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ON ENTIRE SOLUTIONS OF THE EQUATIONS FOR THE DISPLACEMENT FIELDS IN THE DEFORMATION THEORY OF PLASTICITY WITH LOGARITHMIC HARDENING

Abstract. Let $u : \mathbb{R}^2 \to \mathbb{R}^2$ denote an entire solution of the homogeneous Euler-Lagrange equation associated to the energy used in the deformation theory of plasticity with logarithmic hardening. If $|u(x)|$ is of slower growth than $|x|$ as $|x| \to \infty$, then u must be constant. Moreover we show that u is affine if either $\sup\limits_{\mathbb{R}^2} |\nabla u| < \infty$ or $\limsup\limits_{|x|\to\infty} |x|^{-1}|u(x)| < \infty$.

In their paper [5], Frehse and Seregin propose to approximate the Hencky model used in perfect plasticity (cf. [4, 11] or [12]) by a variational problem formulated in terms of the displacement fields, in which the energy density $G(\varepsilon(u))$ is of quadratic growth with respect to the trace of $\varepsilon(u)$ and of $L \log L$-growth with respect to the deviator $\varepsilon^D(u) = \varepsilon(u) - \frac{1}{n}(\operatorname{div} u)\mathbf{1}$ of $\varepsilon(u)$. Here u is a displacement field defined on some region in $\mathbb{R}^n$, $\varepsilon(u)$ denotes the symmetric part of the Jacobian matrix of u and $\mathbf{1}$ is the unit matrix. Modulo physical constants we have in the case of logarithmic hardening

$$G(\varepsilon) = h\left(|\varepsilon^D|\right) + \frac{1}{2}\,(\operatorname{trace} \varepsilon)^2 \tag{1}$$

for symmetric $(n \times n)$–matrices ε, where

$$h(t) = t\ln(1+t), \quad t \geqslant 0\,. \tag{2}$$

Frehse and Seregin discuss solvability of the associated boundary value problems in suitable weak spaces and prove smoothness of local solutions at least in the case that $n = 2$. Later Seregin and the first author (see [7]) established partial regularity in the $3D$ case.

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A related problem arises in the study of certain models describing the flow of generalized Newtonian fluids, for which the stress-strain relation takes the form

$$T^D = DH(\varepsilon)\,. \tag{3}$$

If we let

$$H(\varepsilon) = h(|\varepsilon|) \tag{4}$$

with h defined in equation (2), then (3) is the constitutive law for the so-called Prandtl–Eyring fluid, which has been the subject of the paper [7] and also of the monograph [8]. Very recently the authors discussed the behaviour of entire solutions of this fluid model at least in the stationary case for two spatial variables and proved Liouville-type results (see [9]). The purpose of the present paper now is the investigation of planar entire solutions in the setting of plasticity with logarithmic hardening.

Definition 1. *A field $u : \mathbb{R}^2 \to \mathbb{R}^2$ of class C^1 is an entire local minimizer of the energy*

$$I[v,\Omega] = \int_\Omega G(\varepsilon(v))\,dx \tag{5}$$

with density G defined according to Eqs. (1) *and* (2) *if for any bounded domain $\Omega \subset \mathbb{R}^2$ and all fields $v : \Omega \to \mathbb{R}^2$ such that $\operatorname{spt}(u-v)$ is compactly contained in Ω it holds*

$$I[u,\Omega] \leqslant I[v,\Omega].$$

Remark 1. The smoothness assumption concerning u in Definition 1 is justified by the results in [5].

Remark 2. If u is an entire local I-minimizer, then it holds

$$\int_\Omega DH(\varepsilon^D(u)) : \varepsilon^D(\varphi)\,dx + \int_\Omega \operatorname{div} u \operatorname{div} \varphi\,dx = 0 \tag{6}$$

for any domain $\Omega \subset \mathbb{R}^2$ and all fields $\varphi \in C_0^1(\Omega;\mathbb{R}^2)$. In equation (6) the symbol ":" is the scalar product of matrices and H is introduced in Eq. (4).

Now we can state our main results:

Theorem 1. *Let $u : \mathbb{R}^2 \to \mathbb{R}^2$ denote an entire local I-minimizer (cf. Eq.* (5)*) in the sense of Definition* 1*. If u satisfies the asymptotic condition*

$$\lim_{|x|\to\infty} \frac{|u(x)|}{|x|} = 0, \tag{7}$$

then the displacement field u is a constant vector. In particular, the boundedness of the field implies its constancy.

The next theorem concerns entire solutions satisfying a global Lipschitz condition.

Theorem 2. *Consider an entire local I-minimizer $u : \mathbb{R}^2 \to \mathbb{R}^2$ in the sense of Definition 1. If we know that $|\nabla u| \in L^\infty(\mathbb{R}^2)$, then u must be affine.*

Finally we relax the global boundedness of the gradient by imposing a growth condition on u:

Theorem 3. *If the entire local I-minimizer $u : \mathbb{R}^2 \to \mathbb{R}^2$ satisfies $\limsup\limits_{|x|\to\infty} |x|^{-1}|u(x)| < \infty$, then u must be affine.*

Remark 3. It would be interesting to know what can be said about entire solutions in the $3D$-case. Due to the lack of regularity (cf. [7, 8]) one either has to deal with weak local minimizers or the smoothness of u has to be imposed as a severe extra condition. In the latter case we think that for $n = 3$ condition (7) has to be replaced by $\lim\limits_{|x|\to\infty} \frac{|u(x)|}{\sqrt{|x|}} = 0$ in order to obtain the constancy of u, and this conclusion probably also holds in the case that $\limsup\limits_{|x|\to\infty} \frac{|u(x)|}{\sqrt{|x|}} < \infty$ (compare the proof of Theorem 3).

For the proof of Theorem 1 we need two auxiliary results:

Lemma 1. (Korn-type inequality) *For fields $v : \mathbb{R}^2 \to \mathbb{R}^2$ with compact support it holds*

$$\int\limits_{\mathbb{R}^2} |\nabla v|^2 \, dx \leqslant 2 \int\limits_{\mathbb{R}^2} |\varepsilon^D(v)|^2 \, dx. \tag{8}$$

Korn-type inequalities involving ε^D have been established by Reshetnyak [13] in a much more general setting. Recently Dain rediscovered these estimates in the L^2-setting (see [3]), and the first author together with Bildhauer proved variants in the context of Orlicz–Sobolev spaces (cf. [6]).

The next lemma is essentially due to Giaquinta and Modica (compare Lemma 0.5 in [10]), in the formulation given below it corresponds to Lemma 3.1 in [9].

Lemma 2. *Let f, f_1, …, f_ℓ denote non-negative functions from the space $L^1_{\mathrm{loc}}(\mathbb{R}^2)$ and suppose that we are given exponents $\alpha_1, \ldots, \alpha_\ell > 0$. Then* *we can find a number $\delta_0 > 0$ depending on $\alpha_1, \dots, \alpha_\ell$ as follows: if for $\delta \in (0, \delta_0)$ it is possible to calculate a constant $c(\delta) > 0$ such that the inequality*

$$\int_{Q_R(z)} f\,dx \leqslant \delta \int_{Q_{2R}(z)} f\,dx + c(\delta) \sum_{j=1}^{\ell} R^{-\alpha_j} \int_{Q_{2R}(z)} f_j\,dx$$

holds for any choice of $Q_R(z) := \{x \in \mathbb{R}^2 : |x_i - z_i| < R,\, i = 1, 2\}$, then there is a constant $c > 0$ with the property

$$\int_{Q_R(z)} f\,dx \leqslant c \sum_{j=1}^{\ell} R^{-\alpha_j} \int_{Q_{2R}(z)} f_j\,dx$$

again for all squares $Q_R(z)$.

Remark 4. Of course Lemma 2 extends to $\mathbb{R}^n$, $n \geqslant 3$, replacing squares by cubes, and it is easy to see that estimate (8) remains valid in higher dimensions.

Now we pass to the *proof of Theorem* 1 proceeding in several steps.
Step 1. A growth estimate for the energy

We fix a square $Q_{2R}(x_0)$ and choose $\eta \in C_0^1(Q_{2R}(x_0))$ such that $\eta = 1$ on $Q_R(x_0), 0 \leqslant \eta \leqslant 1, |\nabla \eta| \leqslant c/R$. Then we apply equation (6) by selecting $\varphi = \eta^2 u$. We get with H defined in (4)

$$\int_{Q_{2R}(x_0)} \eta^2 DH\left(\varepsilon^D(u)\right) : \varepsilon^D(u)\,dx + \int_{Q_{2R}(x_0)} \eta^2 (\operatorname{div} u)^2\,dx \tag{9}$$

$$= -2 \int_{Q_{2R}(x_0)} \eta DH\left(\varepsilon^D(u)\right) : (\nabla\eta \otimes u)^D\,dx - 2 \int_{Q_{2R}(x_0)} \eta \operatorname{div} u \nabla\eta \cdot u\,dx$$

$$\leqslant c \left[\int_{Q_{2R}(x_0)} \eta h'\left(|\varepsilon^D(u)|\right) |\nabla\eta||u|\,dx + \int_{Q_{2R}(x_0)} \eta |\operatorname{div} u||\nabla\eta||u|\,dx \right].$$

Using Young's inequality we obtain for any $\delta > 0$

$$\eta h'\left(|\varepsilon^D(u)|\right) |\nabla\eta||u| \leqslant \delta\eta^2 h'\left(|\varepsilon^D(u)|\right) |\varepsilon^D(u)| + \delta^{-1} |\nabla\eta|^2 \frac{h'(|\varepsilon^D(u)|)}{|\varepsilon^D(u)|} |u|^2,$$

$$\eta |\operatorname{div} u||\nabla\eta||u| \leqslant \delta\eta^2 (\operatorname{div} u)^2 + \delta^{-1} |\nabla\eta|^2 |u|^2.$$

Inserting these estimates in inequality (9) and observing that $\frac{h'(t)}{t} \leqslant 2$, we deduce after appropriate choice of δ and recalling the properties of η

$$\begin{aligned} \int\limits_{Q_R(x_0)} G\left(\varepsilon(u)\right)\,dx &= \int\limits_{Q_R(x_0)} \left[H\left(\varepsilon^D(u)\right) + \frac{1}{2}\left(\operatorname{div} u\right)^2\right]\,dx \\ &\leqslant cR^{-2}\int_{Q_{2R}(x_0)-\overline{Q}_R(x_0)} |u|^2\,dx\,. \end{aligned} \tag{10}$$

In particular, if we choose $x_0 = 0$ and abbreviate

$$\Theta(R) := \sup\left\{|x|^{-1}|u(x)| : x \in \mathbb{R}^2 - \overline{Q}_R\right\}\,,$$

then (10) implies

$$\int\limits_{Q_R} G\left(\varepsilon(u)\right)\,dx \leqslant cR^2\Theta(R)^2 \tag{11}$$

with $\lim\limits_{R\to\infty} \Theta(R) = 0$ according to our hypothesis (7).

Step 2. Discussion of the second derivatives

Returning to equation (6) and performing an integration by parts we get for $\alpha = 1,\,2$ and $\varphi \in C_0^1(Q_{\frac{3}{2}R}(x_0))$

$$\begin{aligned} 0 = &\int\limits_{Q_{\frac{3}{2}R}(x_0)} D^2H\left(\varepsilon^D(u)\right)\left(\varepsilon^D(\partial_\alpha u),\,\varepsilon^D(\varphi)\right)\,dx \\ &+ \int\limits_{Q_{\frac{3}{2}R}(x_0)} \operatorname{div}(\partial_\alpha u)\operatorname{div}\varphi\,dx. \end{aligned} \tag{12}$$

In Eq. (12), we choose $\varphi = \eta^2\partial_\alpha u$ (from now on summation with respect to $\alpha = 1, 2$), where η is as in Step 1 with $2R$ replaced by $\frac{3}{2}R$. From (12) we easily obtain by applying the Cauchy-Schwarz inequality to the quantity

$$D^2H\left(\varepsilon^D(u)\right)\left(\eta\varepsilon^D(\partial_\alpha u),\,(\nabla\eta\otimes\partial_\alpha u)^D\right)$$

and appropriate use of Young's inequality (observing the boundedness of $|D^2H(\varepsilon^D(u))|$)

$$\int_{Q_{\frac{3}{2}R}(x_0)} D^2H\left(\partial_\alpha\varepsilon^D(u), \partial_\alpha\varepsilon^D(u)\right)\eta^2\,dx + \int_{Q_{\frac{3}{2}R}(x_0)} \eta^2|\nabla(\operatorname{div} u)|^2\,dx \leqslant c \int_{Q_{\frac{3}{2}R}(x_0)} |\nabla\eta|^2|\nabla u|^2\,dx\,,$$

hence by the properties of η

$$\int_{Q_R(x_0)} D^2H\left(\varepsilon^D(u)\right)\left(\varepsilon^D(\partial_\alpha u), \varepsilon^D(\partial_\alpha u)\right)\,dx + \int_{Q_R(x_0)} |\nabla(\operatorname{div} u)|^2\,dx \leqslant cR^{-2} \int_{Q_{\frac{3}{2}R}(x_0)} |\nabla u|^2\,dx\,, \tag{13}$$

and inequality (13) holds for all squares $Q_R(x_0)$. Note that (13) implies that entire local minimizers having finite Dirichlet integral must be affine. This follows by letting $R \to \infty$ and observing that on the right-hand side of (13) the domain of integration can be replaced by $Q_{\frac{3}{2}R}(x_0) - \overline{Q_R(x_0)}$. In order to control $\int_{Q_{\frac{3}{2}R}(x_0)} |\nabla u|^2\,dx$ we choose $\Psi \in C_0^1(Q_{2R}(x_0))$ such that $0 \leqslant \Psi \leqslant 1$, $\Psi = 1$ on $Q_{\frac{3}{2}R}(x_0)$ and $|\nabla\Psi| \leqslant c/R$. From estimate (8) in Lemma 1, we obtain

$$\begin{aligned}
\int_{Q_{\frac{3}{2}R}(x_0)} |\nabla u|^2\,dx &\leqslant c\left[\int_{Q_{2R}(x_0)} |\nabla(\Psi u)|^2\,dx + \int_{Q_{2R}(x_0)} |\nabla\Psi|^2|u|^2\,dx\right] \\
&\leqslant c\left[\int_{Q_{2R}(x_0)} |\varepsilon^D(\Psi u)|^2\,dx + \int_{Q_{2R}(x_0)} |\nabla\Psi|^2|u|^2\,dx\right] \\
&\leqslant c\left[\int_{Q_{2R}(x_0)} \Psi^2|\varepsilon^D(u)|^2\,dx + R^{-2}\int_{Q_{2R}(x_0)} |u|^2\,dx\right]
\end{aligned}$$

or by the support properties of Ψ

$$\int\limits_{Q_{\frac{3}{2}R}(x_0)} |\nabla u|^2\, dx \leqslant c\left[\int\limits_{Q_{2R}(x_0)} \Psi^2 |\varepsilon^D(u)|^2\, dx + R^{-2} \int\limits_{Q_{2R}(x_0)-\overline{Q_{\frac{3}{2}R}(x_0)}} |u|^2\, dx\right]. \tag{14}$$

In order to proceed we observe

$$\varepsilon^D_{ij}(u) = \frac{1}{2}\left(\frac{\partial u^j}{\partial x_i} + \frac{\partial u^i}{\partial x_j}\right) - \frac{1}{2}(\operatorname{div} u)\delta_{ij}\,,$$

hence by the symmetry of $\varepsilon^D(u)$ and the fact that $\varepsilon^D_{ij}(u)\delta_{ij} = 0$

$$\begin{aligned}
&\int\limits_{Q_{2R}(x_0)} \Psi^2 |\varepsilon^D(u)|^2\, dx \\
&\quad = \frac{1}{2}\int\limits_{Q_{2R}(x_0)} \Psi^2 \left\{\left(\frac{\partial u^i}{\partial x_j} + \frac{\partial u^j}{\partial x_i}\right) - (\operatorname{div} u)\delta_{ij}\right\} \varepsilon^D_{ij}(u)\, dx \\
&\quad = -\int\limits_{Q_{2R}(x_0)} \partial_i\left(\Psi^2 \varepsilon^D_{ij}(u)\right) u^j\, dx\,.
\end{aligned}$$

This yields

$$\int\limits_{Q_{2R}(x_0)} \Psi^2 |\varepsilon^D(u)|^2\, dx \leqslant c\left[\int\limits_{Q_{2R}(x_0)} |\nabla \Psi^2| |u| |\varepsilon^D(u)|\, dx + \int\limits_{Q_{2R}(x_0)} \Psi^2 |\nabla \varepsilon^D(u)| |u|\, dx\right]$$

$$\leqslant c\left[R^{-1}\int\limits_{Q_{2R}(x_0)}|u||\varepsilon^D(u)|\,dx+\delta\int\limits_{Q_{2R}(x_0)}\frac{|\nabla\varepsilon^D(u)|^2}{1+|\varepsilon^D(u)|}\,dx\right.$$
$$\left.+\delta^{-1}\int\limits_{Q_{2R}(x_0)}|u|^2\left(1+|\varepsilon^D(u)|\right)\,dx\right].$$

Let $\omega:=D^2H\left(\varepsilon^D(u)\right)\left(\partial_\alpha\varepsilon^D(u),\partial_\alpha\varepsilon^D(u)\right)$. If we combine (13), (14) and the inequalities from above, we obtain for any $\delta>0$ and all squares $Q_R(x_0)$

$$\int\limits_{Q_R(x_0)}\omega\,dx+\int\limits_{Q_R(x_0)}|\nabla(\operatorname{div}u)|^2\,dx \tag{15}$$
$$\leqslant\ c\left[R^{-4}\int\limits_{Q_{2R}(x_0)}|u|^2\,dx+R^{-2}\delta\int\limits_{Q_{2R}(x_0)}\omega\,dx\right.$$
$$\left.+\ R^{-3}\int\limits_{Q_{2R}(x_0)}|u||\varepsilon^D(u)|\,dx+R^{-2}\delta^{-1}\int\limits_{Q_{2R}(x_0)}|u|^2\left(1+|\varepsilon^D(u)|\right)\,dx\right].$$

Replacing δ by $\delta' R^2$ an application of Lemma 2 yields

$$\int\limits_{Q_R(x_0)}\omega\,dx+\int\limits_{Q_R(x_0)}|\nabla(\operatorname{div}u)|^2\,dx$$
$$\leqslant c\left[R^{-4}\int\limits_{Q_{2R}(x_0)}|u|^2\,dx+R^{-3}\int\limits_{Q_{2R}(x_0)}|u||\varepsilon^D(u)|\,dx\right. \tag{15'}$$
$$\left.+R^{-4}\int\limits_{Q_{2R}(x_0)}|u|^2\left(1+|\varepsilon^D(u)|\right)\,dx\right].$$

Let $x_0=0$ and $R\geqslant 1$. From our hypothesis (7) we obtain $|u(x)|\leqslant cR$ on Q_{2R}. Therefore (15') implies

$$\int\limits_{Q_R}\omega\,dx+\int\limits_{Q_R}|\nabla(\operatorname{div}u)|^2\,dx\leqslant c\left[R^{-4}R^4+R^{-2}\int\limits_{Q_{2R}}|\varepsilon^D(u)|\,dx\right]. \tag{16}$$

Clearly we have $\left(Q^+ := Q_{2R} \cap \left[|\varepsilon^D(u)| \geqslant 1\right], Q^- := \dots\right)$

$$
\begin{aligned}
\int\limits_{Q_{2R}} |\varepsilon^D(u)|\, dx &= \int\limits_{Q^+} |\varepsilon^D(u)|\, dx + \int\limits_{Q^-} |\varepsilon^D(u)|\, dx \\
&\leqslant \Big(\int\limits_{Q^-} 1\, dx\Big)^{1/2} \Big(\int\limits_{Q^-} |\varepsilon^D(u)|^2\, dx\Big)^{1/2} + \frac{1}{\ln 2} \int\limits_{Q^+} H\left(\varepsilon^D(u)\right)\, dx \\
&\leqslant cR\Big(\int\limits_{Q_{2R}} H\left(\varepsilon^D(u)\right)\, dx\Big)^{1/2} + \frac{1}{\ln 2} \int\limits_{Q_{2R}} H\left(\varepsilon^D(u)\right)\, dx,
\end{aligned}
$$

and if we use (10), we find

$$
\int\limits_{Q_{2R}} |\varepsilon^D(u)|\, dx \leqslant cR^2\,. \tag{17}
$$

This shows that the right-hand side of (16) stays bounded as $R \to \infty$, which means

$$
\int\limits_{\mathbb{R}^2} \omega\, dx + \int\limits_{\mathbb{R}^2} |\nabla(\operatorname{div} u)|^2\, dx < \infty\,. \tag{18}
$$

Step 3. Conclusion

We claim that the integral in (18) vanishes. In order to prove this we choose $x_0 = 0$ and return to inequality (13) recalling that in place of (13) we actually have

$$
\int\limits_{Q_R} |\nabla\left(\operatorname{div} u\right)|^2\, dx + \int\limits_{Q_R} \omega\, dx \leqslant cR^{-2} \int\limits_{Q_{\frac{3}{2}R} - \overline{Q}_R} |\nabla u|^2\, dx\,.
$$

Let $\Psi \in C_0^1(Q_{2R} - \overline{Q}_{R/2})$ such that $0 \leqslant \Psi \leqslant 1$ and $\Psi = 1$ on $Q_{\frac{3}{2}R} - \overline{Q}_R$ together with $|\nabla\Psi| \leqslant c/R$. Observing

$$
\int\limits_{Q_{\frac{3}{2}R} - \overline{Q}_R} |\nabla u|^2\, dx \leqslant c\left[\int\limits_{Q_{2R} - \overline{Q}_{R/2}} |\nabla(\Psi u)|^2\, dx + R^{-2} \int\limits_{Q_{2R} - \overline{Q}_{R/2}} |u|^2\, dx\right]
$$

we obtain a variant of (14), in which now the term $\int\limits_{Q_{2R}-\overline{Q}_{R/2}} \Psi^2|\varepsilon^D(u)|^2\,dx$ occurs on the right-hand side. Proceeding as before we get in place of (15)

$$\begin{aligned} &\int\limits_{Q_R} \omega\,dx + \int\limits_{Q_R} |\nabla(\operatorname{div} u)|^2\,dx \qquad (19)\\ &\leqslant c\Bigg[R^{-4}\int\limits_{Q_{2R}-\overline{Q}_{R/2}} |u|^2\,dx + R^{-2}\delta \int\limits_{Q_{2R}-\overline{Q}_{R/2}} \omega\,dx \\ &\quad + R^{-3}\int\limits_{Q_{2R}-\overline{Q}_{R/2}} |u||\varepsilon^D(u)|\,dx + R^{-2}\delta^{-1}\int\limits_{Q_{2R}-\overline{Q}_{R/2}} |u|^2\left(1+|\varepsilon^D(u)|\right)\,dx\Bigg]. \end{aligned}$$

Let $\delta := \frac{1}{2c}R^2$. Inequality (19) then yields

$$\begin{aligned} \int\limits_{Q_R} \omega\,dx + \int\limits_{Q_R} |\nabla(\operatorname{div} u)|^2\,dx &\leqslant \frac{1}{2}\int\limits_{Q_{2R}-\overline{Q}_{R/2}} \omega\,dx \qquad (20)\\ &+ c\Bigg[\Theta^2(R/2) + \Theta(R/2)R^{-2}\int\limits_{Q_{2R}} |\varepsilon^D(u)|\,dx \\ &+ \Theta^2(R/2)R^{-2}\int\limits_{Q_{2R}} \left(1+|\varepsilon^D(u)|\right)\,dx\Bigg], \end{aligned}$$

and if we use (17) and (18) together with the hypothesis that $\lim\limits_{R\to\infty}\Theta(R) = 0$, estimate (20) implies after passing to the limit $R\to\infty$ that ω as well as $\nabla(\operatorname{div} u)$ must vanish, thus $\nabla\varepsilon(u)\equiv 0$. But then it holds $\nabla^2 u \equiv 0$, which means that u is affine and thereby constant on account of our assumption (7). This completes the proof of Theorem 1.

For *proving Theorem* 2 we observe that boundedness of $|\nabla u|$ implies the estimate

$$|u(x)| \leqslant cR,\ x \in Q_{2R}\,,$$

provided $R \geqslant 1$. Using this information we again arrive at inequality (18), and this estimate can be restated in the form (recall (1) and (2))

$$\int_{\mathbb{R}^2} D^2G(\varepsilon(u))(\partial_\alpha\varepsilon(u), \partial_\alpha\varepsilon(u))\,dx < \infty\,.$$

Note that this is also a direct consequence of estimate (13). Using $|\nabla^2 u| \leqslant c|\nabla\varepsilon(u)|$ together with the boundedness of $\varepsilon(u)$ we get

$$\int_{\mathbb{R}^2} \left|\nabla^2 u\right|^2\,dx < \infty\,. \tag{21}$$

Similar to equation (12) it holds ($\alpha = 1, 2$)

$$0 = \int_{Q_{2R}} D^2G(\varepsilon(u))\,(\partial_\alpha\varepsilon(u), \varepsilon(\varphi))\,dx\,,$$

and we may choose $\varphi = \eta^2\partial_\alpha u$ with $\eta \in C_0^1(Q_{2R})$ such that $\eta = 1$ on Q_R, $0 \leqslant \eta \leqslant 1$ and $|\nabla\eta| \leqslant c/R$. We get

$$\begin{aligned}
&\int_{Q_{2R}} D^2G(\varepsilon(u))\,(\partial_\alpha\varepsilon(u), \partial_\alpha\varepsilon(u))\,\eta^2\,dx \\
&\quad= \quad -2\int_{Q_{2R}-\overline{Q}_R} D^2G\,(\varepsilon(u))\,(\partial_\alpha\varepsilon(u), \nabla\eta\otimes\partial_\alpha u)\,\eta\,dx,
\end{aligned}$$

and the boundedness of $\varepsilon(u)$ yields (recall $|\nabla^2 u| \leqslant c|\nabla\varepsilon(u)|$)

$$\begin{aligned}
\int_{Q_R} |\nabla^2 u|^2\,dx &\leqslant cR^{-1}\int_{Q_{2R}-\overline{Q}_R} |\nabla^2 u||\nabla u|\,dx \\
&\leqslant \quad cR^{-1}\Bigg(\int_{Q_{2R}-\overline{Q}_R} \left|\nabla^2 u\right|^2\,dx\Bigg)^{1/2}\Bigg(\int_{Q_{2R}-\overline{Q}_R} \left|\nabla u\right|^2\,dx\Bigg)^{1/2};
\end{aligned}$$

hence, by the boundedness of the gradient,

$$\int_{Q_R} \left|\nabla^2 u\right|^2\,dx \leqslant c\Bigg(\int_{Q_{2R}-\overline{Q}_R} \left|\nabla^2 u\right|^2\,dx\Bigg)^{1/2}. \tag{22}$$

On account of (21) the right-hand side of (22) vanishes as $R \to \infty$, thus $\nabla^2 u \equiv 0$, which proves Theorem 2. □

Let us finally discuss the *proof of Theorem* 3. As remarked in the beginning of the proof of Theorem 2 the growth condition imposed now on u is still sufficient to get

$$\int_{\mathbb{R}^2} D^2G\left(\varepsilon(u)\right)\left(\partial_\alpha\varepsilon(u),\partial_\alpha\varepsilon(u)\right)\,dx<\infty\,. \tag{23}$$

We return to equation (12) choosing $x_0=0$, $R\geqslant 1$, and select η as done after (12). The Cauchy–Schwarz inequality together with Hölder's estimate implies ($\xi_\alpha:=\partial_\alpha u\otimes\nabla\eta^2$)

$$\begin{aligned}&\int_{Q_R} D^2G\left(\varepsilon(u)\right)\left(\partial_\alpha\varepsilon(u),\partial_\alpha\varepsilon(u)\right)\,dx=\int_{Q_R}\omega\,dx+\int_{Q_R}\left|\nabla(\operatorname{div}u)\right|^2\,dx\\
&\leqslant c\Bigg[\Bigg(\int_{Q_{\frac32 R}-\overline{Q}_R}\omega\,dx\Bigg)^{1/2}\Bigg(\int_{Q_{\frac32 R}-\overline{Q}_R}D^2H\left(\varepsilon^D(u)\right)\left(\xi_\alpha^D,\xi_\alpha^D\right)\,dx\Bigg)^{1/2}\\
&\quad+\Bigg(\int_{Q_{\frac32 R}-\overline{Q}_R}\left|\nabla(\operatorname{div}u)\right|^2\,dx\Bigg)^{1/2}\Bigg(\int_{Q_{\frac32 R}-\overline{Q}_R}|\nabla u|^2|\nabla\eta|^2\,dx\Bigg)^{1/2}\Bigg],\end{aligned}$$

hence we find

$$\begin{aligned}&\int_{Q_R}\omega\,dx+\int_{Q_R}\left|\nabla(\operatorname{div}u)\right|^2\,dx\\
&\leqslant cR^{-1}\Bigg(\int_{Q_{\frac32 R}-\overline{Q}_R}|\nabla u|^2\,dx\Bigg)^{1/2}\Bigg\{\int_{Q_{\frac32 R}-\overline{Q}_R}\omega\,dx+\int_{Q_{\frac32 R}-\overline{Q}_R}\left|\nabla(\operatorname{div}u)\right|^2\,dx\Bigg\}^{1/2},\end{aligned} \tag{24}$$

and with (23) and (24) our claim will follow by letting $R\to\infty$ as soon as we can show

$$\int_{Q_R}|\nabla u|^2\,dx\leqslant cR^2 \tag{25}$$

for all $R\geqslant 1$. For proving (25) we recall that by (14) it holds

$$\int_{Q_{\frac32 R}}|\nabla u|^2\,dx\leqslant c\left\{\int_{Q_{2R}}\Psi^2\left|\varepsilon^D(u)\right|^2\,dx+R^{-2}\int_{Q_{2R}-\overline{Q}_{\frac32 R}}|u|^2\,dx\right\} \tag{26}$$

with Ψ defined after (13), and from (26) in combination with our growth assumption imposed on u we infer that it remains to bound the quantity $\int_{Q_{2R}} \Psi^2 \left|\varepsilon^D(u)\right|^2 dx$ in terms of R^2. Proceeding as done after inequality (14) we find

$$\begin{aligned}\int_{Q_{2R}} \Psi^2 \left|\varepsilon^D(u)\right|^2 dx &\leqslant c\Big\{ \int_{Q_{2R}} |\nabla\Psi^2||u||\varepsilon^D(u)|\, dx + \int_{Q_{2R}} \Psi^2|u||\nabla\varepsilon^D(u)|\, dx\Big\}\\ &\leqslant c\Big\{\frac{1}{R}\int_{Q_{2R}} |u||\varepsilon^D(u)|\, dx + \Big(\int_{Q_{2R}} \frac{|\nabla\varepsilon^D(u)|^2}{1+|\varepsilon^D(u)|}\, dx\Big)^{1/2}\\ &\quad\times\Big(\int_{Q_{2R}} |u|^2\left(1+|\varepsilon^D(u)|\right) dx\Big)^{1/2}\Big\}.\end{aligned}$$

On Q_{2R} it holds $|u(x)| \leqslant cR$, hence

$$\begin{aligned}&\int_{Q_{2R}} \Psi^2 \left|\varepsilon^D(u)\right|^2 dx \\ &\leqslant\ c\Big\{\int_{Q_{2R}} \left|\varepsilon^D(u)\right| dx + R\Big(\int_{Q_{2R}} \left(1+|\varepsilon^D(u)|\right) dx\Big)^{1/2}\Big\},\end{aligned} \tag{27}$$

where we also made use of (23) to bound the term involving $\nabla\varepsilon^D(u)$. The discussion following inequality (16) shows

$$\begin{aligned}&\int_{Q_{2R}} \left|\varepsilon^D(u)\right| dx \\ &\leqslant\ cR\left(\int_{Q_{2R}} H\left(\varepsilon^D(u)\right) dx\right)^{1/2} + \frac{1}{\ln 2}\int_{Q_{2R}} H\left(\varepsilon^D(u)\right) dx,\end{aligned} \tag{28}$$

and if we go back to estimate (10) (being valid without any growth hypothesis imposed on u) we find that now this inequality yields

$$\int_{Q_{2R}} H\left(\varepsilon^D(u)\right) dx \leqslant cR^2\,. \tag{29}$$

Finally, we combine the inequalities (28) and (29) with the result that

$$\int_{Q_{2R}} \left|\varepsilon^D(u)\right| \, dx \leqslant cR^2 \,, \tag{30}$$

and we see that (27) and (30) imply the correct bound for

$$\int_{Q_{2R}} \Psi^2 \left|\varepsilon^D(u)\right|^2 \, dx.$$

Thus we have established (25) and the proof of Theorem 3 is complete. □

Remark 5. We leave it to the reader to discuss Theorem 1, 2, and 3 for nonlinear Hencky materials, which means that the energy density from equation (1) is replaced by

$$W(\varepsilon) := \Phi\left(|\varepsilon^D|\right) + \frac{1}{2}(\mathrm{tr}\ \varepsilon)^2$$

for a "general" N-function Φ (compare, e.g. [1] for a definition). We refer to the article [2], where one will find natural hypotheses to be imposed on Φ under which one can expect a Liouville-type result.

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