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ON k-ABELIAN AVOIDABILITY

Abstract. We consider a recently defined notion of *k-abelian equivalence* of words by giving some basic results and concentrating on avoidability problems. This equivalence relation counts the numbers of factors of length k for a fixed natural number k. We ask for the size of the smallest alphabet for which k-abelian squares and cubes can be avoided, respectively. For 2-abelian squares this is four – as in the case of *abelian words*, while for 2-abelian cubes we have only strong evidence that the size is two – as it is in the case of *words*. In addition, we point out a few properties of morphisms supporting the view that it might be difficult to find solutions to our questions by simply iterating a morphism.

§1. Introduction

Theory of avoidability is among the oldest and most studied topics in Combinatorics on Words. The first results in this area, or in fact in the whole field, were obtained by Norwegian Axel Thue as early as at the beginning of 20th century [16, 17]. He showed the existence of an infinite binary word, which does not contain any factor three times consecutively, that is the existence of an infinite cube-free word. Similarly, he showed that squares can be avoided in infinite ternary words.

Since late 1960's *abelian*, i.e. commutative, variants of the above problems were studied. Apparently, first nontrivial results were obtained by Evdokimov [6] who showed that commutative squares can be avoided in infinite words over a 25-letter alphabet. The size of the alphabet was reduced to five by Pleasant [15], until the optimal value, four, was found by Keränen [11]. Dekking [5] managed to prove already earlier that the optimal value for the size of the alphabet where abelian cubes are avoidable is three.

We introduce in this paper new variants of the problems by defining repetitions via new equivalence relations which lie properly in between equality and abelian equality. For this relation we use the notion *k-abelian*

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equivalence, where $k \geq 1$ is a natural number. We will give some computational results about k-abelian avoidability and present some relations between morphisms and k-abelian repetitions. It is interesting to recall that results on word avoidability and abelian avoidability are typically obtained by iterating a morphism but for our questions it seems unlikely that the same would hold.

§2. Preliminaries

For the basic terminology of words as well as avoidability we refer to [14] and [4]. The basic notion in this paper, *k-abelian equivalence* of words, is defined as follows.

Definition 1. *Let $k \geq 1$ be a natural number. We say that words u and v in Σ^+ are k-abelian equivalent, in symbols $u \equiv_{a,k} v$, if*

(1) *$pref_{k-1}(u) = pref_{k-1}(v)$ and $suf_{k-1}(u) = suf_{k-1}(v)$, and*
(2) *for all $w \in \Sigma^k$, the number of occurrences of w in u and v coincide, i.e. $\#(w,u) = \#(w,v)$.*

Here pref_{k-1} (resp. suf_{k-1}) is used to denote the prefixes (resp. suffixes) of length $k-1$. The first condition makes the notion a sharpening of abelian equality. Indeed, words *aba* and *bab* have the same factors of length 2 but they are not abelian equivalent. In fact, it is enough to require the words to have either a common prefix of length $k-1$ or a common suffix of length $k-1$. As is easy to see, two words with a common prefix and satisfying the condition 2. also have a common suffix, and vice versa.

It is straightforward to see that $\equiv_{a,k}$ is an equivalence relation and, moreover,

$$u = v \Rightarrow u \equiv_{a,k} v \Rightarrow u \equiv_a v,$$

where $\equiv_a$ denotes the abelian equivalence, and that

$$u = v \Leftrightarrow u \equiv_{a,k} v \quad \forall\, k \geq 1.$$

An optional definition for k-abelian equivalent words would be the following:

Definition 2. *Let $k \geq 1$ be a natural number. Words u and v in Σ^+ are k-abelian equivalent if*

(1) *for all $w \in \Sigma^{k-1}$, the number of occurrences of w in u and v coincide, i.e. $\#(w,u) = \#(w,v)$, and*

(2) *for all* $w' \in \Sigma^k$*, the number of occurrences of* w' *in* u *and* v *coincide, i.e.* $\#(w', u) = \#(w', v)$.

Now, notions like *k-abelian repetitions* are naturally defined. For instance, $w = uv$ is a *k-abelian square* if and only if $u \equiv_{a,k} v$. By combining these two definitions we can easily conclude the next lemma.

Lemma 1. *If an infinite word* w *contains* k*-abelian repetition of order* m *then* w *contains* k'*-abelian repetition of order* m *for each* $1 \leq k' \leq k$.

Proof. Follows from the definitions of k-abelian equivalence straightforwardly: If words u and v are k-abelian equivalent they have common prefixes of length $k' \leq k-1$ (resp. suffixes) and the number of factors of length k and $k-1$ coincide, and thus u and v are also $(k-1)$-abelian equivalent. Inductively, they are k'-abelian equivalent for each $1 \leq k' \leq k$. □

In addition, in the binary case 2- and 3-abelian words are fairly easy to characterize.

Example 1. In a binary alphabet $\Sigma = \{a, b\}$ the characterization of equivalence classes of 2-abelian words, via their representatives, can be given in the form:

$$aa^k b^l (ab)^m a^n \quad \text{or} \quad bb^k a^l (ba)^m b^n,$$

where $k, l, m \geq 0$ and $n \in \{0, 1\}$. And in the same alphabet the characterization of equivalence classes of 3-abelian words can be given in the following form containing eight possible combinations:

$$\left.\begin{array}{l} aaa^k b^l (aabb)^m \\ bbb^k a^l (aabb)^m \\ abb^k a^l (aabb)^m \\ baa^k b^l (aabb)^m \end{array} \text{or} \right\} * \quad \text{connected with} \quad * \left\{ \begin{array}{l} (aab)^g (ab)^h b^i a^j \\ (abb)^g (ab)^h b^i a^j, \end{array} \right. \text{or}$$

where $k, l, m, g, h \geq 0$, $i \in \{0, 1\}$ and $j \in \{0, \ldots, 2-i\}$. Note that here the representation of the equivalence classes is not unambiguous.

The above allows us to estimate the number of the corresponding equivalence classes, see [8]. For binary words of length n the number of 2-abelian equivalence classes is $n^2 - n + 2$, i.e., $\Theta\left(n^2\right)$ and for 3-abelian equivalence classes $\Theta\left(n^4\right)$ (and not $\Theta(n^3)$), respectively. Recently, A. Saarela [10] showed that in general the number of k-abelian equivalence classes of words of length n is polynomial in n but the degree of the polynomial grows exponentially in k (in a fixed but arbitrary alphabet). For $k = 4$ the value of the exponent is eight in a binary alphabet.

§3. Problems on avoidability

Natural variants of the Thue's problems ask for the smallest alphabets where k-abelian squares and cubes can be avoided. A goal of this paper is to point out that these problems are not trivial, even in the case $k = 2$. Before going into our problems we recall Table 1 which summarizes these values in the case of word repetitions and abelian repetitions, and tells the limits of our problems:

Table 1. Avoidability of different types of repetitions in infinite words.

Avoidability of squares				Avoidability of cubes			
	type of rep.				type of rep.		
size of the alph.	$=$	$\equiv_{a,2}$	$\equiv_a$	size of the alph.	$=$	$\equiv_{a,2}$	$\equiv_a$
2	$-$	$-$	$-$	2	$+$	?	$-$
3	$+$	?	$-$	3	$+$	$+$	$+$
4	$+$	$+$	$+$				

Our next example hints that the ordinary method of iterating a morphism might not give answers to our problems.

Example 2. In each of the following known cases where a repetition free infinite word is obtained by iterating a morphism, a 2-abelian cube is found fairly early from the beginning. If anything else is not mentioned see [1] for the reference of the following words.

- Infinite overlap-free Thue-Morse word (by iterating the morphism: $0 \to 01$, $1 \to 10$): $01\overbrace{101001}\overbrace{100101}\overbrace{101001}011...$
- Cube-free infinite word (by iterating the morphism: $0 \to 001$, $1 \to 011$): $001001\overbrace{011001}\overbrace{001011}\overbrace{001011}011...$
- Morphism $0 \to 001011$, $1 \to 001101$, $2 \to 011001$ maps ternary cube-free words to binary cube-free words, see [2], but $001011 \equiv_{a,2} 001101 \equiv_{a,2} 011001$, thus images of all words mapped with this morphism contains 2-abelian cubes.
- A binary overlap-free word w can also be gained in form $w = c_0c_1c_2\ldots$, where c_n means the number of zeros (mod 2) in the binary expansion of n. Again, a 2-abelian cube of length 6 begins as early as from the fifth letter: $w = 0010\overbrace{011010}\overbrace{010110}\overbrace{011010}011...$

- A binary sequence called Kolakoski sequence is cube-free, see [3] and [13], but not 2-abelian cube-free: $122\,\overbrace{112122}\,\overbrace{122112}\,\overbrace{112212}\,112\ldots$ (It is an open question whether the Kolakoski sequence is a morphic word.)

The next Theorems 1 and 2 bring out some properties that the infinite words generated by iterating a morphism have. These also support the view that iterating a morphism may not be a strong enough tool to produce infinite k-abelian repetition free words. For the proofs of theorems we give two lemmas, the latter being an extension of the former. For the clarity, we first introduce and prove this special case.

Lemma 2. *Let h be a 1-free morphism over an alphabet Σ and let w be a word over Σ. Let $n = min\{|h(a)| : a \in \Sigma\}$. If w has 2-abelian equivalent factors u and v then the word $h(w)$ has $(n+1)$-abelian equivalent factors $h(u)$ and $h(v)$.*

Proof. Clearly, both $h(u)$ and $h(v)$ are factors of $h(w)$. Let $\text{pref}_1(u) = \text{pref}_1(v) = x$ and $\text{suf}_1(u) = \text{suf}_1(v) = y$ for some $x, y \in \Sigma$. Now $h(x)$ is a prefix of $h(u)$ and $h(v)$ and similarly $h(y)$ is a suffix of $h(u)$ and $h(v)$ where $|h(x)|, |h(y)| \geq n$. Thus the first condition of Definition 1 of k-abelian equivalent words holds.

Each factor of length $(n+1)$ in $h(u)$ (or $h(v)$) is contained in a factor $h(st)$ where st is a factor of u (or v) and $s, t \in \Sigma$. This follows from the choice of n. In fact, a factor of length $(n+1)$ may be contained in a factor of the form $h(s)$ for some $s \in \Sigma$. In any case, words u and v are 2-abelian equivalent words and thus abelian equivalent words, too. So words u and v have the same number of each letter and the same number of each factor of length two, respectively. Thus the words $h(u)$ and $h(v)$ have the same number of factors $h(s)$ for each $s \in \Sigma$ and $h(st)$ for each $s, t \in \Sigma$. From this we can conclude that the words $h(u)$ and $h(v)$ have the same number of occurrences of each factor of length $(n+1)$. Now the second condition of Definition 1 of k-abelian equivalent words is also satisfied which completes the proof. $\square$

We can generalize the previous lemma by taking k-abelian factors as a starting point.

Lemma 3. *Let h be a 1-free morphism over an alphabet Σ and let w be a word over Σ. Let $n = min\{|h(a)| : a \in \Sigma\}$. If w has k-abelian equivalent* *factors u and v then the word $h(w)$ has $((k-1)n+1)$-abelian equivalent factors $h(u)$ and $h(v)$.*

Proof. The idea of the proof is the same as in the proof of Lemma 2. Now $\text{pref}_{k-1}(u) = \text{pref}_{k-1}(v)$ and $\text{suf}_{k-1}(u) = \text{suf}_{k-1}(v)$ ensuring $h(u)$ and $h(v)$ to have a common prefix (resp. suffix) of length at least $(k-1)n$.

Correspondingly, each factor of length $((k-1)n+1)$ in $h(u)$ (or $h(v)$) is contained in a factor $h(p)$ where p is a factor of u (or v) and $|p| \leq k$. Because the words u and v are k-abelian equivalent the numbers of each factor of length at most k coincide in these words. From these it follows that $h(u)$ and $h(v)$ have the same number of each factor of length $((k-1)n+1)$, and thus the words are $((k-1)n+1)$-abelian equivalent. □

In the following theorems we assume h to be a prefix preserving morphism over an alphabet Σ and $a \in \Sigma$ to be such that $h^\infty(a)$ is well defined. Let $n = \min\{|h(a)| : a \in \Sigma\} > 1$.

Theorem 1. *The following two conditions are equivalent:*

(1) *The infinite word $h^\infty(a)$ contains k-abelian repetition of order m for some $k \geq 2$.*
(2) *The infinite word $h^\infty(a)$ contains k-abelian repetition of order m for each $k \geq 1$.*

Proof. It is clear that the condition 1. follows from 2. straightforwardly.

Let us now prove that the first condition implies the second one. Let $w = u_1 u_2 \cdots u_m$ be a factor of $h^\infty(a)$ such that words u_i are k-abelian equivalent words with each other. Now $h^\infty(a)$ also contains the factor $h(w) = h(u_1)h(u_2)\cdots h(u_m)$. From Lemma 3 we know that words $h(u_i)$ are $((k-1)n+1)$-abelian equivalent words with each other and thus $h^\infty(a)$ contains a $((k-1)n+1)$-abelian repetition of order m. Now we can again apply Lemma 3 for the case $h^\infty(a)$ having a $((k-1)n+1)$-abelian repetition of order m. It gives us that $h^\infty(a)$ has a $((k-1)n^2+1)$-abelian repetition of order m. Repeating this procedure we can conclude that $h^\infty(a)$ has a $((k-1)n^i+1)$-abelian repetition of order m for each $i \in \mathbb{N}$. In addition, from Lemma 1 we know that $h^\infty(a)$ contains k'-abelian repetition of order m for each $1 \leq k' \leq (k-1)n^i+1$. Thus the infinite word $h^\infty(a)$ contains k-abelian repetition of order m for each $k \geq 1$. □

We can also formalize the previous Theorem 1 in the context of k-abelian avoidability.

Theorem 2. *The following two conditions are equivalent:*

(1) *The infinite word $h^\infty(a)$ is k-abelian m-free for some $k \geq 1$.*
(2) *The infinite word $h^\infty(a)$ is k-abelian m-free for each $k \geq 2$.*

Proof. Follows from Theorem 1. □

Next we mention a few consequences of the above. We remark that h was chosen to be a prefix preserving morphism so that $h^\infty(a)$ is well defined and $n = \min\{|h(a)| : a \in \Sigma\} > 1$. Let H be the set of morphisms satisfying these conditions.

Remark 1. If each infinite binary word contained 2-abelian cube then from Theorem 1 would follow that for each $h \in H$ over binary alphabet $h^\infty(a)$ would contain k-abelian cube for all $k \geq 1$. This means that if there exists a binary morphism $h \in H$ such that $h^\infty(a)$ is k-abelian cube-free for some $k \geq 2$ then there exists an infinite 2-abelian cube-free word over binary alphabet. The same result can be concluded straightforwardly from Theorem 2.

Remark 2. There exists a 2-uniform prefix preserving morphism over two letter alphabet generating a cube-free binary word, for example Thue-Morse word. However, Thue-Morse word contains 2-abelian cube as shown in Example 2, and thus from Theorem 1 it follows that this word is not k-abelian cube-free for any $k \geq 1$.

Remark 3. We can construct an infinite binary 8-abelian cube-free word as a morphic image of an infinite word generated by iterating a uniform morphism, see [7]. It is easy to see that this word contains 2-abelian cube as a factor, which implies by Theorem 2 that the word cannot be obtained by iterating a binary morphism $h \in H$. This also shows how we can use our theorems for deciding whether some infinite word can be obtained by iterating a morphism $h \in H$.

We will conclude with another general remark. With *Parikh properties* we mean abelian properties of words, that is, the numbers of occurrences of each letter. The *k-generalized Parikh properties* then refer to as numbers of occurrences of factors of length k. As stated in [9] these are closely related in problems defined by morphisms. Problems on 1-free morphisms and k-generalized Parikh properties can be reduced to problems on 1-free morphisms and usual Parikh properties in a bigger alphabet. The action to check the factors of length k is denoted with a mapping $\bigwedge_k : \Sigma^* \to \widehat{\Sigma}^*$

and we use $\bigwedge_k(x) = \widehat{x}$, for convenience. Details of the notions and the proof of the following lemma are given in [9].

Lemma 4. *Let $h : \Sigma^* \to \Sigma^*$ be a 1-free morphism and $k \geq 1$. Then there exists a morphism $\widehat{h} : \widehat{\Sigma}^* \to \widehat{\Sigma}^*$ such that $\bigwedge_k h = \widehat{h} \bigwedge_k$, i.e., the following diagram holds true for all $x \in \Sigma^*$*

$$\begin{array}{ccc} x & \xrightarrow{\wedge_k} & \widehat{x} \\ {\scriptstyle h}\big\downarrow & & \big\downarrow{\scriptstyle \widehat{h}} \\ h(x) & \xrightarrow{\wedge_k} & \widehat{h(x)} = \widehat{h}(\widehat{x}). \end{array}$$

From Lemma 4 we can conclude, for instance, that the avoidability questions on k-abelian repetitions for a morphism h can be reduced to problems on abelian avoidability for a morphism $\widehat{h}$, although in bigger alphabets.

§4. Computational results

We were able to settle the first one of the question marks in Table 1 by computer checking, i.e., we were able to produce the longest ternary 2-abelian square-free word and thus show that it is finite. For more details see [8].

Theorem 3. *The longest ternary word which is 2-abelian square-free has length 537, which shows that there does not exist an infinite 2-abelian square-free word over a ternary alphabet. This longest word is unique up to the permutations of the alphabet, $\Sigma = \{a, b, c\}$:*

abcbabcacbacabacbabcbacabcbabcabacabcacbacabacbabcbacbcacbabcacbcabcba
bcabacbabcbacbcacbacabacbabcbacabcbabcabacabcacbacabacbabcbacbcacbacab
acbcabacabcacbcabcbacbcacbacabacbabcbacbcacbabcacbcabcbabcabacbabcbacb
cacbacabacbabcbacabcbabcabacabcacbacabacbabcbacbcacbacabacbcabacabcacb
cabcbabcabacabcacbacabacbabcbacabcbabcabacabcacbcabcbabcabacbabcbacbca
cbabcacbcabcbabcabacabcacbcabcbacbcacbacabacbcabacabcacbcabcbabcabacab
cacbacabacbabcbacabcbabcabacabcacbcabcbabcabacbabcbacbcacbabcacbcabcba
bcabacabcacbacabacbabcbacabcbabcabacabcacbabcba

Now we can combine the results of Theorem 1 and Theorem 3 to once again point out the difficulties we have with words generated by iterating a morphism. The set H is the same as in the previous section.

Remark 4. Theorem 3 shows that each infinite word over three letter alphabet contains a 2-abelian square. From Theorem 1 it follows that for each $h \in H$ over ternary alphabet $h^{\infty}(a)$ contains k-abelian square for all $k \geq 1$. This means that k-abelian square-free word over ternary alphabet cannot be generated by iterating a morphism $h \in H$ over ternary alphabet for any $k \geq 1$.

Remark 5. There exists a 13-uniform prefix preserving morphism over three letter alphabet generating a squre-free ternary word, see [12]. Nevertheless, as mentioned in Remark 4 this word is not k-abelian square-free for any $k \geq 1$.

We return to analyze our computational results more closely.

Example 3. We constructed each ternary 2-abelian square-free word and analyzed the sizes of the sets containing words from length 1 to 537, respectively. There exist 404 286 words of length 105 which form the biggest set. The number of words grows monotonically from 3 up to 403 344 for lengths from 1 to 103. The sizes of the sets containing ternary 2-abelian square-free words are shown in Figure 1.

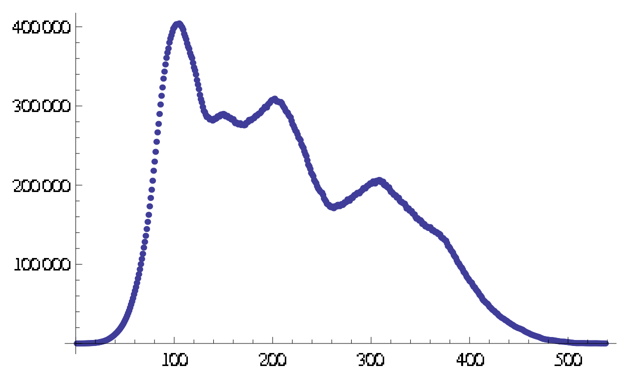

Fig. 1. The number of 2-abelian square-free words with respect to their lengths.

To solve the other question mark we also did some computer checking – and obtained evidence that the answer is likely to be different compared to the first one.

Example 4. With a computer we were able to construct a binary word of more than 100 000 letters that still avoids 2-abelian cubes. This shows that there exist, at least, very long binary 2-abelian cube-free words.

Example 5. Similarly as for the ternary 2-abelian square-free words, we can search for the numbers of binary 2-abelian cube-free words of different lengths. The numbers of such words with lengths from 1 to 60 grow approximately with a factor 1.3 at each increment of the length, see Figure 2. So that the number of binary 2-abelian cube-free words of length 60 is already 478 456 030. And already, with length 12 there exist more binary 2-abelian cube-free words (254) than ternary 2-abelian square-free words (240).

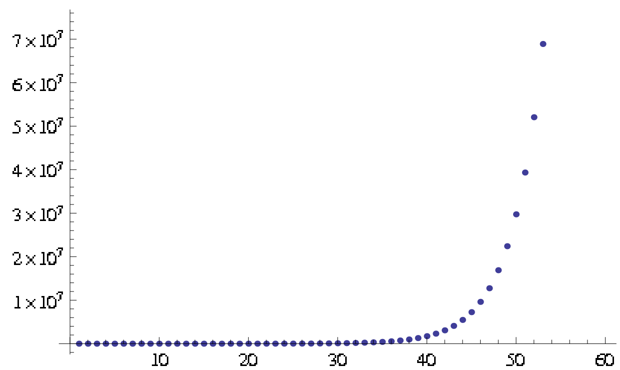

Fig. 2. The numbers of 2-abelian cube-free words of length n.

We also chose some binary 2-abelian cube-free prefixes and counted the numbers of binary 2-abelian cube-free words having these fixed prefixes. In this way we can check how many suitable extensions the chosen 2-abelian cube-free word has. As a result, we found examples of binary 2-abelian cube-free words with a property that the number of their extensions again grows approximately with a factor 1.3 when increasing the length of extensions by one. In Figure 3 this is done fore a fixed prefix of length 2000.

These examples support the conjecture that there would exist an infinite binary word that avoids 2-abelian cubes.

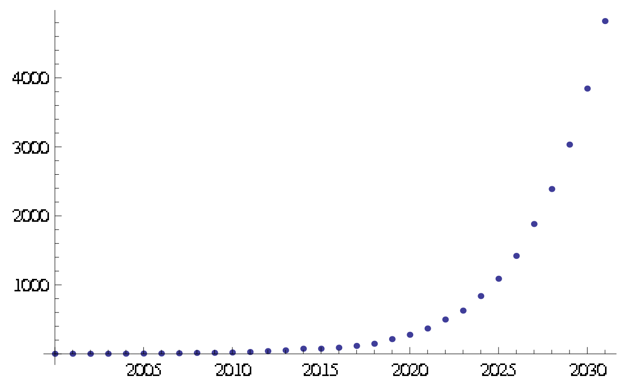

Fig. 3. The numbers of 2-abelian cube-free words of lengths 2000-2031 having a fixed prefix of length 2000.

§5. Conclusions and open problems

There exist only finitely many 2-abelian square-free words over a ternary alphabet meaning that the smallest alphabet avoiding 2-abelian squares is four-letter alphabet. In the case of 2-abelian cubes we have not proved that a binary alphabet would avoid such a repetition but we have strong evidence for that. As a conclusion, our two considered problems would behave differently: one like words and the other like abelian words.

Thus, it remains open to show whether there exist an infinite binary word avoiding 2-abelian cubes. We could also ask for the value of k, if exists, enabling the avoidability of k-abelian squares in a ternary alphabet. As was mentioned, this cannot be obtained by iterating a morphism h having $|h(a)| > 1$ for all $a \in \Sigma$. Similarly, we could ask for the avoidability of k-abelian cubes in binary alphabet for different values of k. In fact, the questions about 2-abelian cube-freeness and k-abelian cube-freeness are closely related in the context of morphism iteration. If we could find a morphism g over $\{a, b\}$, with $|g(a)|, |g(b)| > 1$, such that the iteration of it would generate an infinite word avoiding k-abelian cubes for some $k > 1$ then we would have an infinite word avoiding k-abelian cubes for all $k \geq 1$, especially in the case $k = 2$.

Nevertheless, iterating morphisms is not the only way to produce infinite words. As was mentioned in Remark 3, there exists an infinite binary 8-abelian cube-free word that is obtained as a morphic image of an infinite word generated by iterating a uniform morphism. Anyhow, this word cannot be obtained by only iterating a binary morphism g' over $\{a, b\}$, with $|g'(a)|, |g'(b)| > 1$, because the word contains 2-abelian cubes.

An interesting remark is also that there exist infinite words over ternary alphabet avoiding usual word squares but they are not k-abelian square-free for any $k \geq 1$. Similarly, there exist infinite words over binary alphabet avoiding usual word cubes but they are not k-abelian cube-free for any $k \geq 1$.

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