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PARAMETRIC PAINLEVÉ EQUATIONS

Abstract. The parametric Painlevé equations are those ODEs whose general solutions can be presented in the parametric form in terms of the Painlevé functions. Most of these ODEs do not possess the Painlevé property. By considering similarity solutions of the short pulse equation and its decoupled generalization we derive a non-trivial example of the parametric Painlevé equation related with the third Painlevé equation. We also discuss some analytic properties of this equation describing the structure of movable singularities.

1. Introduction

Many ODEs can be solved explicitly in a parametric form in terms of elementary or special functions. Many such examples can be found in the handbook [9]. Since the Painlevé functions gain now the status of special functions it is reasonable to have some knowledge about the parametric Painlevé equations, those equations whose general solutions can be presented in a parametric form in terms of the Painlevé functions.

In this note, I would like to present one nontrivial example of the parametric Painlevé equation which is related with the similarity solutions of the so-called short pulse equation (SPE) [1], known also as the cubic Rabelo equation describing the pseudospherical surfaces [11],

$$u_{xt} = u + \frac{1}{6}\big(u^3\big)_{xx}, \tag{1}$$

where, $u = u(x,t)$ and the subscripts denote differentiation with respect to the corresponding variables. We also consider a parametric Painlevé equation related with a natural generalization of equation (1). This example related with the parametrization of the solutions by the third Painlevé function P_3. Although the corresponding parametric Painlevé equation is in some sense equivalent to P_3 it has absolutely different analytic and

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transformation properties and represent an interesting example of the non-Painlevé equation whose moving singularities of regular type can be studied in a greater detail comparing to the other ODEs without the hidden Painlevé structure.

It seems that all such parametric Painlevé equations could be related with the isomonodromy deformations of ODEs with rational coefficients, where the deformation parameter is introduced in a "wrong way." We know that the canonical (natural) choice of the deformation parameters is: positions of poles (of the rational coefficients) and parameters defining formal asymptotic expansions of the general solutions in proper neighborhoods of the poles. Sometimes it might be needed to consider isomonodromy deformations with respect to parameters chosen in a different ("wrong") way, because it is dictated by the original setting of the problem. Clearly, in this case one can make a change of variables from a given set of parameters to the canonical one and arrive at the standard situation. This change of variables is nothing but the parametrization of the system describing the isomonodromy deformations in terms of the "wrong" parameter(s) into the canonical ("Painlevé") ones. The purpose of this note is to consider an interesting example of such transformation. On this way we arrive at a non-Painlevé type ODE and a change of variables that maps it to the third Painlevé equation. This change of variables is not obvious to find by other direct methods, so that the fact of a relation between the ODEs are not easy to guess.

One interesting particular class of the parametric Painlevé equations is considered by Fokas and Yang [3]. As far as I can judge from [3] the original idea of Yang was to recast the Painlevé transcendents as the functions of their Hamiltonians, in this case the "old time" considered now as the function of the canonical variables and Hamiltonian, where the latter treated as the "new time", plays the role of the Hamiltonian for new parameterized Painlevé equations. Since we know that the Painlevé Hamiltonians satisfy nonlinear ODEs, Yang's idea can be viewed as a "one-variable" hodograph-type transformation of the latter ODEs for Hamiltonians with the further substitution of the result to the Painlevé equations. This approach becomes more fruitful when considering generalization of the above mentioned idea for the Garnier systems.

At the same time, our general point of view suggests to consider equations obtained by Fokas and Yang as a very special case of the parametric

Painlevé equations. One can obtain many other parametric Painlevé equations even with functional parameters related with a given Painlevé equation that has nothing to do with the original Hamiltonian interpretation given in [3]. On the other hand equations obtained in [3] have the simple form and might have some remarkable analytic properties, so they might be interesting objects for further studies.

In our paper, we consider a parametric Painlevé equation related with the third Painlevé equation. In [3], examples related with P_3 is not considered. Our starting point is a relation of our parameterized Painlevé equation with the similarity solutions of SPE (1), however, some Hamiltonian structures also appear in our parameterizations.

Let us agree about the notation. Primes as usual are used to denote differentiation of the functions of one argument. Differentiations of the functions of several arguments are denoted by corresponding literal subscript (without the primes). The numeral subscripts means labels that serve to distinguish similar but different objects. For this purpose we use also "hats" ant tildes. All the square roots of the same quantity have an arbitrary but the same branch.

2. The short pulse and sine-Gordon equations

In this section, we collected some basic known facts from the theory of SPE which we need for our construction of the parametric Painlevé equation.

The zero curvature representation for Equation (1) was found by A. Sakovich and S. Sakovich [10]:

$$\frac{\partial}{\partial x}\psi = U\psi, \quad \frac{\partial}{\partial t}\psi = V\psi, \tag{2}$$

$$U = \lambda \begin{pmatrix} 1 & u_x \\ u_x & -1 \end{pmatrix}, \quad V = \frac{\lambda u^2}{2} \begin{pmatrix} 1 & u_x \\ u_x & -1 \end{pmatrix} + \frac{u}{2} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} + \frac{\sigma_3}{4\lambda}, \tag{3}$$

where λ is the spectral parameter and $\sigma_3 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ is the Pauli matrix.

This Lax pair immediately suggests an explicit invertible map relating equation (1) with the sine-Gordon equation,

$$v_{XT} = \sin v. \tag{4}$$

This fact was established by A. Sakovich and S. Sakovich in [10], see also [11]. Since the authors of these works used direct methods, we recall the relation between equations (1) and (4) by presenting an explicit mapping of the Wadati–Konno–Ichicawa type [12] pair (2), (3) to the (AKNS-type) zero curvature representation for the sine-Gordon equation [13],

$$\frac{\partial}{\partial X}\Psi=\begin{pmatrix}\lambda & -v_X/2\\ v_X/2 & -\lambda\end{pmatrix}\Psi,\quad \frac{\partial}{\partial T}\Psi=\frac{1}{4\lambda}\begin{pmatrix}\cos v & \sin v\\ \sin v & -\cos v\end{pmatrix}\Psi,\tag{5}$$

Looking at the large λ asymptotics of the matrices U and V (see Eqs. (3)) it is immediate to notice that the *eigenvalues* of the matrices

$$\begin{pmatrix}1 & u_x\\ u_x & -1\end{pmatrix}\quad\text{and}\quad \frac{u^2}{2}\begin{pmatrix}1 & u_x\\ u_x & -1\end{pmatrix}\tag{6}$$

should satisfy the following compatibility condition

$$\frac{\partial}{\partial t}\sqrt{1+u_x^2}=\frac{\partial}{\partial x}\left(\frac{u^2}{2}\sqrt{1+u_x^2}\right).\tag{7}$$

Equation (7) is nothing but Eq. (1) written in the form of a conservation law. This conservation law suggests that we can define new variables X and T, the reciprocal variables, by the following one-forms:

$$dX=\sqrt{1+u_x^2}dx+\frac{u^2}{2}\sqrt{1+u_x^2}dt,\tag{8}$$

$$dT=-dt.\tag{9}$$

The reciprocal conservation law (e.g., see [14]) for (7) reads,

$$\frac{\partial}{\partial T}\frac{1}{\sqrt{1+u_x^2}}=\frac{\partial}{\partial X}\frac{u^2}{2},\tag{10}$$

where the functions u and u_x are treated as the functions of X and T. In terms of the original variables x and t, Eq. (10) is equivalent to the original SPE (1), however, as we see below, it also produces a useful relation between equations (1) and (4). Now using variables X and T we define the following explicit gauge transformation between solutions of the Lax pairs (2), (3), and (5):

$$\psi(x,t)=R\Psi(X,T),$$

$$R=\frac{\sqrt{1+\sqrt{1+u_x^2}}}{\sqrt{2\sqrt{1+u_x^2}}}\begin{pmatrix}1 & -\frac{u_x}{1+\sqrt{1+u_x^2}}\\ \frac{u_x}{1+\sqrt{1+u_x^2}} & 1\end{pmatrix},\tag{11}$$

where the function $v(X,T)$ in system (5) is given by the following equations:

$$\sin v = \frac{u_x}{\sqrt{1+u_x^2}}, \quad \cos v = -\frac{1}{\sqrt{1+u_x^2}}, \quad v_X = -\frac{u_{xx}}{\left(\sqrt{1+u_x^2}\,\right)^3}. \tag{12}$$

To invert this transformation one has to find the function $u(x,t)$ in terms of $v(X,T)$. For this purpose, it is convenient to exploit the reciprocal law (10) and substitute in the right-hand side the second equation in (12) to find,

$$\sin v\, v_T = u u_X \tag{13}$$

Taking into account that

$$u_X = u_x \frac{\partial x}{\partial X} = \frac{u_x}{\sqrt{1+u_x^2}} = \sin v$$

Eq. (13) can be rewritten

$$u(x,t) = v_T(X,T).$$

The same equation can be reproduced via a more cumbersome calculation by differentiating the first equation in (12) with respect to T.

3. Similarity reduction

It is easy to find similarity reduction of (1),

$$u(x,t) = w(z)/t, \quad z = xt, \tag{14}$$

where the function $w(z)$ is the general solution of the following ODE,

$$zw'' = w + \frac{1}{6}(w^3)''. \tag{15}$$

Introducing a new dependent variable $w_1 \equiv w_1(z)$ by the equation $w = \sqrt{6}\, w_1{}''$ one rewrites equation (15) in the following form,

$$(w_1'')^3 - zw_1'' + 2w_1' + w_1 = 0, \tag{16}$$

where the constants of integration are omitted because they can be removed by a linear (with respect to z) shift of w_1, which does not effect on w.

Resolving equation (15) with respect to w'' we find

$$w'' = \frac{2w(w'^2+1)}{2z - w^2}. \tag{17}$$

Introducing variable $w_2 = w^2/(2z)$ we rewrite the latter equation in the following form

$$w_2'' = \left(\frac{1}{2w_2} + \frac{1}{1-w_2}\right){w_2'}^2 + \frac{3w_2-1}{1-w_2}\cdot\frac{w_2'}{z} + \frac{2w_2}{z(1-w_2)} + \frac{w_2(w_2+1)}{2z^2(1-w_2)}. \tag{18}$$

As follows from the analysis presented in Chap. XIV of the Ince book [4] Eqs. (17) and (18) cannot be transformed by the help of the fractional-linear transformations of the dependent variables to any of the members of the list of 50 equations of the Painlevé type presented in that chapter, so that they do not possess the Painlevé property. Below in this section we discuss the "non-Painlevé" structure of these equations.

There are three special solutions of equation (15) in terms of the elementary functions: $w \equiv 0$ and $w = \pm iz + C_1$, where $C_1 \in \mathbb{C}$ is a parameter: its general solution is a transcendental function expressible in terms of the general solution of the special case of the third Painlevé equation in the trigonometric form,

$$ZV'' + V' = \sin V. \tag{19}$$

This statement follows from the fact that the transformation relating equations (1) and (4) described in Sec. 1 enjoys the similarity reduction: more precisely, in the similarity case (14), not only Eq. (9) but also Eq. (8) can be integrated explicitly: to see this one can consider differential dZ and after straightforward manipulations with the help of Eqs. (8) and (9) arrive at the following equation,

$$dZ + \sqrt{1+w'^2}\,dz = \frac{dt}{t}\left(Z - \left(w(z)^2/2 - z\right)\sqrt{1+w'(z)^2}\right).$$

Thus, since the left-hand side of this equation depends only on z, Z, one finds the following first integral and corresponding differential equation:

$$Z = \left(w(z)^2/2 - z\right)\sqrt{1+w'(z)^2}, \quad Z = XT, \tag{20}$$

$$dZ = -\sqrt{1+w'(z)^2}\,dz. \tag{21}$$

Sure, equation (21) follows from equation (20) with the help of equation (15). Therefore the similarity solutions (14) are mapped (modulo solutions of the equation $(w')^2 + 1 = 0$) to the similarity solution $v(X,T)$ of equation (4),

$$v(X,T) \equiv V(Z), \quad Z = XT,$$

which solves equation (19).

Applying the similarity reduction (14) in Eqs. (12) one finds, with the help of Eq. (15), a "parametric" representation of general solution $V(Z)$ of Eq. (19) in terms of general solution, $w(z)$ of Eq. (15):

$$\sin V(Z) = \frac{w'(z)}{\sqrt{1 + w'(z)^2}}, \quad \cos V(Z) = -\frac{1}{\sqrt{1 + w'(z)^2}}, \tag{22}$$

$$V'(Z) = \frac{w(z)}{(z - w^2(z)/2)\sqrt{1 + w'(z)^2}}, \tag{23}$$

where Z is given by Eq. (20).

Since the general solution of Eq. (19) is considered now as a well-established special function it is reasonable to invert formulas (22), (23), and (20) to find the function $w(z)$ in terms of $V(Z)$:

$$w = -ZV'(Z) \equiv -p, \tag{24}$$

$$z = Z \cos V(Z) + (ZV'(Z))^2/2 \equiv \tau H(p, q, \tau), \tag{25}$$

where

$$H(p, q, \tau) = \frac{p^2}{2\tau} + \cos q \tag{26}$$

is the time-dependent Hamiltonian of equation (19) with the following definition of the canonical variables:

$$\tau \equiv Z, \quad q \equiv V = V(Z), \quad p \equiv ZV'(Z). \tag{27}$$

Equations (24)–(27) represents our function $w(z)$ in terms of the Hamiltonian variables of the third Painlevé equation (19), so it looks similar to the objects considered in the work [3], though not exactly: our independent variable has the additional factor τ.

As it was mentioned above the function $w(z)$ does not have the Painlevé property. The parametric representation (24), (25) allows one to study more precisely the "non-Painlevé structure" of the function $w(z)$ with the help of the known asymptotic results for the function $V(Z)$. We need some preliminary definitions to formulate the result. On the complex z-plane define the parabola,

$$\Pi := \left\{ z \in \mathbb{C} : \ 2\Re z + 1 = (\Im z)^2 \right\}. \tag{28}$$

The completion of the parabola to the complex plane is the union of two domains:

$$\mathbb{C} \setminus \Pi = \mathcal{P} \cup \mathcal{NP},$$

where the domain $\mathcal{NP}$ contains the positive semi-axis. It seems (the conjecture) that the domain $\mathcal{P}$ either does not contain movable singular points of $w(z)$ rather than poles: so it can be called the Painlevé domain. The set the regular singular points of $w(z)$ are moving in the closure, $\mathcal{NP} \cup \Pi$, of the domain $\mathcal{NP}$, which can be called the non-Painlevé domain. To prove this statement we recall that the function $V(Z)$ apart of the poles moving in the cylinder $\mathbb{C} \setminus \{0\}$, has a regular singular point at 0 and an irregular singular point at ∞. The transformation (25) preserve the point at infinity and "make a directed (along the positive semi-axis) explosion" of the origin. The latter statement means that solutions with branching at $Z = 0$, with the leading behavior $V \asymp \sigma \ln Z$, $|\Im\sigma| \leq 1$ is mapped to the solution $w(z)$ with the branching point at $z = \sigma^2/2$, which "moves" in the domain $\mathcal{NP} \cup \Pi$. At the same time, the transformation (24) does not change the type of the singular point.

4. Isomonodromy deformations

It is known [2], that the similarity solutions of integrable PDEs can be treated as solutions of the isomonodromy class [5]. This means that one can characterize these solutions by attaching to their Lax pair an additional linear PDE that contains a differentiation with respect to the spectral parameter. In particular, the similarity solutions considered in Sec. 3 can be characterized by the following equation

$$\lambda\psi_\lambda = x\psi_x - t\psi_t, \tag{29}$$

where ψ is the isomonodromy solution of the Lax pair (2), with only two singular point at $\lambda = 0$ and $\lambda = \infty$. Equation (29) implies that the function ψ, modulo a scalar factor, can be presented as a function of the similarity variables, μ and z, which are the first integrals of the following system of the first order PDEs:

$$\frac{d\lambda}{\lambda} = \frac{dt}{t} = \frac{dx}{x} \quad \Rightarrow \quad \mu = \lambda/t, \quad z = xt.$$

Now denoting $\phi = \phi(\mu, z) = \psi(\lambda, x, t)$ for the isomonodromy function ψ (29) and using the Lax pair (2) we arrive at the following Fuchs–Garnier pair [7]:

$$\frac{d}{d\mu}\phi = \widehat{A}\phi, \quad \frac{d}{dz}\phi = \widehat{U}\phi, \tag{30}$$

$$\widehat{A} = -\frac{Z}{\sqrt{1+w'^2}}\begin{pmatrix}1 & w'\\ w' & -1\end{pmatrix} - \frac{w}{2\mu}\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix} - \frac{\sigma_3}{4\mu^2}, \quad \widehat{U} = \mu\begin{pmatrix}1 & w'\\ w' & -1\end{pmatrix},$$

where we use the notation introduced in (14) and (20).

The compatibility condition for the Fuchs–Garnier pair implies that w solves Eq. (15). To establish a relation of the function $w(z)$ in this approach with the third Painlevé function we can map the Fuchs–Garnier pair (30) to the corresponding Fuchs–Garnier pair for the third Painlevé equation in Appendix C of [6]. Instead of presenting this transformation we consider a slight generalization of our deformation problem:

$$\frac{d\Phi}{d\lambda} = \left(A + \frac{B}{\lambda} + \frac{C}{\lambda^2}\right)\Phi \equiv \mathcal{A}\Phi, \quad \frac{d\Phi}{dy} = \lambda A_1 \Phi, \tag{31}$$

where $A, A_1, B, C \in \mathrm{sl}_2(\mathbb{C})$, and y is a parameter, which is assumed to be an analytic function of the matrix elements of $A,\ A_1,\ B,\ C$. The fact that all coefficient matrices in system (31) are traceless is not a restriction in our case, since from the compatibility condition one deduce that

$$\partial_y \mathrm{tr}\, B = \partial_y \mathrm{tr}\, C = 0 \quad \text{and} \quad \partial_y \mathrm{tr}\, A = \mathrm{tr}\, A_1,$$

so that this requirement can be fulfilled via a transformation of Φ by a scalar factor. A further study of the compatibility condition for system (31) shows that $A = fA_1$, where f is an arbitrary (analytic) function of y and, possibly some of the matrix elements of $A,\ A_1,\ B,\ C$, which are also assumed to be analytic functions of y. Since our variable y is not fixed yet we can redefine (normalize) it as follows

$$y \to \widetilde{y}, \quad f dy = d\widetilde{y}, \tag{32}$$

which means that without "loss of generality" we can assume that $f = 1$ and keep the same notation. After that we can parameterize our system (31) as follows:

$$A = A_1 = \begin{pmatrix}a(y) & b'(y)\\ c'(y) & -a(y)\end{pmatrix}, \quad B = 2\kappa\begin{pmatrix}\beta & -b(y)\\ c(y) & -\beta\end{pmatrix}, \quad C = \kappa\sigma_3,$$

where κ, $\beta \in \mathbb{C}$ are parameters and the functions $a(y)$, $b(y)$, and $c(y)$ solve the system of ODEs:

$$a' = a + 2\kappa (bc)', \tag{33}$$

$$b'' = (1 - 4\kappa\beta) b' - 4\kappa a b, \tag{34}$$

$$c'' = (1 + 4\kappa\beta) c' - 4\kappa a c. \tag{35}$$

Our purpose now is to show that on one hand system (33)–(35) can be integrated in terms of the complete third Painlevé equation ($\mathbb{P}_3$) and on the other hand it can be viewed as a generalization of Eq. (15). For this purpose it is convenient to notice that the generating function for the first integrals for system (33)–(35) reads,

$$\partial_y \mathcal{A}^2 = \{\mathcal{A}, A\} \equiv \mathcal{A}\, A + A\, \mathcal{A}. \tag{36}$$

Taking into account definition of $\mathcal{A}$, see (31), we can rewrite Eq. (36) as follows:

$$\partial_y A^2 = 2A^2, \quad \partial_y \{B, A\} = \{B, A\}, \quad \partial_y (\{C, A\} + B^2) = \{C, A\}. \tag{37}$$

In terms of the functions $a(y)$, $b(y)$, $c(y)$, the first two relations in (37) generate the following first integrals:

$$\sqrt{a^2 + b'c'} = e^y, \tag{38}$$

$$2\beta a + c b' - b c' = -2\alpha\, e^y, \tag{39}$$

while the last relation in system (37) is equivalent to Eq. (33).

We begin with a generalization of Eq. (17). Define variable $\widetilde{z}$, $d\widetilde{z} = a dy$, and integrate Eq. (33),

$$\widetilde{a} = \widetilde{z} + 2\kappa\, \widetilde{b}\widetilde{c}, \tag{40}$$

where

$$\widetilde{a}(\widetilde{z}) = a(y), \quad \widetilde{b}(\widetilde{z}) = b(y), \quad \text{and} \quad \widetilde{c}(\widetilde{z}) = c(y). \tag{41}$$

Now we can rewrite system (34), (35) in terms of the tilde-variables as follows:

$$\left(\widetilde{z} + 2\kappa\, \widetilde{b}\widetilde{c}\right) \widetilde{b}'' = -2\kappa \left(2\beta + (\widetilde{b}\widetilde{c})'\right) \widetilde{b}' - 4\kappa\, \widetilde{b}, \tag{42}$$

$$\left(\widetilde{z} + 2\kappa\, \widetilde{b}\widetilde{c}\right) \widetilde{c}'' = -2\kappa (-2\beta + (\widetilde{b}\widetilde{c})') \widetilde{c}' - 4\kappa\, \widetilde{c}. \tag{43}$$

Note that if $\alpha = \beta = 0$, then Eq. (39) implies that $\widetilde{c} = \widetilde{C}\widetilde{b}$, where $\widetilde{C} \in \mathbb{C}$ is a parameter. Now we put $\widetilde{C} = 1$, $\kappa = -1/4$, and $\widetilde{z} = z$, $\widetilde{b} = \widetilde{c} = w(z)$

and arrive at Eq. (17). In the case $\widetilde{b} \equiv 0$ or $\widetilde{c} \equiv 0$, the system reduces to the Bessel equation.

To cope with the general case we rewrite Eqs. (38) and (39) in tilde-variables as follows:

$$e^{y} = (\widetilde{z} + 2\kappa\,\widetilde{b}\,\widetilde{c})\sqrt{1 + \widetilde{b}\,'\,\widetilde{c}\,'}, \tag{44}$$

$$\widetilde{c}\,'\,\widetilde{b} - \widetilde{b}\,'\,\widetilde{c} = -2\alpha\sqrt{1 + \widetilde{b}\,'\,\widetilde{c}\,'} + 2\beta. \tag{45}$$

Equations (40), (41), and (44) provide us with the parametric representation of solution of system (33), (34), and (35) in terms of solution of the "tilde"-system (42), (43).

Integration of the system (42), (43) can be reduced to a second order ODE, which generalizes equation (17), in the following way. Define the polar coordinates

$$\widetilde{b} \equiv \widetilde{w}(\widetilde{z})e^{i\widetilde{\theta}(\widetilde{z})}, \quad \widetilde{c} \equiv \widetilde{w}(\widetilde{z})e^{-i\widetilde{\theta}(\widetilde{z})}. \tag{46}$$

As follows from Eq. (45), the function $\widetilde{\theta} = \widetilde{\theta}(\widetilde{z})$ can be obtained via integration of the following relation

$$-i\widetilde{\theta}\,' = \frac{\beta\widetilde{w} - \alpha\sqrt{(\widetilde{w}^2 + \alpha^2)\left(1 + \widetilde{w}'^{\,2}\right) - \beta^2}}{\widetilde{w}\left(\widetilde{w}^2 + \alpha^2\right)},$$

where any branch of the square root can be taken and the function $\widetilde{w} = \widetilde{w}(\widetilde{z})$ is the general solution of the second order ODE,

$$\begin{aligned}&\left(\widetilde{w}'' + 4\kappa\widetilde{w}\frac{\widetilde{w}'^{\,2} + 1}{\widetilde{z} + 2\kappa\widetilde{w}^2} + \frac{\alpha^2(\widetilde{w}'^{\,2} + 1) - \beta^2}{\widetilde{w}(\widetilde{w}^2 + \alpha^2)} + \frac{2\beta^2\widetilde{w}}{(\widetilde{w}^2 + \alpha^2)^2}\cdot\frac{\widetilde{z} - 2\kappa\alpha^2}{\widetilde{z} + 2\kappa\widetilde{w}^2}\right)^2\\ &= \frac{4\alpha^2\beta^2}{(\widetilde{w}^2 + \alpha^2)^4}\left(\frac{\widetilde{z} - 2\kappa\alpha^2}{\widetilde{z} + 2\kappa\widetilde{w}^2}\right)^2\big((\widetilde{w}^2 + \alpha^2)(\widetilde{w}'^{\,2} + 1) - \beta^2\big).\end{aligned} \tag{47}$$

As we see if $\alpha\beta \neq 0$, then Eq. (47) is quadratic with respect to the second derivative. Most probably there should be another way to reduce system (42), (43) to the second order ODE which is linear with respect to the second derivative. At this stage, we consider Eq. (47) as a generalization of Eq. (17). In the case $\alpha\beta = 0$, Eq. (47) reduces to the ODEs which are linear with respect to the second derivative:

$$\alpha = 0 \quad \Rightarrow \quad \widetilde{w}'' = -4\kappa\widetilde{w}\,\frac{\widetilde{w}'^{\,2}+1}{\widetilde{z}+2\kappa\widetilde{w}^2} - \frac{\beta^2}{\widetilde{w}^3}\cdot\frac{\widetilde{z}-2\kappa\widetilde{w}^2}{\widetilde{z}+2\kappa\widetilde{w}^2}, \tag{48}$$

$$\beta = 0 \quad \Rightarrow \quad \widetilde{w}'' = -4\kappa\widetilde{w}\,\frac{\widetilde{w}'^{\,2}+1}{\widetilde{z}+2\kappa\widetilde{w}^2} - \frac{\alpha^2}{\widetilde{w}}\cdot\frac{\widetilde{w}'^{\,2}+1}{\widetilde{w}^2+\alpha^2}. \tag{49}$$

Now we turn to the reduction of system (33)–(35) to $\mathbb{P}_3$. The easiest way to find it is to map system (31) into the Fuchs–Garnier pair for $\mathbb{P}_3$ via the following transformation of

$$\Phi(y,\lambda) \to \widehat{\Phi}(t_p,\nu), \quad e^y = \frac{t_p^2}{4}, \quad \lambda = \frac{2}{\nu t_p},$$

where ν is a new spectral parameter and t_p is the argument of $\mathbb{P}_3$; and than, after a proper rescaling with κ, use a corresponding parametrization of the resulting system given either in [6] or in [8]. In the latter reference instead of $\mathbb{P}_3$ parametrization is given in terms of the degenerate (confluent) fifth Painlevé equation ($\mathbb{P}_5'$). The latter equation is known to be (bi-rationally) equivalent to $\mathbb{P}_3$.

Below we present parametrization in terms of $\mathbb{P}_5'$, without considering the mapping $\Phi \to \widehat{\Phi}$; instead we just directly reproduce the result of [8] in our current notation. Turning back to the integrals (38) and (39) and denoting

$$\widehat{z} = e^y, \quad \widehat{a}(\widehat{z}) = a(y), \quad \widehat{b}(\widehat{z}) = b(y), \quad \widehat{c}(\widehat{z}) = c(y), \tag{50}$$

we can rewrite them in the new variables as follows

$$\widehat{a} = \widehat{z}\sqrt{1-\widehat{b}'\widehat{c}'}, \tag{51}$$

$$\widehat{c}'\widehat{b} - \widehat{b}'\widehat{c} = 2\beta\sqrt{1-\widehat{b}'\widehat{c}'} + 2\alpha. \tag{52}$$

The system (34), (35) takes the following form:

$$\widehat{z}\widehat{b}'' = -4\kappa\beta\widehat{b}' - 4\kappa\sqrt{1-\widehat{b}'\widehat{c}'}\,\widehat{b}, \tag{53}$$

$$\widehat{z}\widehat{c}'' = 4\kappa\beta\widehat{c}' - 4\kappa\sqrt{1-\widehat{b}'\widehat{c}'}\,\widehat{c}. \tag{54}$$

In the case $\widehat{b} \equiv 0$ or $\widehat{c} \equiv 0$, the system reduces to the Bessel equation. If $\widehat{b}'\widehat{c}' = 1$ then solution is given in terms of the elementary functions:

$$\widehat{b}' = b_0\widehat{z}^{-4\kappa\beta}, \quad \widehat{c}' = \widehat{z}^{4\kappa\beta}/b_0,$$

where $b_0 \in \mathbb{C}$ is a parameter. Otherwise, the system is equivalent to the degenerate (confluent) fifth Painlevé equation. To see this we introduce new variables $\widehat{V} = \widehat{V}(\widehat{z})$ and $\widehat{\theta} = \widehat{\theta}(\widehat{z})$:

$$\widehat{b}' \equiv \frac{2\sqrt{-\widehat{V}}}{1-\widehat{V}}\, e^{i\widehat{\theta}}, \quad \widehat{c}' \equiv \frac{2\sqrt{-\widehat{V}}}{1-\widehat{V}}\, e^{-i\widehat{\theta}}, \quad \sqrt{1-\widehat{b}'\widehat{c}'} \equiv \frac{1+\widehat{V}}{1-\widehat{V}}.$$

The functions $\widehat{b}$ and $\widehat{c}$ can be calculated as follows:

$$\widehat{b} = \frac{e^{i\widehat{\theta}}}{\sqrt{-\widehat{V}}} \left(\frac{\widehat{z}\,\widehat{V}'}{4\kappa(1-\widehat{V})} + \frac{\alpha+\beta}{2} - \frac{\alpha-\beta}{2}\,\widehat{V} \right),$$

$$\widehat{c} = \frac{e^{-i\widehat{\theta}}}{\sqrt{-\widehat{V}}} \left(\frac{\widehat{z}\,\widehat{V}'}{4\kappa(1-\widehat{V})} - \frac{\alpha+\beta}{2} + \frac{\alpha-\beta}{2}\,\widehat{V} \right).$$

The function $\widehat{\theta}$ can be obtained via integration of the relation,

$$i\widehat{z}\,\widehat{\theta}' = \kappa\frac{1-\widehat{V}}{\widehat{V}} \left(\alpha+\beta+(\alpha-\beta)\widehat{V} \right),$$

where $\widehat{V}$ is the general solution of the degenerate (confluent) fifth Painlevé equation,

$$\begin{aligned} \widehat{V}'' &= \left(\frac{1}{2\widehat{V}} + \frac{1}{\widehat{V}-1} \right) \widehat{V}'^2 - \frac{\widehat{V}'}{\widehat{z}} \\ &+ \frac{\left(\widehat{V}-1\right)^2}{\widehat{z}^2} \left(2\kappa^2(\alpha-\beta)^2\widehat{V} - \frac{2\kappa^2(\alpha+\beta)^2}{\widehat{V}} \right) + \frac{8\kappa}{\widehat{z}}\widehat{V}. \end{aligned} \tag{55}$$

As long as the solution of the "hat"- or "tilde"-system is obtained one can construct the solution of system (33), (34), (35), by making use of the formulae (50), or (41) and (44), respectively. It is also immediate to get the general solution of the "hat"-system in terms of the "tilde"-one. Since equation (44) can be rewritten as follows,

$$\widehat{z} = (\widetilde{z} + 2\kappa\widetilde{b}\,\widetilde{c})\sqrt{1+\widetilde{b}'\,\widetilde{c}'}. \tag{56}$$

We would like however express the solution of the "tilde"-system in terms of the "hat"-system, since the general solution of the latter system is given in terms of the Painlevé functions. For this purpose we have to invert relation (56). To do it one proves the following identity,

$$\sqrt{1+\widetilde{b}'\,\widetilde{c}'} = -\frac{1}{\sqrt{1-\widehat{b}'\,\widehat{c}'}}.$$

Using it one finds

$$\widetilde{z} = -\widehat{z}\sqrt{1 - \widehat{b}'\,\widehat{c}'} - 2\kappa\widehat{b}\widehat{c}. \tag{57}$$

The later equation can be rewritten in terms of the Hamiltonian function for Eq. (55):

$$\widetilde{z} = \widehat{z}\,\widehat{\mathcal{H}}\,(\widehat{q}\,(\widehat{z}),\,\widehat{p}\,(\widehat{z}),\,\widehat{z}), \tag{58}$$

$$\widehat{\mathcal{H}}\,(\widehat{p},\,\widehat{q},\,\widehat{z}) \equiv 2\kappa\frac{\widehat{p}^{\,2}}{\widehat{z}} - \frac{2\kappa}{\widehat{z}}\,(\beta\coth\widehat{q} + \alpha/\sinh\,\widehat{q}\,)^2 - \cosh\widehat{q}$$

$$\widehat{q}\,(\widehat{z}) \equiv \ln\frac{1+\sqrt{\widehat{V}}}{1-\sqrt{\widehat{V}}}, \quad \widehat{p}\,(\widehat{z}) \equiv \frac{\widehat{z}\,\widehat{V}\,'}{4\kappa\sqrt{\widehat{V}}(1-\widehat{V})} = \frac{\widehat{z}\,\widehat{q}'\,(\widehat{z})}{4\kappa},$$

where $\widehat{q}$ and $\widehat{p}$ are the canonical variables and variable $\widehat{z}$ is the time. The Hamiltonian function, $\widehat{H}(\widehat{z})$, in the original variables reads,

$$\widehat{H}(\widehat{z}) \equiv \widehat{\mathcal{H}}\,(\widehat{q}\,(\widehat{z}),\widehat{p}\,(\widehat{z}),\,\widehat{z}) = \frac{\widehat{z}\,\widehat{V}\,'^2}{8\kappa\widehat{V}(1-\widehat{V})^2} - \frac{\kappa}{2\widehat{z}\,\widehat{V}}\left(\alpha+\beta-(\alpha-\beta)\widehat{V}\right)^2 - \frac{1+\widehat{V}}{1-\widehat{V}}. \tag{59}$$

To complete parametrization of Eq. (47) in terms of P_5' (55),

$$\widetilde{w}^2 = -\left(\frac{\widehat{z}\,\widehat{V}\,'}{4\kappa\sqrt{\widehat{V}}(1-\widehat{V})}\right)^2 + \frac{1}{4\widehat{V}}\left(\alpha+\beta-(\alpha-\beta)\widehat{V}\right)^2. \tag{60}$$

So, Eqs. (58), (59), and (60) gives parametrization of the general solution of Eq. (47) in terms of P_5'. Note that in fact system (46) define function $\widetilde{w}$ up to the sign. This results in the fact that all Eqs. (47), (48), and (49) can be rewritten in a rational form with respect to $\widetilde{w}^2$.

It is easy to derive a generalization of SPE (1) whose similarity solutions are described by the parametric third Painlevé equation (47). Consider a "decoupled" analog of the Zakharov–Shabat pair (2), (3):

$$\frac{\partial}{\partial x}\widetilde{\psi} = \widetilde{U}\widetilde{\psi}, \quad \frac{\partial}{\partial t}\widetilde{\psi} = \widetilde{V}\widetilde{\psi}, \tag{61}$$

$$\widetilde{U} = \lambda\widetilde{f}\begin{pmatrix}\widetilde{g} & \widetilde{u}_x\\ \widetilde{v}_x & -\widetilde{g}\end{pmatrix}, \quad \widetilde{V} = \lambda\begin{pmatrix}\widetilde{g} & \widetilde{u}_x\\ \widetilde{v}_x & -\widetilde{g}\end{pmatrix} + \frac{1}{2}\begin{pmatrix}0 & -\widetilde{u}\\ \widetilde{v} & 0\end{pmatrix} + \frac{\sigma_3}{4\lambda}. \tag{62}$$

The compatibility condition for (61), (62) reads:

$$\begin{aligned}
&\frac{\partial}{\partial t}\widetilde{g} = \frac{\partial}{\partial x}\left(\widetilde{f}\widetilde{g} - \frac{\widetilde{u}\widetilde{v}}{2}\right),\\
&\widetilde{u}_{xt} = (\widetilde{f}\widetilde{u})_x + \widetilde{g}\widetilde{u},\\
&\widetilde{v}_{xt} = (\widetilde{f}\widetilde{v})_x + \widetilde{g}\widetilde{v}.
\end{aligned} \tag{63}$$

So, we get three equations for four functions: $\widetilde{f}$, $\widetilde{g}$, $\widetilde{u}$, $\widetilde{v}$. One of these functions can be taken arbitrary, say, if we put $\widetilde{g} \equiv 1$, then $\widetilde{f} = \widetilde{u}\widetilde{v}/2$ and (63) takes the form of the decoupled SPE (1):

$$\begin{aligned}
\widetilde{u}_{xt} &= \widetilde{u} + \frac{1}{2}(\widetilde{u}\widetilde{v}\widetilde{u}_x)_x,\\
\widetilde{v}_{xt} &= \widetilde{v} + \frac{1}{2}(\widetilde{u}\widetilde{v}\widetilde{v}_x)_x.
\end{aligned}$$

The similarity reduction

$$\lambda\widetilde{\psi}_\lambda = x\widetilde{\psi}_x - t\widetilde{\psi}_t + \frac{\beta}{2}\sigma_3\widetilde{\psi},$$

reduces the Zakharov–Shabat pair (61), (62) to the Fuchs–Garnier one of the type (31). The above similarity reduction of $\widetilde{\psi}$ can be rewritten in terms of the functions $\widetilde{f}$, $\widetilde{g}$, $\widetilde{u}$, and $\widetilde{v}$ as follows:

$$\begin{aligned}
\widetilde{g}(x,t) = \widehat{g}(z), \quad \widetilde{f}(x,t) = t^2\widehat{f}(z), \quad z = xt,\\
\widetilde{u}(x,t) = t^{-1+\beta}\widehat{u}(z), \quad \widetilde{v}(x,t) = t^{-1-\beta}\widehat{v}(z).
\end{aligned}$$

5. Further Remarks

In this paper, we obtained a parametric P_3 or P_5' equation (47). Although this equation is, in some sense, equivalent to the corresponding Painelevé equation it has absolutely different analytic and transformation properties: The equation does not have Painlevé property but the complex plain can be divided into two domains in one of them "travel" regular singularities of the type that P_3 has at the origin, while in the complimentary domain the parametric Painlevé equation might even have Painlevé property, or at least, possess traveling singularities of a simpler type. In case the existence of the Painlevé type domain would be confirmed then one can pose in this domain the connection problems.

If we use formulae for the Bäcklund transformations for P_5' from the [8], then we find that the action of these transformations on the parametric

Painlevé equations shifts both dependent and independent variables. This feature was also mentioned for the parametric Painlevé equations considered in [3]. It would be interesting to check what types functional-difference and functional-differential equations satisfy "Bäcklund" iterations of the parametric Painlevé functions.

¿From our derivation we see that one can actually find infinitely many parametric Painlevé equations associated with the given Painlevé equation; the ambiguity is hidden in the function f (see Eq. (32)) which we put 1 "without loss of generality." Although there are many parametric Painlevé equations, which is good since it widen application of the Painlevé functions, the problem of classification of rational parameterizations for rational ODEs (rational parametric Painlevé equations) might have an explicit solution. More precisely, the problem can be formulated as follows find all equations of the form $y^{''} = R(y, y', z)$ or $y^{''2} = R(y, y', z)$ where R is a rational function of all its arguments with a nontrivial dependence on z, such that the solution $y = y(z)$ has the following parametric representation:

$$y = R_1(Y, Y', Z), \quad z = R_2(Y, Y', Z), \tag{64}$$

where R_1 and R_2 are the functions of the same type as R, i.e., rational with respect to all their arguments and the function $Y(Z)$ is the function of the Painlevé type. It is natural to call two such equations equivalent if their general solutions are related via birational transformations in the same sense as for the Painlevé equations. To exclude from that definition the Painlevé functions and/or their inverses we can require that one or both functions R_i, $i = 1, 2$ should have a nontrivial dependence on Y or Y'.

We can define a notion of equivalency of parameterizations: Say, if we have another parametrization $y = \widetilde{R}_1(\widetilde{Y}, \widetilde{Y}', \widetilde{Z})$ and $z = \widetilde{R}_2(\widetilde{Y}, \widetilde{Y}', \widetilde{Z})$, where $\widetilde{Y}(\widetilde{Z})$ is a solution of some Painlevé equation than the latter parametrization is equivalent to the one given by Eq. (64) if the functions $Y(Z)$ and $\widetilde{Y}(\widetilde{Z})$ are related with a rational transformation $\widetilde{Y} = r(Y, Y', Z)$, $\widetilde{Z} = r_0(Z)$ with some rational functions r_1 and r_0.

Our consideration shows that some "stationary" singular points of these equations can "blow up" producing the non-Painlevé domains. Co-existence of the Painlevé and non-Painlevé domains could be a characteristic feature for a "direct" detection of such equations: it can be called a partial Painlevé property. On the other hand some simpler examples of rational parametric Painlevé equations related with the first and second Painlevé equations show existence of regular singularities with the rational branching moving in the whole complex plain. Understanding of the analytical properties of such functions, in particular, the connection formulae, might be an interesting subject for further studies.

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