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NOTE ON BOUNDED SCALE-INVARIANT QUANTITIES FOR THE NAVIER–STOKES EQUATIONS

Abstract. In this note, we show that if the velocity field $v \in L_\infty(BMO^{-1})$, then all scaled energy quantities are bounded. An interesting consequence is that each axially symmetric solution to the Navier–Stokes belonging to $L_\infty(BMO^{-1})$ is smooth.

§1. Main result

Let us consider the classical Navier–Stokes equations

$$\partial_t v + v\cdot\nabla u - \Delta v = -\nabla q, \qquad \operatorname{div} v = 0 \tag{1.1}$$

in the unit parabolic cylinder $Q = B\times]-1,0[$. A pair (v,q) is said to be a *suitable weak solution* to the Navier–Stokes equations in Q if the following conditions hold:

$$v \in W^{1,0}_2(Q)\cap L_{2,\infty}(Q), \qquad q \in L_{\frac32}(Q); \tag{1.2}$$

the Navier–Stokes equations are satisfied in the sense of distributions; for a.a. $t_0 \in]-1,0[$, the inequality

$$\int\limits_B \varphi^2(x,t_0)|v(x,t_0)|^2\,dx + 2\int\limits_{-1}^{t_0}\int\limits_B \varphi^2|\nabla v|^2\,dx\ dt$$

$$\leqslant \int\limits_{-1}^{t_0}\int\limits_B \Big(|v|^2(\Delta\varphi^2+\partial_t\varphi^2) + v\cdot\nabla\varphi^2(|v|^2+2q)\Big)\,dx\,dt \tag{1.3}$$

holds for all $\varphi \in C^\infty_0(B\times]-1,1[)$, see, for details, [1] and [2].

Our main assumption is that there exists a skew symmetric matrix d with the following properties

$$v = \operatorname{div} d, \qquad d \in L_\infty(-1,0;BMO(B;\mathbb{M}^{3\times3})). \tag{1.4}$$

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In what follows, we shall denote by c all positive constants depending only on $\Gamma = \|b\|_{L_\infty(BMO)} =: \|b\|_{L_\infty(-1,0;BMO(B))}$.

$B(x,r)$ is a ball of radius r in $\mathbb{R}^n$ centered at the point x. For a space-time point $z=(x,t)$, we denote by $Q(z,r)$ the usual parabolic cylinder so that $Q(z,r) = B(x,r)\times]t-r^2,t[$. We are also going to use the following abbreviations:

$$B(r) = B(0,r), \quad B = B(1), \quad Q(r) = Q(0,r), \quad Q = Q(1).$$

Mean values will be denoted as

$$[f]_{x,r} = \frac{1}{|B(r)|}\int\limits_{B(x,r)} f(y)dy, \qquad (u)_{z_0,r} = \frac{1}{|Q(r)|}\int\limits_{Q(z_0,r)} u(z)\,dz.$$

We also recall that

$$\|d\|_{BMO(\Omega;\mathbb{M}^{3\times3})} = \sup\left\{\frac{1}{|B(0,r)|}\int\limits_{B(x,r)} |d-[d]_{x,r}|\;dx \,\middle|\, B(x,r)\subset\Omega\right\}$$

is a semi-norm of the space $BMO(\Omega;\mathbb{M}^{3\times3})$

As usual, we let

$$C(z_0,R) = \frac{1}{R^2}\int\limits_{Q(z_0,R)} |v|^3\,dz, \qquad D(z_0,R) = \frac{1}{R^2}\int\limits_{Q(z_0,R)} |q|^{\frac{3}{2}}\,dz,$$

$$E(z_0,R) = \frac{1}{R}\int\limits_{Q(z_0,R)} |\nabla v|^2\,dz,$$

$$A(z_0,R) = \frac{1}{R}\sup_{t_0-R^2<t<t_0}\int\limits_{B(x_0,R)} |v(x,t)|^2\,dx.$$

Our main result is as follows.

Proposition 1.1. *Let v and q be a suitable weak solution to the Navier–Stokes equations in Q and let v satisfy condition* (1.4). *Then scaled energy quantities are bounded in the following sense*

$$\sup_{z_0\in\overline{Q}(1/2),\,0<r<1/4} A(z_0,R) + D(z_0,R) + E(z_0,r) + C(z_0,r) < \infty.$$

Proposition 1.1 has an interesting consequence. Assume that v is axially symmetric (with respect to x_3-axis) solution to the Navier Stokes equations:

$$v \in L_3(Q), \qquad q \in L_{\frac{3}{2}}(Q)$$

and

$$v \in L_\infty(-1,-a;B) \cap L_\infty(-1,0;\{r < |x| < 1\})$$

for any $a \in]0,1[$ and for some $r \in]0,1[$.

As it has been explained in [6], this is more or less a general situation, i.e., any problem of local regularity for suitable weak solutions can reduced to such a case.

Corollary 1.2. *Assume that v and q satisfy the above two conditions and let in addition v obeys condition* (1.4). *Then the origin $z = 0$ is a regular point of v, i.e., $v \in L_\infty(Q(\varrho))$ for some positive ϱ.*

The latter result has been established recently in [3] for the Cauchy problem. Our proof is a direct consequence of Proposition 1.1 and results and methods developed in [7] and [6], see also [5]. Indeed, since scaled energy quantities are bounded, we can argue as in [7] and [6] to obtain to the following scale-invariant point-wise estimate

$$\sqrt{x_1^2 + x_2^2}|v(x,t)| \leqslant C$$

for any $z \in \overline{Q}(1/4)$. The rest is exactly as in [6] and based on the application of a Liouville type theorem for axially symmetric solutions from [4]. This completes the proof of Corollary 1.2.

§2. Auxiliary estimates

2.1. The first estimate. Let $0 < R < \varrho < 1$ and φ be a suitable non-negative cut-off function. Then we have

$$\int\limits_{B(x_0,R)} |v|^2\,dx \leqslant \int\limits_{B(x_0,\varrho)} |v|^2 \varphi\,dx = \int\limits_{B(x_0,\varrho)} v \cdot \operatorname{div}(d - [d]_{x_0,\varrho})\varphi\,dx$$

and after integration by part

$$\int\limits_{B(x_0,R)} |v|^2\,dx \leqslant \int\limits_{B(x_0,\varrho)} |d - [d]_{x_0,\varrho}|\Big(|\nabla v| + \frac{1}{\varrho - R}|v|\Big)\,dx.$$

So, we find

$$\int\limits_{B(x_0,R)} |v|^2\,dx \leqslant c\varrho^{\frac{3}{2}}\Big(\int\limits_{B(x_0,\varrho)} \Big[|\nabla v|^2 + \frac{|v|^2}{(\varrho-R)^2}\Big]\,dx\Big)^{\frac{1}{2}}. \qquad (2.1)$$

2.2. The second estimate. A Rough version. By the known multiplicative inequality,

$$\int\limits_{B(x_0,R)} |v|^3\,dx \leqslant c\Big(\int\limits_{B(x_0,R)} |v|^2\,dx\Big)^{\frac{3}{4}}\Big(\int\limits_{B(x_0,R)} \Big[|\nabla v|^2 + \frac{|v|^2}{R^2}\Big]\,dx\Big)^{\frac{3}{4}}.$$

Then applying (2.1), we have

$$\int\limits_{B(x_0,R)} |v|^3\,dx \leqslant c\varrho^{\frac{9}{8}}\Big(\int\limits_{B(x_0,\varrho)} \Big[|\nabla v|^2 + \frac{|v|^2}{(\varrho-R)^2}\Big]\,dx\Big)^{\frac{9}{8}}. \qquad (2.2)$$

2.3. The second estimate. We have

$$\int\limits_{B(x_0,R)} |v|^3\,dx \leqslant \int\limits_{B(x_0,\varrho)} |v|^3\varphi\,dx = \int\limits_{B(x_0,\varrho)} v\cdot \mathrm{div}(d-[d]_{x_0,\varrho})|v|\varphi\,dx.$$

Then, for some $s\in]1,4/3[$, integration by parts gives the following estimates:

$$\int\limits_{B(x_0,R)} |v|^3\,dx \leqslant c\int\limits_{B(x_0,\varrho)} |d-[d]_{x_0,\varrho}|\Big(\frac{|v|^2}{\varrho-R}+|v||\nabla v|\Big)\,dx$$

$$\leqslant \frac{c}{\varrho-R}\Big(\int\limits_{B(x_0,\varrho)} |d-[d]_{x_0,\varrho}|^3\,dx\Big)^{\frac{1}{3}}\Big(\int\limits_{B(x_0,\varrho)} |v|^3\Big)^{\frac{2}{3}}$$

$$+c\Big(\int\limits_{B(x_0,\varrho)} |d-[d]_{x_0,\varrho}|^{s'}\,dx\Big)^{\frac{1}{s'}}\Big(\int\limits_{B(x_0,\varrho)} |v|^s|\nabla v|^s\Big)^{\frac{1}{s}}$$

$$\leqslant I_1 + I_2,$$

where $s' = s/(s-1)$,

$$I_1 = c\frac{\varrho}{\varrho-R}\Big(\int\limits_{B(x_0,\varrho)} |v|^3\Big)^{\frac{2}{3}},$$

and

$$I_2 = c(s)\varrho^{\frac{3}{s'}}\Big(\int\limits_{B(x_0,\varrho)} |v|^s|\nabla v|^s\Big)^{\frac{1}{s}}.$$

Since I_1 can be estimated with the help of (2.2), we focus on evaluation of I_2. By the choice of s, $2s/(2-s) < 6$ and thus, after application of Hölder inequality, one may use the multiplicative inequality

$$I_2 \leqslant c(s)\varrho^{\frac{3}{s'}}\Big(\int\limits_{B(x_0,\varrho)} |\nabla v|^2\,dx\Big)^{\frac{1}{2}}\Big(\int\limits_{B(x_0,\varrho)} |v|^{\frac{2s}{2-s}}\,dx\Big)^{\frac{2-s}{2s}}$$

$$\leqslant c(s)\varrho^{\frac{3}{s'}}\Big(\int\limits_{B(x_0,\varrho)} |\nabla v|^2\,dx\Big)^{\frac{1}{2}}\Big(\int\limits_{B(x_0,\varrho)} |v|^2\Big)^{\frac{1}{2}(1-\frac{3}{s'})}\Big(\int\limits_{B(x_0,\varrho)} \Big[|\nabla v|^2+\frac{|v|^2}{\varrho^2}\Big]\,dx\Big)^{\frac{1}{2}\frac{3}{s'}}.$$

Selecting $r \in]\varrho, 1[$ and making use of (2.1), we find

$$I_2 \leqslant c(s)\varrho^{\frac{3}{s'}}\Big(\int\limits_{B(x_0,\varrho)} |\nabla v|^2\,dx\Big)^{\frac{1}{2}} r^{\frac{3}{4}(1-\frac{3}{s'})}\Big(\int\limits_{B(x_0,r)} \Big[|\nabla v|^2 + \frac{|v|^2}{(r-\varrho)^2}\Big]\,dx\Big)^{\frac{1}{4}(1-\frac{3}{s'})}$$

$$\times\Big(\int\limits_{B(x_0,\varrho)} \Big[|\nabla v|^2 + \frac{|v|^2}{\varrho^2}\Big]\,dx\Big)^{\frac{1}{2}\frac{3}{s'}}.$$

To simplify the latter estimate, let us assume that $R < 1/2$ and put $\varrho = \frac{3}{2}R$, $r = 2R$. This gives the following:

$$I_2 \leqslant c(s)R^{\frac{3}{4}+\frac{3}{4s'}}\Big(\int\limits_{B(x_0,2R)} \Big[|\nabla v|^2 + \frac{|v|^2}{R^2}\Big]\,dx\Big)^{\frac{3}{4}+\frac{3}{4s'}}.$$

Now, by (2.1), we have the second estimate which reads

$$\int\limits_{B(x_0,R)} |v|^3\,dx \leqslant c(s)\Big\{R^{\frac{3}{4}}\Big(\int\limits_{B(x_0,2R)} \Big[|\nabla v|^2 + \frac{|v|^2}{R^2}\Big]\,dx\Big)^{\frac{3}{4}}$$

$$+R^{\frac{3}{4}+\frac{3}{4s'}}\Big(\int\limits_{B(x_0,2R)} \Big[|\nabla v|^2 + \frac{|v|^2}{R^2}\Big]\,dx\Big)^{\frac{3}{4}+\frac{3}{4s'}}\Big\}. \tag{2.3}$$

§3. Boundedness of scaled energy quantities

Fixing s so that $s' = 6$ and integrating inequality (2.3), we show

$$C(z_0,R) \leqslant c\Big\{E^{\frac{3}{4}}(z_0,2R)+A^{\frac{3}{4}}(z_0,2R)+E^{\frac{7}{8}}(z_0,2R)+A^{\frac{7}{8}}(z_0,2R)\Big\} \quad (3.1)$$

provided $Q(z_0,2R)\subset Q$.

So, it remains to add to (3.1) energy inequality estimate

$$A(x_0,R/2)+E(x_0,R/2)\leqslant c\Big[C^{\frac{2}{3}}(z_0,R)+C(z_0,R)+D(z_0,R)\Big] \quad (3.2)$$

and

$$D(z_0,r)\leqslant c\Big[\Big(\frac{r}{R}\Big)D(z_0,R)+\Big(\frac{R}{r}\Big)^2C(z_0,R)\Big] \quad (3.3)$$

for any $0<r<R$.

Let z_0 be any point of the closed set $\overline{Q}(1/2)$. Let $0<r<1/4$ be fixed and number $\vartheta\in]0,1/8[$ be defined later. Using Young inequality, we derive from (3.2) and (3.3)

$$\begin{aligned}\mathcal{E}(\vartheta r) &= A(z_0,\vartheta r)+E(z_0,\vartheta r)+D(z_0,\vartheta r)\\ &\leqslant c\Big[1+C(z_0,2\vartheta r)+D(z_0,2\vartheta r)\Big]+D(z_0,\vartheta r)\\ &\leqslant c\Big[\vartheta D(z_0,r)+1+a(\vartheta)(C(z_0,2\vartheta r)+C(z_0,r/2))\Big]\end{aligned}$$

with a positive constant a depending on ϑ. Now, we can use (2.3) and Young inequality to conclude that

$$\mathcal{E}(\vartheta r)\leqslant c\vartheta\mathcal{E}(\vartheta r)+a_1(\vartheta).$$

It remains to pick up ϑ so that $c\vartheta<1/2$. After usual iterations of the latter inequality and applications of known arguments, we arrive at the required estimate

$$\sup\{\mathcal{E}(z_0,r)+C(z_0,r):\, z_0\in\overline{Q}(1/2),\, 0<r<1/4\}<\infty.$$

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