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DEGENERACY OF SOME DERIVED CATEGORIES

Abstract. We study derived categories for the category of the modules over some generalized rings. In particular, the cases of $\mathcal{O}_{\mathbb{R}}$ and of $\mathbb{F}_{1^n}$ are considered. It is shown that these derived categories are degenerate. The degeneracy means that every isomorphism in such a category can be detected on the π_0- and π^0-levels.

Introduction

M. J. Shai Haran in [1] and N. Durov in [2] propose certain generalizations of the algebraic geometry. In this paper we adhere to Durov's version.

When studying a new geometry it is natural to apply methods, well working for the classical prototype. In particular, it would be desirable to get a generalized version of derived categories, related to the category of the $\mathcal{O}_X$-modules, and it would be desirable to get corresponding derived functors as well. Partly it is done in [2], where the categories $D^{\leq 0}(\mathcal{O}_X\text{-Mod})$ and $D^-(\mathcal{O}_X\text{-Mod})$ are defined, and the derived functors Lf^* for the inverse image functor f^* are constructed. The approach, used there, leads often to very rich categories. For example, $D^{\leq 0}(\mathbb{F}_0\text{-Mod})$ is equivalent to the unstable homotopy category.

The present paper is a result of studying derived constructions for the test case of the structure morphism $\widehat{\operatorname{Spec}\mathbb{Z}} \to \operatorname{Spec}\mathbb{F}_{1^2}$. As to the inverse image and categories of $D^{\leq 0}$-type, it appears that the existing theory [2, Th. 9.7.12] doesn't look adequate for the important particular case of generalized rings, similar to $\mathcal{O}_{\mathbb{R}}$. Corresponding results are given below in the first section.

As to categories of $D^{\geq 0}$- or D^+-type and to derived direct images, the situation appears to be even more unclear. In particular, the second section below contains results, displaying a degeneracy of one possible approach to the problem on the $\mathbb{F}_1$-level already.

Key words and phrases: field with one element, generalized ring, derived category, direct image, simplicial, cosimplicial, Archimedean, Dold–Kan.

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§1. Derived categories for rings of $\mathcal{O}_\mathbb{R}$-type

Let R be a generalized ring, R-Mod be the category of the left R-modules (see [2]), and sR-Mod be the category of the simplicial R-modules. By definition the derived category $D^{\leq 0}(R)$ is the localization of the category sR-Mod with a class of weak equivalences $W(sR\text{-Mod})$. Here a morphism of simplicial R-modules is considered to be a weak equivalence if the morphism of simplicial sets, obtained by forgetting the R-module structure, is a weak equivalence.

1.1. The stable derived category $D^-(\mathcal{O}_\mathbb{R})$. Let $\mathcal{O}_\mathbb{R} = \mathbb{Z}_\infty$, where the generalized ring $\mathbb{Z}_\infty$ is defined in [2]. The following result was obtained by my student Svyatoslav Pimenov in [3].

1.1.1. Theorem (Pimenov)**.** *Let $i : \mathcal{O}_\mathbb{R} \to \mathbb{R}$ be the natural inclusion; then the functors of the derived direct and inverse images in the diagram*

$$D^-(\mathcal{O}_\mathbb{R}) \underset{Ri_*}{\overset{Li^*}{\rightleftarrows}} D^-(\mathbb{R})$$

are quasi-inverse equivalences of categories.

We are going to explain the notions used in this statement. Occasionally we note some important distinctions between the generalized case and the classical case.

Let $\rho : S \to R$ be a morphism of generalized rings, $\rho^* : R\text{-Mod} \to S\text{-Mod}$ be the corresponding inverse image functor, and $\rho_* : S\text{-Mod} \to R\text{-Mod}$ be the corresponding direct image functor. Then the derived functor $L\rho^* : D^{\leq 0}(R) \to D^{\leq 0}(S)$ and the derived functor $R\rho_* : D^{\leq 0}(S) \to D^{\leq 0}(R)$ are understood in the meaning of Quillen [4, 4.1].

Contrary to the classical case, the direct image functor ρ_* is not right exact [2, 0.4.27], generally speaking. It is somewhat surprising because we are still in the affine world! In fact, let $R = \mathbb{F}_0$, $S = \mathbb{Z}$, and ρ be the unique homomorphism. Then the natural homomorphism $\rho_* M \coprod \rho_* M \to \rho_*(M \coprod M)$ is not an isomorphism for $M = \mathbb{Z}(1)$. Nevertheless, the derived functor $R\rho_*$ coincides with the original functor ρ_* in the sense that the natural functor morphism $\gamma_S \circ s\rho_* \to R\rho_* \circ \gamma_R$ is an isomorphism. Here $s\rho_* : sR\text{-Mod} \to sR\text{-Mod}$ is the natural extension of ρ_* to the categories of the simplicial modules, whereby γ_S and γ_R are the corresponding localizing functors for the classes of the weak equivalences. The mentioned coincidence can be seen from the existence of certain model structure on

sR-Mod. Namely, the class of weak equivalences of this structure coincides with the above mentioned class $W(sR\text{-Mod})$, and the forgetting functor $sR\text{-Mod} \to \mathrm{s}\mathbf{S}\mathrm{et}$ maps every acyclic fibration to an acyclic fibration [2, 0.8.8]. Thus, in the generalized situation the derived direct image does not improve, generally speaking, the situation with the (non)exactness of the direct image functor!

Next, let R be an $\mathbb{F}_1$-algebra, i.e., a generalized ring with a central zero. In this case, by definition, the category $D^-(R)$ is constructed of the category $D^{\leq 0}(R)$ by means of formal inverting the suspension functor [2, 8.6.12] (a reason for inverting the pointed suspension only one can learn in [5, p. 594]). Let ρ be a morphism of $\mathbb{F}_1$-algebras. The functors $D^-(R) \to D^-(S)$ and $D^-(R) \to D^-(S)$, induced by the functors $L\rho^* : D^{\leq 0}(R) \to D^{\leq 0}(S)$ and $R\rho_* : D^{\leq 0}(R) \to D^{\leq 0}(S)$, are denoted by the same symbols $L\rho^*$ и $R\rho_*$. Exactly such a situation occurs in the Theorem 1.1.1.

Pimenov's proof of his theorem is based on the fact that the additive structure on the category $D^-(\mathcal{O}_\mathbb{R})$, arising due to the stability, coincides with the additive structure, induced from the additive category of the $\mathbb{R}$-vector spaces.

1.2. The unstable derived category $D^{\leq 0}(\mathbb{F}_\infty)$. The generalized ring $\mathbb{F}_\infty$ is the residue field at the archimedean point of $\mathbb{Q}$. First, consider a smaller generalized ring $\mathbb{FN}_\infty$. Informally speaking, $\mathbb{FN}_\infty$ is the residue field at the archimedean point of $\mathbb{N}$. This generalized ring is generated over $\mathbb{F}_1$ by one binary operation $\wedge$, subjected to the relations

$$x \wedge x = x, x \wedge y = y \wedge x, (x \wedge y) \wedge z = x \wedge (y \wedge z) \tag{1}$$

and to the relation $0 \wedge x = 0$ (see [6]).

1.2.1. Theorem. *Let R be a generalized ring and let $R(2)$ contain an operation $\wedge$, subjected to the relations (1). Let $|X|$ be a simplicial set obtained from $X \in sR$-*Mod *by forgetting the R-module structure; then each connected component of $|X|$ is weakly equivalent to the point.*

Proof. It is enough to show that $\pi_n(|X|, x_0) = 1$ for each $x_0 \in |X|_0$ and $n \geq 1$. Without loss of generality we can assume that $|X|$ is a Kan set. In fact, take a fibered replacement for X in the model structure from [2, 0.8.8]) on the category sR-Mod and use the fact that in this case the forgetting functor transfers every fibered object to a Kan set and every weak equivalence to a weak equivalence. Supposing $\bar{\gamma} \in \pi_n(|X|, x_0)$, let us prove that $\bar{\gamma} = 1$. Since $|X|$ is a Kan set and every simplicial set is a cofibration, then due to a Quillen theorem [4, Cor.1, 1.16] the class $\bar{\gamma} \in \pi_n(|X|, x_0)$ can be represented by a spheroid, i.e., by a simplex $\gamma \in |X|_n$, subjected to the relations $d_0\gamma = \cdots = d_n\gamma = \zeta_{n-1}$. Here ζ_n is the only n-simplex with the support in x_0 (of course, ζ_n is a degenerate simplex), and the image of γ in $\pi_n(|X|, x_0)$ coincides with $\bar{\gamma}$.

We claim that

$$\gamma \sim \gamma_1^2, \tag{2}$$

where $\gamma_1 = \gamma \wedge \zeta_n$, and the symbol $\sim$ means the equality of the corresponding classes in $\pi_n(|X|, x_0)$. Actually, consider the $(n+1)$-simplex $\sigma = s_{n-1}(\gamma) \wedge s_n(\gamma)$ (here s_j is the corresponding degeneration operator) and compute its boundary. We get $d_i(\sigma) = d_i(s_{n-1}(\gamma)) \wedge d_i(s_n(\gamma)) = \zeta_n \wedge \zeta_n = \zeta_n$ for $0 \le i \le n-2$. Besides,

$$\begin{aligned} d_{n-1}(\sigma) &= d_{n-1}(s_{n-1}(\sigma)) \wedge d_{n-1}(s_n(\sigma)) = \gamma \wedge \zeta_n, \\ d_n(\sigma) &= d_n(s_{n-1}(\sigma)) \wedge d_n(s_n(\sigma)) = \gamma \wedge \gamma = \gamma, \\ d_{n+1}(\sigma) &= d_{n+1}(s_{n-1}(\sigma)) \wedge d_{n+1}(s_n(\sigma)) = \zeta_n \wedge \gamma. \end{aligned}$$

Thus, the membrane σ gives the relation (2).

Applying the same routine to γ_1 instead of γ, we get the relation

$$\gamma_1 \sim \gamma_2^2, \tag{3}$$

where $\gamma_2 = \gamma_1 \wedge \zeta_n$. However, $\gamma_1 \wedge \zeta_n = \gamma \wedge \zeta_n \wedge \zeta_n = \gamma \wedge \zeta_n = \gamma_1$ and therefore $\gamma_2 \sim \gamma_1$. But then (3) implies $\gamma_1^2 \sim \gamma_1$. This relation and the fact that π_n is a group imly the relation $\gamma_1 \sim 1$. Taking into account (2), we get $\gamma \sim 1$. □

1.2.2. Corollary. *Let $R = \mathbb{F}_\infty$ or $R = \mathbb{F}\mathbb{N}_\infty$, and X be a simplicial R-module. Then each connected component of $|X|$ is weakly equivalent to the point.*

§2. Derived categories for direct images

Let R be a generalized ring. We have mentioned already that the unstable derived category $D^{\le 0}(R)$ leads to a very substantial theory of the derived inverse image. In addition, the cosimplicial Dold–Kan correspondence says that the category of the cosimplicial R-modules cR-Mod is equivalent to the category $\mathrm{Ch}^{\ge 0}(R)$ for classical R. So, it seems natural to choose appropriate class of weak equivalences for cR-Mod and to consider the corresponding homotopy category as $D^{\ge 0}(R)$.

2.1. Model structures on $c\mathbf{C}$ and Quillen's Theorem. Let $\mathbf{C}$ be a category, and let $\mathrm{c}\mathbf{C} = \mathrm{Funct}(\mathbf{\Delta}, \mathbf{C})$ be the category of the cosimplicial objects in $\mathbf{C}$. Here $\mathbf{\Delta}$ is as usual the category of standard simplexes, i.e., the category of the finite nonempty well-ordered sets and of the (not strictly) increasing maps. Following [2, 8.4.7], we are going to choose a model structure on $\mathrm{c}\mathbf{C}$. Due to the dualization

$$\mathrm{c}\mathbf{C} = (s\mathbf{C}^{op})^{op}$$

it is sufficient to construct a model structure on $s\mathbf{D}$ for $\mathbf{D} = \mathbf{C}^{op}$.

A method for constructing such a structure is given in Quillen's Theorem [4, II, 4.1]. As for model structures let us note, in order to avoid misunderstanding, that we deal with the modern version of this notion. It is more restrictive about the completeness of categories than original Quillen's version (see [2, 8.1.3]). A corresponding modification of Quillen's theorem is given in [2, 8.4.5]. In our case it means that the category $\mathbf{C}$ is assumed to be complete with respect to all the limits and the colimits. For instance, the category R-Mod satisfies this requirement.

When dealing with model structures we denote the classes of the weak equivalences, of the cofibrations, the fibrations, the acyclic cofibrations (both a cofibration and a weak equivalence), and the acyclic fibrations by (w), (c), (f), (ac), and (af), correspondingly.

We are going to apply the Quillen theorem to categories with projective generator. In other words, let's suppose that an injective cogenerator I of $\mathbf{C}$ is chosen and fixed. In such a situation Quillen constructs classes (w), (c), (f). If I is a cosmall object or can be equipped with a group structure in $\mathbf{C}$, the Quillen theorem says that the constructed classes form a model structure.

Note that Quillen's model structure does not depend on the choice of projective generator in the simplicial case and of injective cogenerator in the cosimplicial case. In the simplicial case it can be verified as follows. Call a projective generator to be admissible, if $\mathbf{C}$ with the classes (w), (f), and (c) from [4, II, 4.1] is a simplicial model category. It is clear that the direct sum of admissible generators is an admissible generator itself. It is clear as well that if the direct sum of copies of a projective generator is admissible then the original projective generator id also admissible. Next, if P_1 and P_2 are admissible projective generators then the classes $(\mathrm{w})_1$ and $(\mathrm{f})_1$, determined with P_1, obviously contain the classes (w) and (f), determined with $P = P_1 \sqcup P_2$. In addition, there is an epimorphism $P_1^{(X)} \to P$. Hence (w) $\supset$ $(\mathrm{w})_1$, (f) $\supset$ $(\mathrm{f})_1$ and we are through.

2.2. Quillen's model structure on $c\mathbb{F}_0$ -Mod **and** $c\mathbb{F}_1$ -Mod**.** This model structure is constructed by the method from 2.1. Let F be a generalized ring.

Suppose additionally that there exists such a homomorphism $\rho : F \to K$ in a classical ring K that $\rho_* K$ is an injective cogenerator of F-Mod. Let us recall that by definition an F-module I is a cogenerator iff $X \mapsto \mathrm{Hom}(X, I)$ is a strict functor, i.e., it does not glue arrows. In other words, Hom to a cogenerator is an embedding into the set category (but of course not onto a full subcategory).

In particular, $\mathbb{F}_0$ and $\mathbb{F}_1$ can be taken as F. For instance, an $\mathbb{F}_0$-module (i.e., a set) I is injective iff it is nonempty, and I is a cogenerator iff $\mathrm{card}\, I > 1$. So, for $F = \mathbb{F}_0$ we can take ρ to be the only homomorphism to any classical ring. Similarly, for $F = \mathbb{F}_1$ we can take ρ to be the only homomorphism to any classical ring K with $\mathrm{card}\, K > 2$.

Fix K, ρ and set $I = \rho_* K$. The F-module I induces a model structure on $s(F\text{-Mod}^{op})$ as in 2.1. Taking into account the equality $cF\text{-Mod} = (s(F\text{-Mod}^{op}))^{op}$, we get a model structure on cF-Mod. Here, by definition, an arrow $f : X^\bullet \to Y^\bullet$ of cosimplicial F-modules is a cofibration (a weak equivalence) iff the arrow

$$f^* : \mathrm{Hom}_F(Y, I) \to \mathrm{Hom}_F(X, I)$$

is a cofibration (a weak equivalence) of simplicial sets. We get actually a model structure due to [4, II, Theorem 4.1] and to the fact that the structure of additive group on K induces the structure of an (F-Mod)-group on $\rho_* K$ (use the left exactness of ρ_*).

Let $X^\bullet$ be a cosimplicial F-module. Set

$$\pi^0(X^\bullet) = \mathrm{Ker}\,[(\partial_0, \partial_1) : X^0 \to X^1],$$

where ∂_0 and ∂_1 are the face operators induced by the corresponding maps $[0] \to [1]$, i.e., by the maps $[0] \to [1]$, transferring $0 \mapsto 1$ and $0 \mapsto 0$, correspondingly. As usual, $[n]$ denotes the well-ordered set $\{0 < \cdots < n\}$.

2.2.1. Theorem. *Let $F = \mathbb{F}_0$ or $F = \mathbb{F}_1$, and let $f : X^\bullet \to Y^\bullet$ be any morphism of cosimplicial F-modules; then f is a weak equivalence iff $\pi^0(f)$ is an isomorphism.*

Proof. Below we distinguish explicitly the points where specific features of $\mathbb{F}_0$ and $\mathbb{F}_1$ are used. In fact, only the three following properties of F are important there. First of all, there exists an embedding of F into a classical field. Assume such an embedding to be taken above as $\rho : F \to K$. The second, much more special, property of F says that the natural map

$$N_1 \cup \cdots \cup N_n \to N_1 \sqcup \cdots \sqcup N_n \text{ is a bijection,} \tag{4}$$

where $N_1, \ldots, N_m$ are arbitrary submodules of any F-module M, and $N_1 \sqcup \cdots \sqcup N_n$ coincides with the image of the direct sum by definition.

The third property is "partial flatness" of the extension $\rho : F \to K$ (this extension is not completely flat because it does not transfer the products to the products). This property affirms that the natural morphism

$$\rho^*(N_1 \cap N_2) \to \rho^* N_1 \cap \rho^* N_2 \text{ is an isomorphism} \tag{5}$$

for every submodules N_1 and N_2 of every F-module M.

Note that we restrict ourselves with modules over a generalized ring. In this case every module is a set, equipped with operations. So, there is no difficulty to interpret the images and the inverse images.

Now we are going to prove the main assertion of the theorem directly. For this purpose we restate the property of being a weak equivalence in terms of complexes. Namely, consider the diagram

$$\begin{array}{ccc} cF\text{-Mod} & \xrightarrow{D_I^{cs}} & sF\text{-Mod} \\ \Big\downarrow{\scriptstyle \rho_c^*} & & \Big\uparrow{\scriptstyle \rho_*^s} \\ cK\text{-Mod} & \xrightarrow{D_K^{cs}} & sK\text{-Mod} \\ \Big\downarrow{\scriptstyle NQ} & & \Big\downarrow{\scriptstyle NS} \\ \mathrm{Ch}^{\geq 0}(K\text{-Mod}) & \xrightarrow{D_K^{\geq 0, \leq 0}} & \mathrm{Ch}^{\leq 0}(K\text{-Mod}). \end{array} \tag{6}$$

Here ρ_c^* and ρ_*^s are the direct and inverse image functors, $D_I^{cs}(X)_n = \mathrm{Hom}_F(X^n, I)$ (of course, the inner Hom is meant),

$$D_K^{cs}(X)_n \mapsto \mathrm{Hom}_K(X^n, K), \quad D_K^{\geq 0, \leq 0}(A)_n \mapsto \mathrm{Hom}_K(A^n, L),$$

NQ and NS are the normalized complex functors for the Dold–Kan correspondence (see 2.2.2). The diagram (6) is commutative up to appropriate functor isomorphisms. The top square is commutative because of the the adjointness of the direct and inverse images. The bottom square is commutative in view of (17).

In what follows we need certain model structures on the categories of (6). We mean Quillen's model structures on cF-Mod, cK-Mod, cF-Mod, sK-Mod, the injective model structure on $\mathrm{Ch}^{\geq 0}(K\text{-Mod})$, and the projective model structure on $\mathrm{Ch}^{\leq 0}(K\text{-Mod})$.

Let f be a morphism of cosimplicial F-modules. We claim that

$$f \text{ is a weak equivalence iff } NQ \circ \rho_c^* f \text{ is a quasi-isomorphism.} \tag{7}$$

We prove the criterion (7) in a few steps.

First of all, let g be a morphism of the category cK-Mod and u be a morphism of the category sK-Mod. We claim that

$$f \in (\mathrm{w}), (\mathrm{ac}), (\mathrm{c}) \quad \text{iff } D_I^{cs} f \in (\mathrm{w}), (\mathrm{af}), (\mathrm{f}), \quad \text{correspondingly,} \tag{8}$$

$$g \in (\mathrm{w}), (\mathrm{ac}), (\mathrm{c}) \quad \text{iff } D_K^{cs} g \in (\mathrm{w}), (\mathrm{af}), (\mathrm{f}), \quad \text{correspondingly,} \tag{9}$$

$$u \in (\mathrm{w}), (\mathrm{af}), (\mathrm{f}) \quad \text{iff } \rho_*^s u \in (\mathrm{w}), (\mathrm{af}), (\mathrm{f}), \quad \text{correspondingly.} \tag{10}$$

These assertions hold by definition of Quillen's model structures (see 2.1).

Next, (8), (9), (10) and the commutativity of the diagram (6) imply that

$$f \in (\mathrm{w}), (\mathrm{ac}), (\mathrm{c}) \text{ iff } \rho_c^* f \in (\mathrm{w}), (\mathrm{ac}), (\mathrm{c}), \text{ correspondingly.} \tag{11}$$

Further, we claim that

$$g \in (\mathrm{w}), (\mathrm{ac}), (\mathrm{c}) \text{ iff } D_K^{\geq 0, \leq 0} \circ NQg \in (\mathrm{w}), (\mathrm{af}), (\mathrm{f}), \text{ correspondingly.} \tag{12}$$

Taking into account the commutativity of the diagram (6), this follows (9) and the fact that the projective model structure on sK-Mod coincides with the model structure induced by the equivalence of categories NS (see [2, 8.5.8]).

In addition, let v be a morphism in the category of the complexes $\mathrm{Ch}^{\geq 0}(K\text{-Mod})$. We claim that

$$\text{if } D_K^{\geq 0, \leq 0} v \text{ is a quasi-isomorphism, then } v \text{ is a quasi-isomorphism.} \tag{13}$$

It is valid due the assumption K to be a field. Finally, taking into account the commutativity of the diagram (6), the criterion (7) follows (11), (12) and (13), applied to $v = NQ \circ \rho_c^* f$.

Complete the proof of the theorem 2.2.1 with the help of the criterion (7). For this purpose consider something like nonlinear cohomology. Namely, associate a couple of F-modules $B^n \subset Z^n$ with $X^\bullet$ and $n > 0$, taking

$$Z^n(X^\bullet) = \partial_0^{-1}(\operatorname{Im} \partial_1 \sqcup \dots \sqcup \operatorname{Im} \partial_n) \quad \text{and} \quad B^n(X^\bullet) = \operatorname{Im} \partial_0 \sqcup \dots \sqcup \operatorname{Im} \partial_n .$$

Note that in the linear case, i.e., for classical ring F,

$$H^n(NQ(X^\bullet)) = Z^n(X^\bullet)/B^n(X^\bullet).$$

This equality can be easily seen from the description of NQ in 2.2.2.

Let's prove that the linear cohomology can be obtained from the non-linear ones with the scalar extension. To be more precise,

$$Z^n(\rho^* X^\bullet) = \rho^*(Z^n(X^\bullet)) \quad \text{and} \quad B^n(\rho^* X^\bullet) = \rho^*(B^n(X^\bullet)).$$

These equalities follow the fact that ρ^* respects all the operations, involved in the definition of cohomology. In fact, ρ^* commutes with all the coproducts, being a left adjoint functor. Next, ρ^* preserves the split monomorphisms and the split epimorphisms, being a functor. So, taking preimage is the only involved operation under question. However, in our concrete case we can control the peimage due to (5). In fact, the operator ∂_0 is split by the (co)degeneration $\sigma_0 : \{0 < \cdots < n\} \to \{0 < \cdots < n-1\}$, $n \mapsto n-1, \ldots, 1 \mapsto 0, 0 \mapsto 0$, i.e.,

$$\sigma_0 \partial_0 = \mathrm{id}\,. \tag{14}$$

In particular, ∂_0 is an injection and (5) can be applied.

It remains to verify that $Z^n(X^\bullet) = B^n(X^\bullet)$ for $n > 0$. Let $z \in Z^n(X^\bullet)$, i.e., $\partial_0 z \in \mathrm{Im}\,(\partial_1 \sqcup \cdots \sqcup \partial_n)$. Using the property (4) of the generalized ring F, we see that there exist $i \neq 0$ and $x \in X^n$ such that

$$\partial_i x = \partial_0 z \tag{15}$$

(just the lack of composites is the true reason for the degeneracy of the derived categories under consideration). Let us deduce from this equality that

$$z \in B^n(X^\bullet) = \mathrm{Im}\,(\partial_0 \sqcup \cdots \sqcup \partial_{n-1}).$$

If $i > 1$, then

$$z = \sigma_0 \partial_0 z = \sigma_0 \partial_i x = \partial_{i-1} \sigma_0 x \in \mathrm{Im}\,\partial_{i-1},$$

where the first equality is a consequence of the identity (14), the second equality is a consequence of (15), and the third is a consequence of the identity $\sigma_0 \partial_i = \partial_{i-1} \sigma_0$, which holds for $i > 1$.

If $i = 1$ on the other hand, then

$$z = \sigma_0 \partial_0 z = \sigma_0 \partial_1 x = x = \sigma_1 \partial_1 x = \sigma_1 \partial_0 z = \partial_0 \sigma_0 z \in \mathrm{Im}\,\partial_0.$$

Here the first equality is a consequence of the identity (14), the second equality is a consequence of (15), the third equality follows $\sigma_0 \partial_1 = \mathrm{id}$, the fourth equality follows the identity $\sigma_1\partial_1 = \mathrm{id}$, the fifth equality follows (15), and the sixth equality follows the identity $\sigma_1\partial_0 = \partial_0\sigma_0$. This completes the proof of the theorem 2.2.1. □

The theorem 2.2.1 is a result of joint activity with N. Durov. In essential, it is his proof to be given above.

2.2.2. The Dold–Kan correspondence and the duality. Let **A** be an abelian category. The Dold–Kan correspondence is an equivalence of the categories

$$\mathrm{Ch}^{\leq 0}(\mathbf{A}) \underset{NS \text{ or } NQ}{\overset{K}{\rightleftarrows}} s\mathbf{A}\ ,$$

where the isomorphic functors NS and NQ are usually denoted by N and N'. We prefer to use this nonstandard notation because it clear a space for indices and is more informative. Namely, S and Q stand here to denote a subcomplex and a quotient complex for the full chain complex $C_\bullet = C_\bullet(A)\colon \cdots \to C_2 \xrightarrow{d} C_1 \xrightarrow{d} C_0$, where $C_i = A_i$, $d = \partial_0 - \partial_1 +$. The normalized complexes NS and NQ are quite geometric and easily reproducible. Namely, $NQ = C/D$, where the subcomplex D is generated by the images of all the degenerations. The complex NS consists of all the simplexes subjected to the conditions $\partial_1 = \partial_2 = \cdots = 0$. Note that if such a simplex σ is degenerate, then $\sigma = 0$. Actually, the degeneracy means, that σ can be led through a surjection of standard simplexes $\Delta(n) \xrightarrow{\delta} \Delta(m) \xrightarrow{\sigma'} X$ $(m < n)$. In that case there exist at least two faces of $\Delta(n)$, which are mapped surjectively onto $\Delta(m)$. But the composition $\sigma'\delta$ vanishes at least on one of these two faces due to the degeneracy of σ. Hence $\sigma' = 0$ and $\sigma = 0$. Thus we see that NS consists of all the "essentially nondegenerate simplexes". The subcomplex seems to be more natural since it makes use of the semisimplicial structure only.

The dual Dold–Kan correspondence is an equivalence of the categories

$$\mathrm{Ch}^{\geq 0}(\mathbf{A}) \underset{NQ \text{ or } NS}{\overset{K}{\rightleftarrows}} cs\mathbf{A}$$

and is obtained with a dualization of the original correspondence. It can be explained as follows. Suppose we have a simplicial construction, related to any category **A**, and we are looking for its cosimplicial counterpart. In this case we take the simplicial construction for $\mathbf{A}^{op}$ and come back to **A** with the dualization "*op*". For instance, the subcomplex $NS_\bullet$ turns into the quotient complex $NQ^\bullet$. Thus, the following structures are associated to every cosimplicial object A.

(1) The full cochain complex $C^\bullet : C^0 \xrightarrow{d} C^1 \xrightarrow{d} \dots$, $C^i = A^i$, $d = \partial_0 - \partial_1 + \dots$.
(2) The cochain complex of degenerations $D^\bullet : D^0 \xrightarrow{d} D^1 \xrightarrow{d} \dots$, where D^n is the coimage of A^n under the morphism $A^n \to \prod s_j A^n$, $d = \partial_0 - \partial_1 + \dots$.
(3) The normalized subcomplex $NS^\bullet = \mathrm{Ker}\,[A^\bullet \to D^\bullet]$. In other words, $NS^i = \cap \mathrm{Ker}\, s_j$.
(4) The normalized quotient complex $NQ^n = C^n / [\mathrm{Im}\,\partial_1 + \dots + \mathrm{Im}\,\partial_n]$.

Assume $\mathbf{B}$ to be an abelian category and $\Phi : \mathbf{B} \to \mathbf{A}$ to be an additive functor. Then there is the tautological functor isomorphism $\mathrm{Ch}^{\leq 0}(\Phi) \circ C_\bullet \to C_\bullet \circ s\Phi$. Because of the additivity of Φ this tautological morphism induces a functor morphism

$$\phi : \mathrm{Ch}^{\leq 0}(\Phi) \circ NS \to NS \circ s\Phi, \qquad (16)$$

where the functors map $s\mathbf{B}$ to $\mathrm{Ch}^{\leq 0}(\mathbf{A})$. Since the normalized subcomplex NS is defined in terms of kernels, so ϕ is an isomorphism for every left exact functor Φ. Applying these considerations to $\mathbf{B} = \mathbf{A}^{op}$ and $\Phi = \mathrm{Hom}(*, I)$ for an injective object $I \in \mathbf{A}$, we see that in this case the morphism ϕ of (16) is an isomorphism. Since the dualization of $s\mathbf{B}$ lead to $c\mathbf{A}$, and the dualization of NS lead to NQ, then Φ induces a functor isomorphism

$$\psi : D_K^{\geq 0, \leq 0} \circ NQ \to NS \circ D_K^{cs}, \qquad (17)$$

where the functors map $c\mathbf{A}$ to $\mathrm{Ch}^{\leq 0}(\mathbf{A})$.

2.3. Unary extensions and cosimplicial modules. Informally speaking, the theorem 2.2.1 says that the homotopy category, obtained from the cosimplicial sets with Quillen's model structure, does not fit well to construct an substantial theory of higher direct images in generalized algebraic geometry, at least for morphisms to $\mathrm{Spec}\,\mathbb{F}_1$. The theorem 2.3.1 can be thought as a supplement to 2.2.1. In particular, it shows that the base change by means of roots of unity does not improve the situation.

In the theorem 2.3.1 we deal with Quillen's model structure on the categories cR-Mod and cS-Mod, where R and S are generalized rings. Therein it is meant that there exist admissible injective cogenerators in the categories R-Mod and S-Mod. Note (see 2.1) that Quillen's model structure does not depend on the choice of such a cogenerator.

2.3.1. Theorem (N. Durov). *Let $\rho : R \to S$ be an unary homomorphism of generalized rings (see [2, 5.1.15]). Suppose additionally that the category R-Mod contains an injective cogenerator I, which either is cosmall or admits a cogroup structure. In other words, I, being considered in the category $(R\text{-Mod})^{op}$, satisfies one of the two additional Quillen conditions. Let f be a morphism in the category cS-Mod. Then f is a weak equivalence (a cofibration, an acyclic cofibration) iff $\rho_* f$ is a weak equivalence (a cofibration, an acyclic cofibration) in cR-Mod.*

Proof. Let I be an admissible injective cogenerator in R-Mod. Because the unarity of ρ the direct image functor $\rho_* : S\text{-Mod} \to R\text{-Mod}$ does have a right adjoint functor $\rho^! : R\text{-Mod} \to S\text{-Mod}$ [2, 5.3.15]. In this case $\rho^! I$ is an injective cogenerator, satisfying one of two additional Quillen's conditions. A proof of this and further verification can be easily obtained from the adjointness $\mathrm{Hom}_S(f, \rho^! I) \simeq \mathrm{Hom}_R(\rho_* f, I)$. □

2.3.2. Corollary. *Let f be a morphism of cosimplicial $\mathbb{F}_{0^n}$-modules ($\mathbb{F}_{1^n}$-modules, corresp.). Then f is a weak equivalence (a cofibration, an acyclic cofibration) iff $\rho_* f$ is a weak equivalence (a cofibration, an acyclic cofibration) of cosimplicial sets (pointed sets, corresp.), where ρ is the only homomorphism $\mathbb{F}_0 \to \mathbb{F}_{0^n}$ ($\mathbb{F}_1 \to \mathbb{F}_{1^n}$, corresp.).*

2.4. Conclusion. The theorem 2.2.1 says, roughly speaking, that there is no higher cohomology over $\mathrm{Spec}\,\mathbb{F}_0$. However, the canonical generator of $H^1(\mathbf{P}^1_{\mathbb{Z}}, \mathcal{O}(-2))$ looks to be so fundamental, and its description with the help of the Čech complex looks to be so combinatorial that the lack of similar phenomenon over $\mathbb{F}_0$ is somewhat surprising.

Below we try to understand better what are reasons for higher cohomology not to exist in the nonlinear case and how essential they are. We start with an accurate description of Čech complexes in the linear situation.

2.4.1. Coverings and cosimplicial objects. Let F be a sheaf of abelian groups on a topological space X and let $\mathcal{U} = \{U_i\}_{i \in I}$ be an open covering of X. One can associate to $\mathcal{U} = \{U_i\}_{i \in I}$ a few complexes from $\mathrm{Ch}^{\geq 0}(\mathbb{Z})$. Each of them can be called a Čech complex. Let $U = \coprod U_i$ and let $p : U \to X$ be the direct sum of the natural inclusions. Below it is convenient to think that the morphism p belongs to the site, where the topology lives on. Since often it is not the case, we explain how to get over this inconsistence. For example, one can extend the site, appending to it all the local isomorphisms. Another, more traditional way, relies on the possibility to mimic all the necessary constructions with the help of easy and obvious means. For instance, one can use $\prod F(U_{i,j})$ with $U_{i,j} = U_i \cap U_j$ instead of $F(U \times_X U)$. So, consider the simplicial space $\mathcal{X} = \mathcal{X}(\mathcal{U})$ over X, given by the diagram $\cdots U \times_X U \times_X U \Rrightarrow U \times_X U \rightrightarrows U$.

Informally speaking, the fiber $\mathcal{X}_x$ of the bundle $\mathcal{X} \to X$ over a point x is the gruppoidofication (or Kanification Ex^∞) of the simplex, whose vertices correspond to that covering elements, which contain x. Applying the functor of F-sections, we get a cosimplicial abelian group $F(\mathcal{X})$: $F(U) \rightrightarrows F(U \times_X U) \Rrightarrow F(U \times_X U \times_X U) \ldots.$

The most natural, but no means the most economical, Čech complex is the full complex $C = C(U, F)$. By definition $C^n = F(\mathcal{X}_n)$ and $\partial = \partial_0 - \partial_1 + \ldots.$ Thus the full Čech complex $C^\bullet$ of a covering appears to be the full complex of the corresponding cosimplicial abelian group $F(\mathcal{X})$. In practice often more economical versions of the Čech complex are used. For example, one can take the subcomplex $C' \subset C$, consisting of all those $f \in F(\mathcal{X}_n)$, which are skewsymmetric relatively the natural action of the symmetric group S_n on $\mathcal{X}_n = U \times_X \cdots \times_X U$. Every choice of a total ordering $\leq$ of I leads to a even more economical complex $(C'')^n = \prod F(U_{i_0,\ldots,i_n})$, where the multi-index $(i_0, \ldots, i_n)$ runs over all the strictly increasing sequences, and the differential is given by the alternate sum of the restrictions.

The complexes C, C', and C'' can be thought as forms of one complex in the sense that they are quasi-isomorphic.

2.4.2. Example. Consider the $\mathcal{O}_X$-module $\mathcal{O}_X(-2)$ on $\mathbf{P}^1/\mathbb{F}_0$. Only one of all the versions of the Čech complex (see 2.4.1) is able to survive in the general nonlinear situation. It is just $F(\mathcal{X})$. Thus, the nonlinear version of the Čech complex for $\mathcal{O}_X(-2)$ is given, anyway, by the co(semi)simplicial set

$$S^\bullet : \ S_0 \times S_1 \rightrightarrows S_{0,0} \times S_{0,1} \times S_{1,0} \times S_{1,1} \qquad .$$

$$\Rrightarrow S_{0,0,0} \times S_{0,0,1} \times \cdots \times S_{1,1,0} \times S_{1,1,1} \cdots$$

Here $S_{0,\ldots,0} = \{-2, -3, \ldots\} = -2 - \mathbb{N}$, $S_{1,\ldots,1} = \{0, 1, \ldots\} = \mathbb{N}$, and $S_{i,j,\ldots} = \mathbb{Z}$, if the multi-index $(i, j, \ldots)$ contains both 0 and 1. The face operators in $S^\bullet$ are given by the natural inclusions.

We are going to understand why $S^\bullet$ does not contain the data, representing, in the linear case, the fundamental cohomological class from

$H^1(\mathbf{P}^1_{\mathbb{Z}}, \mathcal{O}_X(-2))$. For this purpose consider the co(semi)simplicial abelian group

$$A^\bullet:\ A_0 \times A_1 \Longrightarrow A_{0,0} \times A_{0,1} \times A_{1,0} \times A_{1,1} \qquad ,$$

$$\Rrightarrow A_{0,0,0} \times A_{0,0,1} \times \cdots \times A_{1,1,0} \times A_{1,1,1} \cdots$$

where $A_{i_0,\ldots,i_n} = \Gamma(U_{i_0,\ldots,i_n}, \mathcal{O}_X(-2))$. Consider the corresponding (full) complex C as well (see 2.2.2). The simplest cochain in $C^1 = A^1$, representing the fundamental class, has the form

$$f = 0 \times s \times (-s) \times 0, \text{ where } s = t_0^{-1} t_1^{-1}.$$

Writing down its nonlinear counterpart is prevented by the lack of 0 and (-1). Add them, i.e., extend the scalars step by step in the tower $\mathbb{F}_0 \subset \mathbb{F}_1 \subset \mathbb{F}_{1^2}$. After that f becomes well-defined, and its cofaces have the form

$$\partial_0(f) = 0 \times s \times (-s) \times 0 \times 0 \times s \times (-s) \times 0,$$
$$\partial_1(f) = 0 \times s \times 0 \times s \times (-s) \times 0 \times (-s) \times 0,$$
$$\partial_2(f) = 0 \times 0 \times s \times s \times (-s) \times (-s) \times 0 \times 0.$$

We see that

$$\partial_0(f) = \partial_1(f) - \partial_2(f).$$

Thus, neither $\partial_1(f)$ nor $\partial_2(f)$ lie in the image of ∂_0, but their compose $\partial_1(f) - \partial_2(f)$ does not. In other words, the reason to exist for the linear fundamental class is the existence of nontrivial linear relation between $\partial_0(f), \partial_1(f)$ and $\partial_2(f)$. These elements can be seen in the nonlinear world, but the relation can not.

2.4.3. The fundamental class and Euler's sequence. There is another way to see the canonical generator in $H^1(\mathbf{P}^1, \mathcal{O}(-2)) = \operatorname{Ext}^1(\mathcal{O}_X, \mathcal{O}_X(-2))$. Namely, consider the extension

$$\mathcal{O}(-2) \to V(-1) \to \mathcal{O}, \tag{18}$$

obtained by the twist of the canonical Euler sequence, where $\mathbf{P}^1 = \mathbf{P}(V)$. What of this does not work in the nonlinear case?

Consider the graded generalized ring $A = \mathbb{F}_0[t_0, t_1]$, where t_0 and t_1 are unary variables with the degree one, and set $\mathbf{P}^1 = \operatorname{Proj} A$. In such a case, as before, we have locally trivial $\mathcal{O}_X$-modules with the rank one $\mathcal{O}(-2)$

and $\mathcal{O}$. As for $V(-1)$, so there are several versions of this $\mathcal{O}_X$-module. For example, we can take $V = \mathcal{O}_X \sqcup \mathcal{O}_X$ and get the sequence

$$\mathcal{O}(-2) \rightrightarrows V(-1) \to \mathcal{O},$$

as a counterpart of the sequence (18). On the other hand, we can take $V = \mathcal{O}_X \times \mathcal{O}_X$ and get

$$\mathcal{O}(-2) \to V(-1) \rightrightarrows \mathcal{O}.$$

In any case we see that some additivity is necessary to expect nontrivial derived direct images. Of course, in the nonlinear case we can mimic the additivity, considering, for example, arguments of high arity. But the prospective of such approach is not clear for me, especially concerning its usefulness.

In the linear situation higher cohomology rather help to compute usual direct images than hinder such computations. In the nonlinear situation the computation of direct images keeps to be very important problem. However, the paradigm of the linear world, apparently, is not quite adequate to this aim.

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