$$

Again, using the symmetry with respect to permutations of the integration variables x_1, \dots, x_K , we can transform the determinant in this expression as

$$\det_{1 \leq r, s \leq K} \{ \sin(j_r x_r) \sin(l_s x_r) \} \longrightarrow \det_{1 \leq r, s \leq K} \{ \sin(l_s x_r) \} \prod_{r=1}^K \sin(j_r x_r) \\ \longrightarrow \frac{1}{K!} \det_{1 \leq r, s \leq K} \{ \sin(j_s x_r) \} \det_{1 \leq r, s \leq K} \{ \sin(l_s x_r) \}. \quad (5.13)$$

Using the character of the irreducible representation of the symplectic Lie algebra corresponding to a partition λ ,

$$\text{sp}_\lambda(x_1, x_2, \dots, x_K) := \frac{\det_{1 \leq j, k \leq K} (x_j^{\lambda_k + K - k + 1} - x_j^{-(\lambda_k + K - k + 1)})}{\det_{1 \leq j, k \leq K} (x_j^{K - k + 1} - x_j^{-(K - k + 1)})}, \quad (5.14)$$

we can express (5.12) in the form

$$P_{j_1, \dots, j_K; l_1, \dots, l_K}^{(0, \infty)}(t) = \frac{1}{\pi^K K!} \int_{-\pi}^{\pi} dx_1 \cdots \int_{-\pi}^{\pi} dx_K e^{-t \sum_{m=1}^K (1 - \cos x_m)} \\ \times \left(\det_{1 \leq r, s \leq K} \{\sin s x_r\} \right)^2 \text{sp}_{\lambda}(e^{ix_1}, e^{ix_2}, \dots, e^{ix_K}) \\ \times \text{sp}_{\mu}(e^{ix_1}, e^{ix_2}, \dots, e^{ix_K}), \quad (5.15)$$

where $\lambda_r = j_r - K + r - 1$ and $\mu_r = l_r - K + r - 1$. The determinant in the above integrand can be evaluated using the identity (for a proof, see [27])

$$\det_{1 \leq r, s \leq K} \{\sin s x_r\} = 2^{K(K-1)} \prod_{r=1}^K \sin x_r \\ \times \prod_{1 \leq j < k \leq K} \sin \frac{x_j - x_k}{2} \sin \frac{x_j + x_k}{2}. \quad (5.16)$$

For the conditional probability we finally obtain the expression

$$P_{j_1, \dots, j_K; l_1, \dots, l_K}^{(0, \infty)}(t) = \frac{2^{2K(K-1)}}{\pi^K K!} \int_{-\pi}^{\pi} dx_1 \cdots \int_{-\pi}^{\pi} dx_K e^{-t \sum_{m=1}^K (1 - \cos x_m)} \\ \times \prod_{r=1}^K \sin^2 x_r \prod_{1 \leq j < k \leq K} \sin^2 \frac{x_j - x_k}{2} \sin^2 \frac{x_j + x_k}{2} \\ \times \text{sp}_{\lambda}(e^{ix_1}, e^{ix_2}, \dots, e^{ix_K}) \text{sp}_{\mu}(e^{ix_1}, e^{ix_2}, \dots, e^{ix_K}). \quad (5.17)$$

In the limit $t \rightarrow \infty$, we can approximate the integrals in the above expression with the integrals

$$\int_{-\infty}^{\infty} dx_1 \cdots \int_{-\infty}^{\infty} dx_K e^{-\frac{1}{2}t \sum_{m=1}^K x_m^2} \prod_{1 \leq j < k \leq K} (x_j^2 - x_k^2)^2 \prod_{j=1}^K x_j^2 \\ = \frac{\prod_{m=1}^K (2m)!}{(2\pi)^{K/2} t^{K(2K+1)/2}}. \quad (5.18)$$

For a proof of (5.18), see [26]. For the leading asymptotic term of the generating function, we thereby find

$$P_{j_1, \dots, j_K; l_1, \dots, l_K}^{(0, \infty)}(t) \sim A_{j_1, \dots, j_K; l_1, \dots, l_K} t^{-\gamma}. \quad (5.19)$$

Here the scaling exponent is given by

$$\gamma = \frac{K(2K+1)}{2} \quad (5.20)$$

and the amplitude is

$$A_{j_1, \dots, j_K; l_1, \dots, l_K} = \frac{\prod_{m=1}^K (2m)!}{2^{K(K+1)} \pi^{\frac{3K}{2}} K!} \text{sp}_\lambda(1, \dots, 1) \text{sp}_\mu(1, \dots, 1), \quad (5.21)$$

in which

$$\text{sp}_\lambda(1, \dots, 1) = \prod_{1 \leq r < s \leq K} (j_r^2 - j_s^2) \prod_{m=1}^{K-1} \frac{[2(K-m)+1]!}{m!(K+m)!}. \quad (5.22)$$

A similar expression can be found for $\text{sp}_\mu(1, \dots, 1)$ with the j_r 's replaced by l_r 's.

We thus find that the scaling exponent of the multi-grain sandpile problem considered here is not equal to the one found previously for the “lock-step” version of vicious walkers, for which $\gamma = K(K-1)/4$ [13, 14], but corresponds to the “random-turns” version of vicious walkers [13, 17]. The connection to the “random-turns” version of vicious walkers is also valid for the discrete-time correlations, as discussed below.

§6. SCALING OF DISCRETE-TIME MULTIPLE-GRAIN CORRELATIONS

For definiteness, here we consider in detail the case of the infinite lattice in the horizontal direction; the case of the semi-infinite lattice can be considered similarly, and below we outline the results for both cases.

Using the relation between the continuous and discrete conditional probabilities, see (4.3), in the case of the infinite lattice we find from (5.1) the representation

$$G_{j_1, \dots, j_K; l_1, \dots, l_K}^{(-\infty, \infty)}(n) = \frac{1}{(2\pi)^K K^n} \int_{-\pi}^{\pi} dx_1 \cdots \int_{-\pi}^{\pi} dx_K \left(\sum_{m=1}^K \cos x_m \right)^n \times \det_{1 \leq r, s \leq K} \left\{ e^{i(l_s - j_r) x_r} \right\}. \quad (6.1)$$

We are interested here in the large n limit with the l_r 's and j_r 's kept fixed. In order to apply the standard saddle-point approximation, we express the first factor of the integrand in the above equation in the form

$\exp\{n \log(\sum_m \cos x_m)\}$, and thereby obtain the following system of saddle-point equations:

$$\frac{\sin x_r}{\sum_{m=1}^K \cos x_m} = 0, \quad r = 1, \dots, K. \quad (6.2)$$

It is obvious that the solutions to this system of equations satisfy $\sin x_r = 0$ ($r = 1, \dots, K$) with the restriction that $\sum_m \cos x_m \neq 0$. Requiring that the matrix of second derivatives

$$\frac{\partial^2}{\partial x_r \partial x_s} \log \left(\sum_{m=1}^K \cos x_m \right) = -\frac{\cos x_r}{\sum_{m=1}^K \cos x_m} \delta_{rs} - \frac{\sin x_r \sin x_s}{\left(\sum_{m=1}^K \cos x_m \right)^2} \quad (6.3)$$

for the solution of (6.2) is a negative definite matrix, we find that the steepest descent corresponds to the solution for which $\cos x_r = 1$ ($r = 1, \dots, K$), i.e., the main contribution to the integrals in (6.1) comes from near the points $x_r = 0$ ($r = 1, \dots, K$), similarly to the case of the continuous conditional probability.

Therefore, replacing the first factor of the integrand in (6.1) by its approximation near the origin of the integration variables,

$$\left(\sum_{m=1}^K \cos x_m \right)^n \propto K^n \exp \left\{ -\frac{n}{2K} \sum_{m=1}^K x_m^2 \right\}, \quad (6.4)$$

and transforming the second factor of the integrand as in Sec. 5.1, we find that, as $n \rightarrow \infty$, the leading order form of the discrete multi-grain probability in the case of the infinite lattice can be expressed as

$$G_{j_1, \dots, j_K; l_1, \dots, l_K}^{(-\infty, \infty)}(n) \sim \frac{s_\lambda(1, 1, \dots, 1) s_\mu(1, 1, \dots, 1)}{(2\pi)^K K!} \times \int_{-\infty}^{\infty} dx_1 \cdots \int_{-\infty}^{\infty} dx_K e^{-(n/2K) \sum_{m=1}^K x_m^2} \prod_{1 \leq r < s \leq K} (x_r - x_s)^2. \quad (6.5)$$

Clearly, this expression is the same as (5.5) with t replaced by n/K . It is easy to check that a similar result is valid for the semi-infinite lattice, now using the procedure of Sec. 5.2. Hence, the leading terms of the large n limits of the discrete multi-grain probabilities can be obtained from those of the large t limits of the continuous probabilities simply by the replacement $t \mapsto n/K$.

We have thus shown that, as $n \rightarrow \infty$ for fixed l_r 's and j_r 's, the discrete multi-grain conditional probabilities scale as

$$G_{j_1, \dots, j_K; l_1, \dots, l_K}(n) \sim B_{j_1, \dots, j_K; l_1, \dots, l_K} n^{-\gamma} \quad (6.6)$$

with

$$B_{j_1, \dots, j_K; l_1, \dots, l_K} = K^\gamma A_{j_1, \dots, j_K; l_1, \dots, l_K}, \quad (6.7)$$

where the exponent γ and the amplitude $A_{j_1, \dots, j_K; l_1, \dots, l_K}$ are given by (5.9) and (5.10), respectively, in the case of the infinite lattice, and by (5.20) and (5.21), respectively, in the case of the semi-infinite lattice.

Therefore, the obtained results imply that the discrete forms of the conditional probabilities scale exactly as the continuous ones, as they should. In particular, this implies that these results also coincide with the known scaling properties of the random-turns version of vicious walkers [16].

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