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ESTIMATES OF DEVIATIONS FROM EXACT SOLUTION OF THE STOKES PROBLEM IN THE VORTICITY-VELOCITY-PRESSURE FORMULATION

Abstract. Vorticity-velocity-pressure formulation for the stationary Stokes problem in 2D is considered. We analyze the corresponding generalized formulation, establish sufficient conditions that guarantee existence of the generalized solution and deduce estimates of the difference between the exact solution (i.e., exact velocity, vorticity, and pressure) and an arbitrary approximating function (velocity, vorticity, pressure) that belongs to the corresponding functional class and satisfies the boundary conditions. For this purpose we use the method suggested in [10, 12], which is based on transformations of the integral identity that defines the corresponding generalized solution.

§1. Statement of the problem

We consider the stationary Stokes problem in a bounded one-connected domain $\Omega \in \mathbb{R}^2$: find a vector-valued function $u = (u_1, u_2)$ (velocity) and a scalar function p (pressure) satisfying the relations

$$-\Delta u + \nabla p = f \quad \text{in} \quad \Omega, \tag{1.1}$$

$$\operatorname{div} u = 0 \quad \text{in} \quad \Omega. \tag{1.2}$$

We assume that $f \in L^2(\Omega)$, Ω has a piecewise smooth boundary Γ, which consists of three nonintersecting parts Γ_0, Γ_τ, and Γ_ν. On the corresponding parts, the following boundary conditions are imposed (in particular, such type boundary conditions arise in problems modeling flow of a viscous incompressible fluid in a pipe, see, e.g., [3]):

$$u \cdot \tau = 0, \quad u \cdot \nu = 0 \quad \text{on} \quad \Gamma_0, \tag{1.3}$$

$$u \cdot \tau = 0, \quad p = p_0 \quad \text{on} \quad \Gamma_\tau, \tag{1.4}$$

$$u \cdot \nu = 0, \quad \omega = \omega_0 \quad \text{on} \quad \Gamma_\nu. \tag{1.5}$$

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In (1.3)–(1.5), ν and τ, denote the unit normal (outward) and tangential vectors to Γ, respectively, and $\omega = \operatorname{curl} u := u_{2,1} - u_{1,2}$. Here and later on partial derivatives are denoted by comma and subindex, i.e. $u_{t,k} := \frac{\partial u_t}{\partial x_k}$, $t, k = 1, 2$. We assume that $p_0 \in L_2(\Gamma_\tau)$, and $\omega_0 \in L_2(\Gamma_\nu)$. The sets Γ_0, Γ_τ, and Γ_ν may have several disconnected components, which structure we do not specify. However, throughout this paper it is assumed that

- either Γ_0 has internal points or Γ_τ has only one connected component;
- the angles between adjoining components are strictly lesser than π and greater than zero.

§2. Correctness of the problem

The Stokes system (1.3)–(1.5) can be represented in a different form (which is called the "vorticity-velocity-pressure formulation", see, e.g., [1]). Now the problem is to find a scalar valued function ω (vorticity), a vector-valued function u (velocity), and a scalar valued function p (pressure) satisfying in Ω the following relations:

$$\operatorname{curl}^* \omega + \nabla p = f, \tag{2.1}$$

$$\omega = \operatorname{curl} u, \tag{2.2}$$

$$\operatorname{div} u = 0. \tag{2.3}$$

Here $\operatorname{curl}^* \omega = (\omega_{,2}, -\omega_{,1})$ is the operator adjoint to curl. In order to introduce the corresponding generalized solution, we introduce the spaces

$$\mathbb{H} = \{w : \ w \in H(\Omega, \operatorname{div}) \cap H(\Omega, \operatorname{curl}), \ w \cdot \nu|_{\Gamma_0 \cup \Gamma_\nu} = 0, \ w \cdot \tau|_{\Gamma_0 \cup \Gamma_\tau} = 0\},$$

and

$$\overset{\circ}{\mathbb{H}} = \{w \in \mathbb{H} : \quad \operatorname{div} w = 0\},$$

where the boundary conditions are understood in the generalized sense and $H(\Omega, \operatorname{div})$ and $H(\Omega, \operatorname{curl})$ are subspaces of $L^2(\Omega, \mathbb{R}^2)$ that contain the vector valued functions with square summable divergence and rotor, respectively. $\mathbb{H}$ is a Hilbert space endowed with the product $\int\limits_\Omega (\operatorname{div} v \operatorname{div} w + \operatorname{curl} v \operatorname{curl} w)\, dx$. In view of (2.5) this product generates a norm $\|w\|_{\mathbb{H}} = \|\operatorname{curl} w\| + \|\operatorname{div} w\|$.

We multiply (2.1) by $w \in \mathbb{H}$, integrate by parts, and use the boundary conditions (1.3)–(1.5). Then, we arrive at the relation

$$\int_\Omega \omega \operatorname{curl} w \, dx = \int_\Omega [f \cdot w + p \operatorname{div} w] \, dx - \int_{\Gamma_\tau} p_0 w \cdot \nu \, d\Gamma + \int_{\Gamma_\nu} \omega_0 w \cdot \tau \, d\Gamma, \quad (2.4)$$

where $w \cdot \nu \in L_2(\Gamma_\tau)$ and $w \cdot \tau \in L_2(\Gamma_\nu)$ (see (2.17)). This integral identity suggests the following generalized formulation of the problem: find $u \in \overset{\circ}{\mathbb{H}}$, $p \in L_2(\Omega)$, and $\omega \in L_2(\Omega)$ such that $\omega = \operatorname{curl} u$, and (2.4) holds for all $w \in \mathbb{H}$.

Below we prove that it is correct in the sense that a generalized solution exists and is unique. The proof is based upon the estimate of L_2-norm of a function w from $\mathbb{H}$ through L_2-norms of its divergence and rotor. We can consider this inequality as a certain generalization of the Poincaré–Friedrichs inequality.

Theorem 2.1. There exists constant C_{PFr} depending only on Ω, Γ_0, Γ_ν, and Γ_τ such that for all $w \in \mathbb{H}$ the following inequality is valid

$$\int_\Omega |w|^2 \, dx \leqslant C_{PF_r} \left(\int_\Omega |\operatorname{curl} w|^2 \, dx + \int_\Omega |\operatorname{div} w|^2 \, dx \right). \quad (2.5)$$

Proof. If $\Gamma_0 = \Gamma$, then C_{PF_r} can be taken from Friedrichs inequality. Assume that $\Gamma_0 \neq \Gamma$. To justify the inequality for this case, we consider the following two boundary value problems:

$$\begin{cases} \Delta\varphi = \operatorname{div} w \qquad \text{in } \Omega \\ \varphi\big|_{\Gamma_0} = 0, \quad \varphi\big|_{\Gamma_\tau} = 0, \quad \dfrac{\partial\varphi}{\partial\nu}\Big|_{\Gamma_\nu} = 0, \end{cases} \quad (2.6)$$

$$\begin{cases} -\Delta\psi = \operatorname{curl} w \qquad \text{in } \Omega \\ \psi\big|_{\Gamma_\nu} = 0, \quad \dfrac{\partial\psi}{\partial\nu}\Big|_{\Gamma_0} = 0 \quad \dfrac{\partial\psi}{\partial\nu}\Big|_{\Gamma_\tau} = 0. \end{cases} \quad (2.7)$$

□

Generalized solutions of these two problems belong to $H^1(\Omega)$ and satisfy the energy inequalities

$$\|\nabla\varphi\| \leqslant C_\tau \|\operatorname{div} w\|, \quad (2.8)$$

$$\|\nabla\psi\| \leqslant C_\nu \| \operatorname{curl} w\|, \tag{2.9}$$

where C_τ and C_ν are the constants from Sobolev embedding theorem for functions vanishing near Γ_τ or Γ_ν respectively and $\|\cdot\|$ denotes the norm of $L_2(\Omega)$.

By means of φ and ψ, we construct the function $v = w - \nabla\varphi - \operatorname{curl}^* \psi$. Our main goal is to show that

$$\|v\| \leq C\left(\|\operatorname{div} w\| + \|\operatorname{curl} w\|\right). \tag{2.10}$$

Since

$$\|w\| \leq \|v\| + \|\nabla\varphi\| + \|\operatorname{curl}^* \psi\| = \|v\| + \|\nabla\varphi\| + \|\nabla\psi\|,$$

the estimates (2.8), (2.9), and (2.10) imply (2.5).

First of all we show that v belongs to a finite dimensional space. By the construction

$$\begin{cases} \operatorname{div} v = 0, \quad \operatorname{curl} v = 0, \\ v\cdot\tau\big|_{\Gamma_0} = 0, \quad v\cdot\tau\big|_{\Gamma_\tau} = 0, \quad v\cdot\nu\big|_{\Gamma_\nu} = 0. \end{cases} \tag{2.11}$$

Since $\operatorname{curl} v = 0$, there exists a function $g \in H^1(\Omega)$ such that $v = \nabla g$ and g satisfies (in the sense of distributions) the following boundary value problem with mixed Dirichlet–Neumann boundary conditions:

$$\begin{cases} -\Delta g = 0, \\ \dfrac{\partial g}{\partial\tau}\Big|_{\Gamma_0} = 0, \quad \dfrac{\partial g}{\partial\tau}\Big|_{\Gamma_\tau} = 0, \quad \dfrac{\partial g}{\partial\nu}\Big|_{\Gamma_\nu} = 0. \end{cases} \tag{2.12}$$

Our goal is to describe the set of nontrivial solutions of the above problem. Assume that $\operatorname{meas}(\Gamma_0 \cup \Gamma_\tau) > 0$. Any solution of (2.12) that satisfies the condition $\frac{\partial g}{\partial\tau}\big|_{\widetilde{\Gamma}} = 0$ on a certain measurable boundary part $\widetilde{\Gamma}$ satisfies the condition $g|_{\widetilde{\Gamma}} = \mathrm{const}$. Certainly, the constants may be different for disjoint parts of $\widetilde{\Gamma}$. Let the number of disjoint components of $\Gamma_0 \cup \Gamma_\tau$ be N. We denote the corresponding components by Γ_j, $j = 1, \ldots, N$. Then we can construct N different boundary value problems with Dirichle boundary conditions (i.e., for the problem i we set $g\big|_{\Gamma_j} = \delta_{ij}$, where δ_{ij} is the Kronecker symbol). Solutions of these problems we denote by $g^{(i)}$, $i = 1, \ldots, N$. Since the boundary conditions are linearly independent, the corresponding solutions functions $g^{(i)}$ are also lineary independent and any nontrivial solution of (2.12) is a linear combination of $g^{(i)}$.

It is clear that $\widetilde{g} = \sum_{k=1}^{N} g^{(k)}$ is a function, which from one hand satisfies (2.12) and, on the other hand, $\widetilde{g} = 1$ on $\Gamma_0 \cup \Gamma_\tau$. The function $\widetilde{g} \equiv 1$ is the only one that meets such conditions (indeed, if there would be another function $\widehat{g}$ satisfying the same conditions then $\widetilde{g} - \widehat{g}$ would be a harmonic function vanishing on a boundary part with nonzero measure, which is impossible). It is evident that the only solution is the function that identically equal to one. Therefore

$$\sum_{k=1}^{N} g^{(k)}(x) = 1, \quad \text{in} \quad \Omega$$

and, consequently,

$$\sum_{k=1}^{N} \nabla g^{(k)}(x) = 0, \quad \text{in} \quad \Omega. \tag{2.13}$$

All solutions of (2.11) are presented by $\nabla g^{(k)}$. Then (2.13) shows that the dimension of a solutions space of (2.11) is smaller or equal to $N-1$. However, the dimension can not be smaller then $N-1$. Indeed, otherwise we have

$$\sum_{k \in \widehat{N}} c_k \nabla g^{(k)}(x) = 0, \quad \text{in} \quad \Omega.$$

for some $|\widehat{N}| < N-1$ where not all c_k are equal to zero. Therefore,

$$\sum_{k \in \widehat{N}} c_k g^{(k)}(x) = c, \quad \text{in} \quad \Omega.$$

The constant c in the right-hand side of the equality cannot be equal to zero (because $g^{(k)}|_{\Gamma_k} = 1$ for $k = 1, \ldots, N$). On the other hand, it cannot be nonzero (because $g^{(k)}|_{\Gamma_l} = 0$ if $k \in \widehat{N}$, $l \notin \widehat{N}$).

It remains to consider the case $\Gamma = \Gamma_\tau$. In this case, (2.12) has only one nontrivial solution $g = 1$, which means that (2.11) has only trivial solutions.

We require from the sets Γ_0, and Γ_τ that either $|\Gamma_0| > 0$ or Γ_τ has only one connected component. In the second case, the space of solutions of (2.9) is empty.

In the first case v belongs to a finite dimensional space where all the norms are equivalent. In view of this fact, $\|\nabla g\|$ can be estimated throughout $\|\frac{\partial g}{\partial n}\|_{-\frac{1}{2},\Gamma_0}$. We recall that $v = w - \nabla\varphi - \operatorname{curl}^* \psi$, $v = \nabla g$, and $w \cdot \nu = 0$

on Γ_0, wherefrom we find that

$$\frac{\partial g}{\partial \nu}|_{\Gamma_0} = -\frac{\partial \varphi}{\partial \nu}|_{\Gamma_0} + \frac{\partial \psi}{\partial \tau}|_{\Gamma_0}. \tag{2.14}$$

Then

$$\begin{aligned}\left\|\frac{\partial g}{\partial n}\right\|_{-\frac{1}{2},\Gamma_0} &\leqslant \left\|\frac{\partial \varphi}{\partial \nu}\right\|_{-\frac{1}{2},\Gamma_0} + \left\|\frac{\partial \psi}{\partial \tau}\right\|_{-\frac{1}{2},\Gamma_0}\\ &\leqslant C_2(\|\nabla\varphi\| + \|\nabla\psi\|) \leqslant C_2(C_\tau + C_\mu)(\|\operatorname{div} w\| + \|\operatorname{curl} w\|),\end{aligned} \tag{2.15}$$

where C_2 is a constant from the trace embedding theorem ($H^{-1/2}(\Gamma_0) \subset L_2(\Omega)$) and in the last inequality we have used (2.8)–(2.9).

Remark 2.1. There exist a constant C_s such that

$$\|w\|^2_{\frac{1}{2}+s,\Omega} \leqslant C_s(\|\operatorname{curl} w\|^2 + \|\operatorname{div} w\|^2)$$

for all $w \in \mathbb{H}$ and $s < s_0$. Here s_0 is a constant, which depends on angles between adjoining smooth components of the boundary.

Proof. We use the decomposition of w into three parts

$$w = \nabla\varphi + \operatorname{curl}^*\psi + \nabla g.$$

The functions φ and ψ are subject to (2.6), (2.7), for which it is known that (see [7, 5])

$$\|\nabla\varphi\|_{\frac{1}{2}+s,\Omega} + \|\nabla\psi\|_{\frac{1}{2}+s,\Omega} \leqslant C(\|\operatorname{div} w\| + \|\operatorname{curl} w\|).$$

Since the function g (as we have proved in the last theorem) belongs to finite-dimension space, we take into account (2.14), rewrite (2.15), and obtain

$$\begin{aligned}c\|\nabla g\|_{\frac{1}{2}+s,\Omega} &\leqslant \left\|\frac{\partial g}{\partial n}\right\|_{s,\Gamma_0} \leqslant \left\|\frac{\partial \varphi}{\partial \nu}\right\|_{s,\Gamma_0} + \left\|\frac{\partial \psi}{\partial \tau}\right\|_{s,\Gamma_0}\\ &\leqslant C_3(\|\nabla\varphi\|_{\frac{1}{2}+s,\Omega} + \|\nabla\psi\|_{\frac{1}{2}+s,\Omega}) \leqslant C_3 C(\|\operatorname{div} w\| + \|\operatorname{curl} w\|),\end{aligned} \tag{2.16}$$

where C_3 is a constant in the trace embedding ($H^s(\Gamma_0) \subset H^{1/2+s}(\Omega)$). $\square$

Corollary 2.1. If $w \in \mathbb{H}$ then there exist trace w on Γ and $w|_\Gamma \in L_2(\Gamma)$

$$\|w\|^2_\Gamma \leqslant C_\Gamma(\|\operatorname{curl} w\|^2 + \|\operatorname{div} w\|^2). \tag{2.17}$$

For proving existence we first consider (2.4) (where $\omega = \operatorname{curl} u$ and $u \in \overset{\circ}{\mathbb{H}}$) on test functions $w \in \overset{\circ}{\mathbb{H}}$, which leads to the integral identity

$$\int\limits_{\Omega} \operatorname{curl} u \operatorname{curl} w \, dx = \int\limits_{\Omega} f \cdot w \, dx - \int\limits_{\Gamma_\tau} p_0 w \cdot \nu \, d\Gamma + \int\limits_{\Gamma_\nu} \omega_0 w \cdot \tau \, d\Gamma \quad \forall w \in \overset{\circ}{\mathbb{H}}. \quad (2.18)$$

We note that the left-hand side of (2.18) defines an inner product in $\overset{\circ}{\mathbb{H}}$ and the right-hand side of (2.18) is a continuous linear functional over $\overset{\circ}{\mathbb{H}}$. Therefore, there exists $u \in \overset{\circ}{\mathbb{H}}$ such that (2.18) holds for all $w \in \overset{\circ}{\mathbb{H}}$ and $\omega = \operatorname{curl} u$. It remains to determine the pressure.

In the case $|\Gamma_\tau| = 0$, we can take test-functions $w \in \overset{\circ}{\mathbb{H}}$ vanishing on the boundary of Ω. Then, (2.18) can be rewritten in the form

$$\int\limits_{\Omega} \operatorname{curl} u \operatorname{curl} w \, dx - \int\limits_{\Omega} f \cdot w \, dx = 0$$

which means that there exist $p \in L_2(\Omega)$ such that the equation $-\Delta u + \nabla p = f$ holds in the sense of distributions (see [6]), i.e.,

$$\int\limits_{\Omega} (\operatorname{curl} u \operatorname{curl} w - f \cdot w - p \operatorname{div} w) \, dx = 0 \quad \forall w \in H^1_0(\Omega, \mathbb{R}^2) \cap \mathbb{H}, \quad (2.19)$$

where H^1_0 denotes the subspace of H^1 containing the functions vanishing on Γ. Henceforth, we use the following Lemma ([6]).

Lemma 2.1. Let $g \in L_2(\Omega)$, $a \in H^{\frac{1}{2}}(\Gamma, \mathbb{R}^2)$, and the compatibility condition

$$\int\limits_{\Omega} g(x) \, dx = \int\limits_{\Gamma} a \cdot \nu \, ds$$

holds. Then there exists a vector valued function $w_g \in H^1(\Omega, \mathbb{R}^2)$ such that

$$\begin{cases} \operatorname{div} w_g = g & x \in \Omega, \\ w_g = a & x \in \partial\Omega, \\ \|\nabla w_g\|^2 \leqslant \kappa \left(\|g\|^2 + \|a\|^2_{\frac{1}{2},\Gamma} \right). \end{cases} \quad (2.20)$$

Remark 2.2. The constant inverse to κ is the constant C_{LBB} in the so-called Ladyzhenskaya–Babuška–Brezzi condition, which is often used in the analysis of boundary value problems for incompressible viscous fluids.

To obtain (2.4) from (2.18) and (2.19) we use Lemma 2.1, which shows that an arbitrary $w \in \mathbb{H}$ can be presented in the form $w = w_0 + w_1$, where

$$\begin{cases} \operatorname{div} w_0 = 0, \\ w_0\big|_{\partial\Omega} = w. \end{cases} \qquad \begin{cases} \operatorname{div} w_1 = \operatorname{div} w, \\ w_1\big|_{\partial\Omega} = 0. \end{cases}$$

Since $|\Gamma_\tau| = 0$ and $w \in \mathbb{H}$, we see that $w \cdot \nu = 0$ on Γ. Hence, the compatibility condition holds for both problems and, therefore, $w_0 \in \overset{\circ}{\mathbb{H}}$ and $w_1 \in H_0^1 \cap \mathbb{H}$ exists.

Combining (2.18) and (2.19) written for w_1 and w_0 we arrive at (2.4) for w.

In the case $|\Gamma_\tau| > 0$, we use Lemma 2.1 in the following form. We fix some continuous function $\varphi(x)$ such that $\varphi(x) \in [0, 1]$ in Ω, ϕ vanishes on Γ everywhere except Γ_τ, and the trace on Γ_τ belongs to $C_0^\infty(\Gamma_\tau)$. Then for arbitrary $q \in L_2(\Omega)$ we can find a function w_q such that

$$\operatorname{div} w_q = q \quad \text{in} \quad \Omega, \tag{2.21}$$

$$w_q = \nu c_q \, \varphi \quad \text{on} \quad \Gamma, \tag{2.22}$$

where

$$c_q = \left(\int_{\Gamma_\tau} \varphi \, d\Gamma \right)^{-1} \int_\Omega q \, dx,$$

so that the compatibility condition holds.

Consider the functional $G(q) : L_2 \to \mathbb{R}$ defined by the relation

$$G(q) = \int_\Omega \operatorname{curl} u \operatorname{curl} w_q \, dx - \int_\Omega f \cdot w_q \, dx + \int_{\Gamma_\tau} p_0 w_q \cdot \nu \, d\Gamma - \int_{\Gamma_\nu} \omega_0 w_q \cdot \tau \, d\Gamma.$$

We note that the problem (2.21)–(2.22) may have different solutions. If w_{q1} and w_{q2} are such solutions, then $w = w_{q1} - w_{q2} \in \overset{\circ}{\mathbb{H}}$ (indeed $\operatorname{div} w = 0$ and the boundary conditions holds). Hence, (2.18) is valid for w, which means that the values of $G(q)$ do not depend on the particular choice of w_q.

Evidently, $G(q)$ is a linear functional. In view of the estimates

$$|G(q)| \leqslant \|\nabla w_q\| \big(\|\operatorname{curl} u\| + \sqrt{C_{PFr}}\|f\| + \sqrt{C_\Gamma}(\|p_0\| + \|\omega_0\|)\big),$$

$$\|\nabla w_q\| \leqslant \sqrt{\kappa}(\|q\| + c_q\|\nu\varphi\|_{1/2,\Gamma}),$$

$$c_q \leqslant \|q\| \left(\int\limits_{\Gamma_\tau} \varphi \, d\Gamma \right)^{-1} |\Omega|^{1/2},$$

it is bounded. Hence, by Riesz theorem there exists (unique) $p \in L_2(\Omega)$ such that

$$G(q) = \int\limits_\Omega qp \, dx = \int\limits_\Omega \operatorname{div} w_q p \, dx.$$

In other words,

$$\int\limits_\Omega \operatorname{curl} u \operatorname{curl} w_q \, dx - \int\limits_\Omega f \cdot w_q \, dx + \int\limits_{\Gamma_\tau} p_0 w_q \cdot \nu \, d\Gamma - \int\limits_{\Gamma_\nu} \omega_0 w_q \cdot \tau \, d\Gamma = \int\limits_\Omega \operatorname{div} w_q p \, dx. \qquad (2.23)$$

Now, we end up the proof as follows. For an arbitrary $w \in \mathbb{H}$, we construct $w_{\operatorname{div} w}$ (which means that we take $q = \operatorname{div} w$ in (2.21)–(2.22)). Since $w_0 = w - w_{\operatorname{div} w} \in \overset{\circ}{\mathbb{H}}$ we can use it in (2.18). We combine (2.23) and (2.18) written for $w_{\operatorname{div} w}$ and w_0, respectively. Then, we arrive at (2.4) for $w \in \mathbb{H}$.

Let us deduce the energy estimates. We set in (2.4) $w = u$ and obtain

$$\int\limits_\Omega |\omega|^2 \, dx = \int\limits_\Omega f \cdot u \, dx - \int\limits_{\Gamma_\tau} p_0 u \cdot \nu \, d\Gamma + \int\limits_{\Gamma_\nu} \omega_0 u \cdot \tau \, d\Gamma.$$

Using Hölder inequality, Theorem 2.1 and (2.17) we arrive at

$$\|\omega\|^2 \leq 2[C_{PF_r}\|f\|^2 + C_\Gamma(\|p_0\|^2 + \|\omega_0\|^2)]. \qquad (2.24)$$

Since $\operatorname{curl} u = \omega$, $\operatorname{div} u = 0$, we use Theorem 2.1 and obtain

$$\|u\|^2 \leq C_{PFr}\|\omega\|^2 \leqslant 2C_{PF_r}[C_{PF_r}\|f\|^2 + C_\Gamma(\|p_0\|^2 + \|\omega_0\|^2)]. \qquad (2.25)$$

Estimates for the pressure can be derived with the help of Lemma 2.1. We note that if $|\Gamma_\tau| \neq 0$, then $L_2(\Omega)$ should be selected as the space for pressure. If $|\Gamma_\tau| = 0$, $\overset{\circ}{L}_2 = \{f \in L_2 : \int\limits_\Omega f(x)\,dx = 0\}$ is the proper space.

First, we deduce the estimate for the function

$$\widetilde{p} = p - \frac{1}{|\Omega|}\int\limits_\Omega p(x)\,dx \in \overset{\circ}{L}_2\,.$$

By Lemma 2.1, we can find $w_{\widetilde{p}} \in H^1_0(\Omega, \mathbb{R}^2)$ such that $\operatorname{div} w_{\widetilde{p}} = \widetilde{p}$ and $\|\nabla w_{\widetilde{p}}\|^2 \leqslant \kappa\,\|\widetilde{p}\|^2$. Plugging $w_{\widetilde{p}}$ into (2.4), we obtain

$$\int\limits_\Omega \widetilde{p}^2\,dx = \int\limits_\Omega \omega \operatorname{curl} w_{\widetilde{p}}\,dx - \int\limits_\Omega f\cdot w_{\widetilde{p}}\,dx$$

and, therefore,

$$\begin{aligned}\int\limits_\Omega \widetilde{p}^2\,dx &\leqslant \|w\|\|\operatorname{curl} w_{\widetilde{p}}\| + \|f\|\|w_{\widetilde{p}}\| \\ &\leqslant (\|\omega\| + \sqrt{C_{PFr}}\|f\|)\|\operatorname{curl} w_{\widetilde{p}}\| \\ &\leqslant (\|\omega\| + \sqrt{C_{PFr}}\|f\|)\sqrt{\kappa}\|\widetilde{p}\|. \end{aligned} \tag{2.26}$$

We divide both parts by $\|\widetilde{p}\|$, use (2.24), and arrive at the estimate

$$\|\widetilde{p}\|^2 \leqslant 3\kappa\big(C_{PF_r}\|f\|^2 + C_\Gamma(\|p_0\|^2 + \|\omega_0\|^2)\big) \tag{2.27}$$

Uniqueness of the solutions to the Stokes problem in the vorticity-velocity-pressure formulation is a consequence of (2.24), (2.25), and (2.27). Indeed, if one takes $f = 0$, $p_0 = 0$, $\omega_0 = 0$, then by these estimates we conclude that $u = 0$, $\omega = 0$, and $p = \overline{p}$ where $\overline{p} = \frac{1}{|\Omega|}\int\limits_\Omega p(x)\,dx$.

Therefore, (2.4) implies

$$0 = \int\limits_\Omega \overline{p}\operatorname{div} w\,dx = \overline{p}\int\limits_{\Gamma_\tau} w\cdot\nu\,d\Gamma.$$

Taking $w \in \mathbb{H}$ such that $\int\limits_{\Gamma_\tau} w\cdot\nu\,d\Gamma \neq 0$, we find that $\overline{p} = 0$.

Assume that $|\Gamma_\tau| \neq 0$. Then, (2.27) can be improved in a sense that $\widetilde{p}$ can be replaced by p.

We add the following integration by parts formula to the right-hand side of (2.4)

$$-c\int_\Omega \operatorname{div} w\,dx = -c\int_\Gamma w\cdot\nu\,d\Gamma,$$

and find that

$$\int_\Omega \omega \operatorname{curl} w\,dx = \int_\Omega [f\cdot w + (p-c)\operatorname{div} w]\,dx - \int_{\Gamma_\tau}(p_0-c)w\cdot\nu\,d\Gamma + \int_{\Gamma_\nu}\omega_0 w\cdot\tau\,d\Gamma. \quad (2.28)$$

Now, we apply Lemma 2.1 to the following problem

$$\operatorname{div} w_p = p - c \qquad \text{in} \quad \Omega, \quad (2.29)$$

$$w_p = \nu c\,\varphi \qquad \text{on} \quad \Gamma, \quad (2.30)$$

where $\varphi(x)$ is a continuous function such that $1 \geqslant \varphi(x) \geqslant 0$ in Ω, vanishes on Γ everywhere except Γ_τ, and the trace on Γ_τ belongs to $C_0^\infty(\Gamma_\tau)$. If

$$c = (|\Omega| + \int_{\Gamma_\tau}\varphi\,d\Gamma)^{-1}\int_\Omega p\,dx,$$

then the compatibility condition holds. By Lemma 2.1, we know that

$$\|\nabla w_p\|^2 \leqslant \kappa\,(\|p-c\|^2 + c^2\|\nu\varphi\|^2_{\frac{1}{2},\Gamma}). \quad (2.31)$$

Plugging w_p into (2.28), we find that

$$\int_\Omega (p-c)^2\,dx - \int_{\Gamma_\tau}(p_0-c)c\varphi\,d\Gamma = \int_\Omega(\omega\operatorname{curl} w_p - f\cdot w_p)\,dx.$$

Therefore,

$$\|p-c\|^2 + c^2\int_{\Gamma_\tau}\varphi\,d\Gamma \leqslant \|\omega\|\|\operatorname{curl} w_p\| + \|f\|\|w_p\| + c\int_{\Gamma_\tau}p_0\varphi\,d\Gamma,$$

and using Poincaré–Friedrichs inequality (2.5) and Hölder inequality, we arrive at the relation

$$\|p-c\|^2 + c^2 \int\limits_{\Gamma_\tau} \varphi \, d\Gamma \leqslant$$
$$\leqslant (\|\omega\| + \sqrt{C_{PFr}}\|f\|)\sqrt{\kappa}\,(\|p-c\| + c\|\nu\varphi\|_{1/2,\Gamma_\tau}) + c\|p_0\||\Gamma_\tau|^{\frac{1}{2}}, \quad (2.32)$$

where we have estimated $\|\operatorname{curl} w_p\|$ with the help of (2.31).

If we select φ such that $\int\limits_{\Gamma_\tau} \varphi \, d\Gamma > \frac{1}{2}|\Gamma_\tau|$, then, dividing by the square root of the left-hand side of (2.32), we obtain

$$\left(\|p-c\|^2 + c^2 \int\limits_{\Gamma_\tau} \varphi \, d\Gamma\right)^{1/2} \quad (2.33)$$
$$\leqslant (\|\omega\| + \sqrt{C_{PFr}}\|f\|)\sqrt{\kappa}\left(1 + \left(\int\limits_{\Gamma_\tau} \varphi \, d\Gamma\right)^{-1/2} \|\nu\varphi\|_{1/2,\Gamma_\tau}\right) + \sqrt{2}\|p_0\|.$$

Note that

$$\|p\|^2 \leqslant \|p-c\|^2 + c^2|\Omega| \leqslant \left(1 + 2|\Omega||\Gamma_\tau|^{-1}\right)\left(\|p-c\|^2 + c^2 \int\limits_{\Gamma_\tau} \varphi \, d\Gamma\right).$$

These two inequalities together with (2.24) lead to

$$\|p\|^2 \leqslant c(\Omega, \Gamma_\tau, \phi)\big(C_{PFr}\|f\|^2 + C_\Gamma(\|\omega_0\|^2 + \|p_0\|^2)\big), \quad (2.34)$$

where

$$c(\Omega, \Gamma_\tau, \varphi) = 4(1 + 2|\Omega||\Gamma_\tau|^{-1})\kappa\,(1 + |\Gamma_\tau|^{-1}\|\nu\varphi\|^2_{\frac{1}{2},\Gamma_\tau}). \quad (2.35)$$

§3. Estimates of deviations from the exact solution for divergence free functions

Let $v \in \overset{\circ}{\mathbb{H}}$, be an approximation of u, $\widetilde{\omega} = \operatorname{curl} v$ and q be an approximation of p satisfying the prescribed boundary conditions. Estimates of the difference between v and u can be derived with the help of special type transformations of the integral identity (2.4). We note that this method has been earlier applied to the classical statement of the Stokes problem in [11], generalized Stokes problem in [13], evolutionary Stokes problem in

[9], and some classes of nonlinear models in the theory of viscous incompressible fluids in [4].

We insert the function v into the integral identity (2.4) and exploit two integration by parts relations (where $q \in H^1(\Omega)$ and $\widehat{\omega} \in H^1(\Omega, \mathbb{R}^2)$)

$$\int_\Omega \widehat{\omega} \operatorname{curl} w \, dx = \int_\Omega w \cdot \operatorname{curl}^* \widehat{\omega} \, dx + \int_\Gamma \widehat{\omega} w \cdot \tau \, d\Gamma, \quad \forall\, w \in H(\Omega, \operatorname{curl}),$$

$$\int_\Omega \nabla q \cdot w \, dx = -\int_\Omega q \operatorname{div} w \, dx + \int_\Gamma q w \cdot \nu \, d\Gamma, \quad \forall\, w \in H(\Omega, \operatorname{div}),$$

which are naturally linked with the differential operators involved in the problem statement.

We obtain

$$\begin{aligned}\int_\Omega (\omega - \widetilde{\omega}) \operatorname{curl} w \, dx &= \int_\Omega \big((f - \nabla q - \operatorname{curl}^* \widehat{\omega}) \cdot w + (p - q - c) \operatorname{div} w - (\widetilde{\omega} - \widehat{\omega}) \operatorname{curl} w \big) \, dx \\ &\quad - \int_\Gamma [(p - q - c) w \cdot \nu - (\omega - \widehat{\omega}) w \cdot \tau] \, d\Gamma. \end{aligned} \tag{3.1}$$

We set in (3.1) $c = 0$, $w = u - v$, and take into account that

$$\operatorname{curl}(u - v) = \omega - \widetilde{\omega}, \qquad \operatorname{div}(u - v) = 0.$$

We have

$$\int_\Omega |\omega - \widetilde{\omega}|^2 \, dx = \int_\Omega \big((f - \nabla q - \operatorname{curl}^* \widehat{\omega}) \cdot (u - v) - (\widetilde{\omega} - \widehat{\omega})(\omega - \widetilde{\omega}) \big) \, dx.$$

We apply Theorem 2.1 for the function $u - v$, which yields

$$\|u - v\| \le C_{PFr} \|\omega - \widetilde{\omega}\|.$$

Hence,

$$\|\omega - \widetilde{\omega}\| \le M[f, q, \widehat{\omega}, \widetilde{\omega}], \tag{3.2}$$

where

$$M[f, q, \widehat{\omega}, \widetilde{\omega}] = \|\widetilde{\omega} - \widehat{\omega}\| + C_{PFr} \|f - \nabla q - \operatorname{curl}^* \widehat{\omega}\|,$$

which is valid for any $q, \widehat{\omega} \in H^1$, and v satisfying (1.2).

Now we deduce estimates of the deviation from p. For this purpose, we use Lemma 2.1 with w_{pq} such that $\operatorname{div} w_{pq} = p - q - c$, where

$$c = \left(|\Omega| + \int\limits_{\Gamma_\tau} \varphi \, d\Gamma \right)^{-1} \int\limits_{\Omega} (p-q) dx.$$

We use such a function in (3.1) and obtain

$$\int\limits_{\Omega} (\omega - \widetilde{\omega}) \operatorname{curl} w_{pq} \, dx = \int\limits_{\Omega} \big[(f - \nabla q - \operatorname{curl}^* \widehat{\omega}) \cdot w_{pq} \\ - (\widetilde{\omega} - \widehat{\omega}) \operatorname{curl} w_{pq} \big] \, dx + \|p - q - c\|^2 + c^2 \int\limits_{\Gamma_\tau} \varphi \, d\Gamma. \quad (3.3)$$

Therefore,

$$\|p - q - c\|^2 + c^2 \int\limits_{\Gamma_\tau} \varphi \, d\Gamma \\ \leqslant \left(\sqrt{C_{PFr}} \|f - \nabla q - \operatorname{curl}^* \widehat{\omega}\| + \|\omega - \widetilde{\omega}\| + \|\widetilde{\omega} - \widehat{\omega}\| \right) \| \operatorname{curl} w_{pq}\|. \quad (3.4)$$

Then, we use Lemma 2.1 to estimate $\| \operatorname{curl} w_{pq}\|$ as follows

$$\| \operatorname{curl} w_{pq}\|^2 \; \leq \; \kappa \left(\|p - q - c\|^2 + c^2 \|\nu\varphi\|^2_{1/2,\Gamma} \right).$$

By (3.2) we estimate $\|\omega - \widetilde{\omega}\|$ and find that

$$\left(\|p - q - c\|^2 + \frac{1}{2} c^2 |\Gamma_\tau| \right)^{1/2} \; \leq \; \left(\|p - q - c\|^2 + c^2 \int\limits_{\Gamma_\tau} \varphi \, d\Gamma \right)^{1/2} \\ \leqslant \left(2\sqrt{C_{PFr}} \|f - \nabla q - \operatorname{curl}^* \widehat{\omega}\| + 2\|\widetilde{\omega} - \widehat{\omega}\| \right) \\ \times \sqrt{\kappa} \left(1 + \left(\int\limits_{\Gamma_\tau} \varphi \, d\Gamma \right)^{-1/2} \|\nu\varphi\|_{1/2,\Gamma} \right). \quad (3.5)$$

Hence,

$$\|p - q - c\| \leq 2M[f, q, \widehat{\omega}, \widetilde{\omega}] \sqrt{\kappa} (1 + |\Gamma_\tau|^{-1} \|\nu\varphi\|_{1/2,\Gamma}),$$

$$|c| \; \leq 2\sqrt{2} |\Gamma_\tau|^{-1/2} M[f, q, \widehat{\omega}, \widetilde{\omega}] \sqrt{\kappa} (1 + |\Gamma_\tau|^{-1} \|\nu\varphi\|_{1/2,\Gamma}).$$

Since $\|p-q\| \le \|p-q-c\| + c|\Omega|$, we finally arrive at the estimate

$$\|p-q\| \leqslant c(\Omega, \Gamma_\tau, \varphi) M[f, q, \widehat{\omega}, \widetilde{\omega}], \tag{3.6}$$

where $c(\Omega, \Gamma_\tau, \varphi)$ determined by (2.35).

Finally, we note that the deviation estimates discussed in this section can be extended to non solenoidal functions. Let $\overline{v} \in \mathbb{H}$, be an approximation of u, and $\overline{\omega} = \operatorname{curl} \overline{v}$, $w = u - \overline{v}$ and q be the same as before. It is easy to see that

$$\begin{aligned} \|\omega - \overline{\omega}\| &\leqslant \|\omega - \operatorname{curl} v\| + \|\operatorname{curl} v - \operatorname{curl} \overline{v}\| \\ &\le M[f, q, \widehat{\omega}, \operatorname{curl} v] + \|v - \widehat{v}\| + \|\operatorname{curl} v - \operatorname{curl} \overline{v}\|, \end{aligned}$$

where $\overline{v}$ is an arbitrary vector valued function in $\overset{\circ}{\mathbb{H}}$. Since

$$\|\operatorname{curl} v - \widehat{\omega}\| \le \|\operatorname{curl} \overline{v} - \widehat{\omega}\| + \|\operatorname{curl} v - \operatorname{curl} \overline{v}\|,$$

we find that

$$\|\omega - \overline{\omega}\| \leqslant M[f, q, \widehat{\omega}, \operatorname{curl} v] + 2\|\operatorname{curl}(v - \overline{v})\|. \tag{3.7}$$

It is clear that the last term is estimated from above by $\|\nabla(v - \overline{v})\|$, which leads to the estimate

$$\|\omega - \overline{\omega}\| \leqslant M[f, q, \widehat{\omega}, \operatorname{curl} v] + 2\kappa\|\operatorname{div} \overline{v}\|. \tag{3.8}$$

Now the estimate of $\|u - \widehat{v}\|$ readily follows from (2.5). Estimates for the pressure can be deduced with the help of the same method that has been used for solenoidal fields.

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