Записки научных
семинаров ПОМИ
Том 397, 2011 г.

V. Vyalov

ON THE LOCAL SMOOTHNESS OF WEAK SOLUTIONS TO THE MHD SYSTEM NEAR THE BOUNDARY

Abstract. We establish conditions sufficient for local regularity of the siutable weak solutions to the MHD system near the plane part of the boundary.

§1. Introduction

Assume that $\Omega \subset \mathbb{R}^3$ is a C^2 – smooth bounded domain and $Q_T = \Omega \times (0,T)$. In this paper, we investigate e boundary regularity of solutions to the system of magnetohydrodynamics (the MHD equations):

$$\left.\begin{array}{c} \partial_t v + (v\cdot\nabla)v - \Delta v + \nabla p = \operatorname{rot} H \times H \\ \operatorname{div} v = 0 \end{array}\right\} \quad \text{in } Q_T, \tag{1}$$

$$\left.\begin{array}{c} \partial_t H + \operatorname{rot}\operatorname{rot} H = \operatorname{rot}(v\times H) \\ \operatorname{div} H = 0 \end{array}\right\} \quad \text{in } Q_T. \tag{2}$$

Here, unknowns are the velocity field $v : Q_T \to \mathbb{R}^3$, pressure $p : Q_T \to \mathbb{R}$, and the magnetic field $H : Q_T \to \mathbb{R}^3$. We impose the boundary conditions:

$$v|_{\partial\Omega\times(0,T)} = 0, \quad H_\nu|_{\partial\Omega\times(0,T)} = 0, \quad (\operatorname{rot} H)_\tau|_{\partial\Omega\times(0,T)} = 0, \tag{3}$$

where by ν is the outword unit normal to $\partial\Omega$, $H_\nu = H\cdot\nu$, and $(\operatorname{rot} H)_\tau = \operatorname{rot} H - \nu(\operatorname{rot} H\cdot\nu)$. We introduce the following definition:

Definition. *Assume $\Gamma \subset \partial\Omega$. The functions (v,H,p) are called a boundary suitable weak solution to the system* (1), (2) *near $\Gamma_T \equiv \Gamma\times(0,T)$ if*

1) $v \in L_{2,\infty}(Q_T) \cap W^{1,0}_2(Q_T) \cap W^{2,1}_{\frac98,\frac32}(Q_T)$,
   $H \in L_{2,\infty}(Q_T) \cap W^{1,0}_2(Q_T)$,
2) $p \in L_{\frac32}(Q_T) \cap W^{1,0}_{\frac98,\frac32}(Q_T)$,

---

Key words and phrases: MHD equations, partial regularity, weak solutions, boundary regularity.

This research is supported by the Chebyshev Laboratory (Department of Mathematics and Mechanics, St.Petersburg State University) under RF government grant 11.G34.31.0026.

3) $\operatorname{div} v = 0$, $\operatorname{div} H = 0$ *a.e. in* Q_T,
4) $v|_{\partial\Omega} = 0$, $H_\nu|_{\partial\Omega} = 0$ *in the sense of traces,*
5) *for any* $w \in L_2(\Omega)$, *the functions*

$$t \mapsto \int_\Omega v(x,t) \cdot w(x)\, dx \quad \text{and} \quad t \mapsto \int_\Omega H(x,t) \cdot w(x)\, dx$$

are continuous,
6) (v, H) *satisfy the following integral identities: for any* $t \in [0,T]$

$$\int_\Omega v(x,t) \cdot \eta(x,t)\, dx - \int_\Omega v_0(x) \cdot \eta(x,0)\, dx$$
$$+ \int_0^t \int_\Omega \Big(- v \cdot \partial_t \eta + (\nabla v - v \otimes v + H \otimes H) :$$
$$\nabla \eta - (p + \frac{1}{2}|H|^2) \operatorname{div} \eta \Big)\, dx\, dt = 0,$$

for all $\eta \in W^{1,1}_{\frac{5}{2}}(Q_t)$ *such that* $\eta|_{\partial\Omega \times (0,t)} = 0$,

$$\int_\Omega H(x,t) \cdot \psi(x,t)\, dx - \int_\Omega H_0(x) \cdot \psi(x,0)\, dx$$
$$+ \int_0^t \int_\Omega \Big(- H \cdot \partial_t \psi + \operatorname{rot} H \cdot \operatorname{rot} \psi - (v \times H) \cdot \operatorname{rot} \psi \Big)\, dx\, dt = 0,$$

for all $\psi \in W^{1,1}_{\frac{5}{2}}(Q_t)$ *such that* $\psi_\nu|_{\partial\Omega \times (0,t)} = 0$.
7) *For every* $z_0 = (x_0, t_0) \in \Gamma_T$ *such that*

$$\Omega_R(x_0) \times (t_0 - R^2, t_0) \subset Q_T$$

and for any

$$\zeta \in C_0^\infty(B_R(x_0) \times (t_0 - R^2, t_0])$$

such that $\frac{\partial \zeta}{\partial \nu}\Big|_{\partial\Omega} = 0$, *the following "local energy inequality near* Γ_T*" holds:*

$$
\begin{aligned}
&\sup_{t\in(t_0-R^2,t_0)} \int\limits_{\Omega_R(x_0)} \zeta\Big(|v|^2+|H|^2\Big)\,dx \\
&+2\int\limits_{t_0-R^2}^{t_0}\int\limits_{\Omega_R(x_0)} \zeta\Big(|\nabla v|^2+|\operatorname{rot} H|^2\Big)\,dx\,dt \\
&\leqslant \int\limits_{t_0-R^2}^{t_0}\int\limits_{\Omega_R(x_0)} \Big(|v|^2+|H|^2\Big)(\partial_t\zeta+\Delta\zeta)\ dxdt \\
&+\int\limits_{t_0-R^2}^{t_0}\int\limits_{\Omega_R(x_0)} \Big(|v|^2+2\bar{p}\Big)v\cdot\nabla\zeta\,dx\,dt \\
&-2\int\limits_{t_0-R^2}^{t_0}\int\limits_{\Omega_R(x_0)} (H\otimes H):\nabla^2\zeta\,dx\,dt \\
&+2\int\limits_{t_0-R^2}^{t_0}\int\limits_{\Omega_R(x_0)} (v\times H)(\nabla\zeta\times H)\,dx\,dt.
\end{aligned}
\tag{4}
$$

Also, we note that the following identity holds

$$
\int\limits_{t_0-R^2}^{t_0}\int\limits_{\Omega_R(x_0)} (v\times H)(\nabla\zeta\times H)\,dx\,dt
$$

$$
=\int\limits_{t_0-R^2}^{t_0}\int\limits_{\Omega_R(x_0)} (v\cdot\nabla\zeta)|H|^2\,dx\,dt\ -\int\limits_{t_0-R^2}^{t_0}\int\limits_{\Omega_R(x_0)} (v\cdot H)(H\cdot\nabla\zeta)\,dx\,dt.
$$

Here, $L_{s,l}(Q_T)$ is the anisotropic Lebesgue space equipped with the norm

$$
\|f\|_{L_{s,l}(Q_T)}:=\Big(\int\limits_0^T\Big(\int\limits_\Omega |f(x,t)|^s\,dx\Big)^{l/s}dt\Big)^{1/l}.
$$

Henceforth, we use the following notation for the functional spaces:

$$W^{1,0}_{s,l}(Q_T) \equiv L_l(0,T; W^1_s(\Omega)) = \{u \in L_{s,l}(Q_T):\ \nabla u \in L_{s,l}(Q_T)\},$$
$$W^{2,1}_{s,l}(Q_T) = \{u \in W^{1,0}_{s,l}(Q_T):\ \nabla^2 u,\ \partial_t u \in L_{s,l}(Q_T)\},$$
$$\overset{\circ}{W}{}^1_s(\Omega) = \{u \in W^1_s(\Omega):\ u|_{\partial\Omega} = 0\},$$

and the following notation for the norms:

$$\|u\|_{W^{1,0}_{s,l}(Q_T)} = \|u\|_{L_{s,l}(Q_T)} + \|\nabla u\|_{L_{s,l}(Q_T)},$$
$$\|u\|_{W^{2,1}_{s,l}(Q_T)} = \|u\|_{W^{1,0}_{s,l}(Q_T)} + \|\nabla^2 u\|_{L_{s,l}(Q_T)} + \|\partial_t u\|_{L_{s,l}(Q_T)}.$$

Denote $B(x_0,r)$ the open ball in $\mathbb{R}^3$ of radius r centered at x_0 and denote by $B^+(x_0,r)$ the half–ball $\{x \in B(x_0,r) \mid x_3 > 0\}$. For $z_0 = (x_0,t_0)$ denote $Q(z_0,r) = B(x_0,r) \times (t_0 - r^2, t_0)$, $Q^+(z_0,r) = B^+(x_0,r) \times (t_0 - r^2, t_0)$. In this paper we shall use the abbreviations: $B(r) = B(0,r)$, $B^+(r) = B^+(0,r)$ etc, $B = B(1)$, $B^+ = B^+(1)$ etc.

§2. Main results

Our work deals with the criteria of local regularity of suitable weak solutions to the MHD system near the plane part of the boundary. In [9] the following results were obtained.

Theorem 2.1. *There exists an absolute constant $\varepsilon_* > 0$ with the following property. Assume (v,H,p) is a boundary suitable weak solution in Q_T and assume $z_0 = (x_0,t_0) \in \partial\Omega \times (0,T)$ is such that x_0 belongs to the plane part of $\partial\Omega$. If there exists $r_0 > 0$ such that $Q^+(z_0,r_0) \subset Q_T$ and*

$$\frac{1}{r_0^2} \int\limits_{Q^+(z_0,r_0)} \Big(|v|^3 + |H|^3 + |p|^{\frac{3}{2}} \Big) dxdt \ < \ \varepsilon_*,$$

then the functions v and H are Hölder continuous on $\bar{Q}^+(z_0, \frac{r_0}{2})$.

Theorem 2.2. *For any $K > 0$ there exists $\varepsilon_0(K) > 0$ with the following property. Assume (v,H,p) is a boundary suitable weak solution in Q_T and assume $z_0 = (x_0,t_0) \in \partial\Omega \times (0,T)$ is such that x_0 belongs to the plane part of $\partial\Omega$. If*

$$\limsup_{r\to 0} \Big(\frac{1}{r} \int\limits_{Q(z_0,r)} |\nabla H|^2 \, dx\, dt \Big)^{1/2} \ < \ K \tag{5}$$

and

$$\limsup_{r\to 0}\Big(\ \frac{1}{r}\int\limits_{Q(z_0,r)}|\nabla v|^2\,dx\,dt\ \Big)^{1/2}\ <\ \varepsilon_0, \tag{6}$$

then there exists $\rho_*>0$ such that the functions v and H are Hölder continuous on the closure of $Q^+(z_0,\rho_*)$.

Let us comment on the main differences between these theorems. The statement of the theorem 2.1 contains smallness conditions on the three functionals, but these conditions have to hold only for one value of cylinder radius. In Theorem 2.2, we have conditions for all sufficiently small values of radius, but smallness condition (6) is imposed only on velocity v.

To describe more conditions of local regularity we need the following notations

$$\begin{aligned}
E(r)&=\Big(\ \frac{1}{r}\int\limits_{Q^+(r)}|\nabla v|^2\,dx\,dt\ \Big)^{1/2},\\
E_*(r)&=\Big(\frac{1}{r}\int\limits_{Q^+(r)}|\nabla H|^2\,dx\,dt\Big)^{1/2},\\
A(r)&\equiv\Big(\frac{1}{r}\sup_{t\in(-r^2,0)}\int\limits_{B^+(r)}|v|^2\,dy\Big)^{1/2},\\
A_*(r)&\equiv\Big(\frac{1}{r}\sup_{t\in(-r^2,0)}\int\limits_{B^+(r)}|H|^2\,dy\Big)^{1/2},\\
C_q(r)&\equiv\Big(\frac{1}{r^{5-q}}\int\limits_{Q^+(r)}|v|^q\,dy\,dt\Big)^{1/q},\\
F_q(r)&=\Big(\frac{1}{r^{5-q}}\int\limits_{Q^+(r)}|H|^q\,dx\,dt\Big)^{1/q}\\
D(r)&\equiv\Big(\frac{1}{r^2}\int\limits_{Q^+(r)}|p-[p]_{B^+(r)}|^{3/2}\,dy\,dt\Big)^{2/3},\\
D_s(r)&=R^{\frac{5}{3}-\frac{3}{s}}\Big(\int\limits_{-r^2}^{0}\Big(\int\limits_{B^+(r)}|\nabla p|^s\ dy\Big)^{\frac{1}{s}\cdot\frac{3}{2}}\,dt\Big)^{2/3},
\end{aligned} \tag{7}$$

$$C(r) = C_3(r), \quad F(r) = F_3(r), \quad D_*(r) = D_{\frac{36}{35}}(r).$$

Note that in the equations (1) (2), these functionals and statements of the previous theorems are invariant under the scaling transformations

$$\left.\begin{aligned} v_\rho(y,s) &= \rho v(\rho y + x_0, \rho^2 s + t_0), \\ H_\rho(y,s) &= \rho H(\rho y + x_0, \rho^2 s + t_0), \\ p_\rho(y,s) &= \rho^2 p(\rho y + x_0, \rho^2 s + t_0). \end{aligned}\right\} \tag{8}$$

We use the approach wich was originally developed in [4] for the Navier–Stokes equations (and later it was used also in [2]). According to this approach the regularity of solutions follows if one of the functionals (7) is bounded uniformly with respect to r and additionally one of these functionals is small only for a single sufficiently small value of the radius. Our goal is to obtain the same result for the solutions to the MHD system.

The main result of our work is the following theorem, that is "interpolation" of Theorems 2.1 and 2.2.

Theorem 2.3. *For arbitrary $K > 0$ there exists a constant $\varepsilon_1(K) > 0$ with the following property: let (v, H, p) be a suitable weak solution to the MHD system in Q_T and $z_0 = (x_0, t_0) \in \partial\Omega \times (0, T)$, where x_0 belongs to the plane part of $\partial\Omega$. If*

$$\limsup_{r\to 0} \Big(\frac{1}{r^2} \int\limits_{Q^+(z_0,r)} |v|^3 \, dxdt \Big)^{1/3} + \Big(\frac{1}{r^3} \int\limits_{Q^+(z_0,r)} |H|^2 \, dxdt \Big)^{1/2} < K \tag{9}$$

and one of the three following conditions holds

$$\begin{gathered} \liminf_{r\to 0} \Big(\frac{1}{r} \int\limits_{Q^+(z_0,r)} |\nabla v|^2 \, dx\, dt \Big)^{1/2} < \varepsilon_1, \\ \liminf_{r\to 0} \Big(\frac{1}{r} \sup_{-r^2<t<0} \int\limits_{B^+(x_0,r)} |v|^2 \, dx\, dt \Big)^{1/2} < \varepsilon_1, \\ \liminf_{r\to 0} \Big(\frac{1}{r^2} \int\limits_{Q^+(z_0,r)} |v|^3 \, dx\, dt \Big)^{1/3} < \varepsilon_1, \end{gathered} \tag{10}$$

then there exists $\rho_* > 0$ such that the functions v and H are Hölder continuous on the closure of $Q^+(z_0, \rho_*)$.

Note that it is possible to prove a lot of analogues of theorem 2.3. Generally the proof consist from two steps. Step one is the proof of boundedness of energy functionals (7). Usually, this procedure is sufficient to have boundedness condition for one functional depending on v and for one depending on H. In our work, section 4 is focused on this step. Also, we will use estimates for magnetic component H, that can be obtained if we consider equation (2) as the heat equation with lower order terms depending on v. These inequalities are proved in section 3.

The second step, is to prove regularity if all the functionals (7) are bounded and one functionals on v is small for a sufficiently small value of r. This result can be found in section 5.

§3. Estimates of solutions to the heat equation

In this section, we study solutions of the heat equations with the lower order terms:

$$\partial_t H \; - \; \Delta H \; = \; \operatorname{div}(v\otimes H - H\otimes v) \quad \text{in} \quad Q^+.$$
$$v|_{x_3=0} = 0,$$
$$H_3|_{x_3=0} = 0, \quad H_{\alpha,3}|_{x_3=0} = 0, \quad \alpha = 1,2.$$

Namely, we assume the functions (v,H) possess the following properties:

$$\begin{gathered} v,\; H \in W^{1,0}_2(Q^+), \\ v|_{x_3=0} = 0, \quad H_3|_{x_3=0} = 0 \quad \text{in the sense of traces,} \end{gathered} \tag{11}$$

for any $\eta \in C_0^\infty(Q;\mathbb{R}^3)$ such that $\eta_3|_{x_3=0} = 0$ the following integral identity holds

$$\int\limits_{Q^+} \Big(- H\cdot\partial_t\eta + \nabla H : \nabla\eta \Big)\, dxdt \; = \; -\int\limits_{Q^+} G : \nabla\eta\, dx\, dt, \tag{12}$$

here $G \; = \; v\otimes H - H\otimes v$, and

$$\operatorname{div} v = 0, \quad \operatorname{div} H = 0 \quad \text{a.e. in} \quad Q^+. \tag{13}$$

Lemma 3.1. *Assume that the conditions* (11)–(13) *hold. Then, for any* $0 < r \leqslant 1$ *and* $0 < \theta \leqslant 1$ *the following estimate holds*

$$F_2(\theta r) \leqslant c\theta^\alpha F_2(r) + c\theta^{-\frac{3}{2}} C(r) A_*(r). \tag{14}$$

Proof. Denote by v^* and H^* the extensions of functions v and H from Q^+ onto Q. Fix arbitrary $r \in (0,1)$ and let $\zeta \in C^\infty(\bar Q)$ be a cut off function such that $\zeta \equiv 1$ on $Q(r)$ and $\operatorname{supp}\zeta \subset B\times(-1,0]$. Denote $\Pi = \mathbb{R}^3 \times$

$(-1,0)$ and denote by $\widehat{G}$ the function which coincides with G^* on $Q(\frac{r}{2})$ and additionally possesses the following properties: $\widehat{G} \in W_1^{1,0}(\Pi) \cap L_{\frac{18}{11},\frac{6}{5}}(\Pi)$, $\widehat{G}$ is compactly supported in Π, and

$$\|\widehat{G}\|_{L_{\frac{6}{5},2}(\Pi)} \leqslant c\|G^*\|_{L_{\frac{6}{5},2}(Q(\frac{r}{2}))} \leqslant c\|G\|_{L_{\frac{6}{5},2}(Q^+(\frac{r}{2}))} \tag{15}$$

We decompose H^* as

$$H^* = \widehat{H} + \widetilde{H},$$

where $\widehat{H}$ is a solution of the Cauchy problem for the heat equation

$$\begin{cases} \partial_t \widehat{H} - \Delta \widehat{H} = \operatorname{div} \widehat{G} \qquad \text{in} \quad \Pi, \\ \widehat{H}|_{t=-1} = 0, \end{cases} \tag{16}$$

defined by the formula $\widehat{H} = \Gamma * \operatorname{div} \widehat{G} = -\nabla\Gamma * \widehat{G}$, where Γ is the fundamental solution of the heat operator. The function $\widetilde{H}$ satisfies the homogeneous heat equation

$$\partial_t \widetilde{H} - \Delta \widetilde{H} = 0 \qquad \text{in} \quad Q(\tfrac{r}{2}). \tag{17}$$

Take arbitrary $\theta \in (0, \frac{1}{2})$. We estimate $\|H\|_{L_2(Q^+(\theta r))}$ in the following way

$$\|H\|_{L_2(Q^+(\theta r))} \leqslant \|H^*\|_{L_2(Q(\theta r))} \leqslant \|\widehat{H}\|_{L_2(Q(\theta r))} + \|\widetilde{H}\|_{L_2(Q(\theta r))}, \tag{18}$$

For $\|\widehat{H}\|_{L_2(Q(\theta r))}$ we have

$$\|\widehat{H}\|_{L_2(Q(\theta r))} \leqslant c\, \|\widehat{H}\|_{L_2(Q(\frac{r}{2}))}. \tag{19}$$

As $\widetilde{H}$ satisfies (17) by local estimate of the maximum of $\widetilde{H}$ via its L_2–norm we obtain

$$\begin{aligned} \|\widetilde{H}\|_{L_2(Q(\theta r))} &\leqslant c\, \theta^{\frac{5}{2}} \, \|\widetilde{H}\|_{L_2(Q(\frac{r}{2}))} \\ &\leqslant c\, \theta^{\frac{5}{2}} \, (\|H^*\|_{L_2(Q(r))} + \|\widehat{H}\|_{L_2(Q(\frac{r}{2}))}) \end{aligned} \tag{20}$$

So, we need to estimate $\|\widehat{H}\|_{L_2(Q(\frac{r}{2}))}$. As singular integrals are bounded on the anisotropic Lesbegue space $L_{s,l}$ (see, for example, [8]) for the convolution $\widehat{h} = \Gamma * \widehat{G}$ we obtain the estimate

$$\|\widehat{h}\|_{W^{2,1}_{\frac{6}{5},2}(Q(r))} \leqslant c\|\widehat{G}\|_{L_{\frac{6}{5},2}(\Pi)}.$$

On the other hand, from the 3D– parabolic imbedding theorem (see [1])

$$W^{2,1}_{s,l}(Q) \hookrightarrow W^{1,0}_{p,q}(Q), \quad \text{as} \quad 1 - \left(\frac{3}{s} + \frac{2}{l} - \frac{3}{p} - \frac{2}{q}\right) \geqslant 0,$$

for $p=q=2$ and $s=\frac{6}{5}$, $l=2$ and for $\widehat{H}=-\nabla\widehat{h}$ we obtain

$$\|\widehat{H}\|_{L_2(Q(r))} \ \leqslant \ c\ \|\widehat{G}\|_{L_{\frac{6}{5},2}(\Pi)}.$$

We note that the constant c in this inequality does not depend on r. Taking into account (15), we arrive at the estimate

$$\|\widehat{H}\|_{L_2(Q(r))} \ \leqslant \ c\ \|G\|_{L_{\frac{6}{5},2}(Q^+(\frac{r}{2}))}. \tag{21}$$

From the definition of G we obtain

$$\begin{aligned} \|G\|_{L_{\frac{6}{5},2}(Q^+(\frac{r}{2}))} &\leqslant c \left(\int\limits_{-r^2/4}^{0} \|v\otimes H\|^2_{L_{\frac{6}{5}}(B^+(r/2))}\ dt \right)^{\frac{1}{2}} \\ &\leqslant \left(\int\limits_{-r^2}^{0} \|v\|^2_{3,B^+(r)}\|H\|^2_{2,B^+(r)}dt \right)^{\frac{1}{2}} \\ &\leqslant \|H\|_{2,\infty,Q^+(r)} \left(\int\limits_{-r^2}^{0} \|v\|^2_{3,B^+(r)}dt \right)^{\frac{1}{2}} \leqslant r^{\frac{3}{2}}C(r)A_*(r). \end{aligned} \tag{22}$$

Combining inequalities (20)-(22), we will get the statement of lemma. □

Using interpolation inequality (see for example (24) below) for $C(r)$ in the right hand side (14), we obtain inequality (14) in another form

Corollary 3.1. *Assume that conditions* (11)–(13) *hold. Then for any* $0<r\leqslant 1$ *and* $0<\theta\leqslant 1$ *the following estimate holds*

$$F_2(\theta r)\leqslant c\theta^{\alpha}F_2(r)+c\theta^{-\frac{3}{2}}E^{\frac{1}{2}}(r)A^{\frac{1}{2}}(r)A_*(r). \tag{23}$$

§4. Boundedness of energy functionals

In this section we derive estimates of energy functionals which allow us to obtain uniform boundedness (with respect to the radius) of all functionals (7) if boundedness of some of them is known.

Note that one can prove a group of estimates that are the consequences of Hölder inequality, embedding theorem and interpolation inequality.

$$C(r) \ \leqslant \ A^{\frac{1}{2}}(r)E^{\frac{1}{2}}(r), \qquad F(r) \ \leqslant \ A_*^{\frac{1}{2}}(r)[E_*^{\frac{1}{2}}(r)+F_2^{\frac{1}{2}}(r)], \tag{24}$$

$$D(r) \ \leqslant \ cD_1(r), \quad D_1(r) \ \leqslant \ cD_s(r), \quad \forall s>1. \tag{25}$$

First of all we prove the decay estimate for the pressure

Lemma 4.1. *If v, p, H are the suitable weak solution near the boundary to the MHD equations in Q^+, then for any $0 < r \leqslant 1$ и $0 < \theta \leqslant 1$ the following estimate holds:*

$$
\begin{aligned}
D_{\frac{12}{11}}(\theta r) \leqslant c\theta^{\alpha} \Big(D_{\frac{12}{11}}(r) + E(r) \Big) \\
+ c(\theta) \Big(E(r) A^{\frac{1}{2}}(r) C^{\frac{1}{2}}(r) + E_*(r) A_*^{\frac{1}{2}}(r) F^{\frac{1}{2}}(r) \Big) .
\end{aligned}
\tag{26}
$$

Proof. In order to obtain (26) we apply the method developed in [3, 5], see also [6]. Denote $\Pi_r = \mathbb{R}^3_+ \times (-r^2, 0)$. We fix $r \in (0, 1]$ and $\theta \in (0, \frac{1}{2})$ and define a function $g : \Pi_r^+ \to \mathbb{R}^3$ by the formula

$$
g \ = \ \begin{cases} \operatorname{rot} H \times H - (v \cdot \nabla) v, & \text{in } \ Q^+(r), \\ 0, & \text{in } \ \Pi_r^+ \setminus Q^+(r) \end{cases}
$$

Then we decompose v and p as

$$
v \ = \ \widehat{v} + \widetilde{v}, \quad p \ = \ \widehat{p} + \widetilde{p},
$$

where $(\widehat{v}, \widehat{p})$ is a solution of the Stokes initial boundary value problem in a half-space

$$
\begin{cases} \partial_t \widehat{v} - \Delta \widehat{v} + \nabla \widehat{p} \ = \ g, \\ \operatorname{div} \widehat{v} = 0 \end{cases} \quad \text{in} \quad \Pi_r^+,
$$
$$
\widehat{v}|_{t=0} = 0, \quad \widehat{v}|_{x_3 = 0} = 0,
$$

and $(\widetilde{v}, \widetilde{p})$ is a solution of the homogeneous Stokes system in $Q^+(r)$:

$$
\begin{cases} \partial_t \widetilde{v} - \Delta \widetilde{v} + \nabla \widetilde{p} \ = \ 0, \\ \operatorname{div} \widetilde{v} = 0 \end{cases} \quad \text{in} \quad Q^+(r),
$$
$$
\widetilde{v}|_{x_3 = 0} = 0.
$$

For $\nabla \widehat{p}$ and $\nabla \widetilde{p}$ the following estimates hold (see [5], see also [7]):

$$
\begin{aligned}
\|\nabla \widehat{p}\|_{L_{\frac{12}{11}, \frac{3}{2}}(Q^+(r))} + \frac{1}{r} \|\nabla \widehat{v}\|_{L_{\frac{12}{11}, \frac{3}{2}}(Q^+(r))} \\
\leqslant c \ \Big(\ \|H \times \operatorname{rot} H\|_{L_{\frac{12}{11}, \frac{3}{2}}(Q^+(r))} + \|(v \cdot \nabla) v\|_{L_{\frac{12}{11}, \frac{3}{2}}(Q^+(r))} \ \Big),
\end{aligned}
\tag{27}
$$

$$
\|\nabla \widetilde{p}\|_{L_{\frac{12}{11}, \frac{3}{2}}(Q^+(\theta r))} \leqslant c \, \theta^{\alpha} \Big(\frac{1}{r} \|\nabla \widetilde{v}\|_{L_{\frac{12}{11}, \frac{3}{2}}(Q^+(r))} + \|\nabla \widetilde{p}\|_{L_{\frac{12}{11}, \frac{3}{2}}(Q^+(r))} \Big). \tag{28}
$$

On order to estimate the right hand side of (27), we use Hölder and interpolation inequalities

$$\|(v\cdot\nabla)v\|_{\frac{12}{11},\frac{3}{2}} = \left(\int\limits_{-r^2}^{0} \|(v\cdot\nabla)v\|_{\frac{12}{11}}^{\frac{3}{2}}\,dt\right)^{\frac{2}{3}} \leqslant \left(\int\limits_{-r^2}^{0} \|\nabla v\|_2^{\frac{3}{2}}\|v\|_{\frac{12}{5}}^{\frac{3}{2}}\,dt\right)^{\frac{2}{3}}$$

$$\leqslant \|\nabla v\|_2 \left(\int\limits_{-r^2}^{0} \|v\|_{\frac{12}{5}}^{6}\,dt\right)^{\frac{1}{6}} \leqslant \|\nabla v\|_2 \left(\int\limits_{-r^2}^{0} \|v\|_2^{3}\|v\|_3^{3}dt\right)^{\frac{1}{6}}$$

$$\leqslant \|\nabla v\|_2 \|v\|_{2,\infty}^{\frac{1}{2}}\|v\|_3^{\frac{1}{2}} \tag{29}$$

The term $\|H\times \operatorname{rot} H\|_{L_{\frac{12}{11},\frac{3}{2}}(Q^+(r))}$ can be estimated analogously and the right hand side of (28) can be estimated as follows:

$$\begin{aligned}&\frac{1}{r}\|\nabla\widetilde{v}\|_{L_{\frac{12}{11},\frac{3}{2}}(Q^+(r))} + \|\nabla\widetilde{p}\|_{L_{\frac{12}{11},\frac{3}{2}}(Q^+(r))}\\ &\quad\leqslant c\Big(\|\nabla v\|_{2,Q^+(r)} + \|\nabla p\|_{L_{\frac{12}{11},\frac{3}{2}}(Q^+(r))}+\\ &\qquad+\frac{1}{r}\|\nabla\widehat{v}\|_{L_{\frac{12}{11},\frac{3}{2}}(Q^+(r))} + \|\nabla\widehat{p}\|_{L_{\frac{12}{11},\frac{3}{2}}(Q^+(r))}\Big).\end{aligned} \tag{30}$$

Combining inequalities (27)–(30) we obtain the statement of Lemma. □

Theorem 4.1. *Let v, p, H are the suitable weak solution near the boundary to the MHD equations in Q^+ and*

$$C(R) + F_2(R) \leqslant M, \quad 0 < R \leqslant 1.$$

For the functional

$$\mathcal{L}(r) = A^2(r) + E^2(r) + A_*^2(r) + E_*^2(r) + D_{\frac{12}{11}}^{\frac{25}{24}}(r)$$

the following estimate holds:

$$\mathcal{L}(r) \leqslant C(M)(r^\alpha \mathcal{L}(1) + 1).$$

Proof. By the local energy inequality, we have

$$\begin{aligned}\mathcal{L}(\theta r) \leqslant c\big(&C_2^2(2\theta r) + F_2^2(2\theta r) + C^3(2\theta r) + C(2\theta r)D(2\theta r)\\ &+ C^2(2\theta r)F_3(2\theta r) + C(2\theta r)F_3^2(2\theta r) + D_{\frac{12}{11}}^{\frac{25}{24}}(\theta r)\Big).\end{aligned}$$

Now, we estimate each term in the right hand side. Our goal is to prove the following estimate:

$$\mathcal{L}(\theta r) \leqslant \frac{1}{2}\mathcal{L}(r) + C(M). \tag{31}$$

After that, we can use the standard iteration procedure (see [4]) and obtain the statement of the theorem.

Estimates for the first three terms are obvious. In order to estimate the 4th term, we use Young inequality and inequality (25)

$$C(2\theta r)D(2\theta r) \leqslant c\left(D_{\frac{12}{11}}^{\frac{25}{24}}(2\theta r) + M^{25}\right).$$

Now we are going to prove estimate for $D_{\frac{12}{11}}(2\theta r)$. To do this, we use inequalities (24) and (26)

$$\begin{aligned} &D_{\frac{12}{11}}(2\theta r) \leqslant \\ &\leqslant c\theta^{\alpha}\left(D_{\frac{12}{11}}(r) + E(r)\right) + c(\theta)\left(E(r)A^{\frac{1}{2}}(r)C^{\frac{1}{2}}(r) + E_*(r)A_*^{\frac{1}{2}}(r)F_3^{\frac{1}{2}}(r)\right) \\ &\leqslant c\theta^{\alpha}\left(D_{\frac{12}{11}}(r) + E(r)\right) + c(\theta)\left(\mathcal{L}^{\frac{3}{4}}(r)M^{\frac{1}{2}} + \mathcal{L}^{\frac{3}{4}}(r)F_3^{\frac{1}{2}}(r)\right). \end{aligned} \tag{32}$$

We use interpolation inequality to estimate $F_3(r)$

$$\begin{aligned} F_3(r) &\leqslant F_{\frac{10}{3}}^{\frac{5}{6}}(r)F_2^{\frac{1}{6}}(r) \leqslant c\left(A_*^{\frac{2}{5}}(r)\left(E_*^{\frac{3}{5}}(r) + F_2^{\frac{3}{5}}(r)\right)\right)^{\frac{5}{6}} F_2^{\frac{1}{6}}(r) \\ &\leqslant c\left(\mathcal{L}^{\frac{5}{12}}(r)M^{\frac{1}{6}} + \mathcal{L}^{\frac{1}{6}}M^{\frac{2}{3}}\right) \end{aligned} \tag{33}$$

and substitute this relation into (32)

$$\begin{aligned} D_{\frac{12}{11}}(2\theta r) &\leqslant c\theta^{\alpha}\left(D_{\frac{12}{11}}(r) + E(r)\right) \\ &\quad + c(\theta)\left(\mathcal{L}^{\frac{3}{4}}(r)M^{\frac{1}{2}} + \mathcal{L}^{\frac{23}{24}}(r)M^{\frac{1}{12}} + \mathcal{L}^{\frac{5}{6}}(r)M^{\frac{1}{3}}\right). \end{aligned}$$

Then, we obtain

$$D_{\frac{12}{11}}^{\frac{25}{24}}(2\theta r) \leqslant c\theta^{\alpha}\mathcal{L}(r) + c(\theta)\left(\mathcal{L}^{\frac{575}{576}}(r)M^{k_1} + \mathcal{L}^{\frac{125}{144}}(r)M^{k_2}\right).$$

Since the right hand side of the last inequality contain $\mathcal{L}(r)$ in the degree smaller then 1, choosing θ sufficiently small and using Young inequality we obtain an estimate (31).

To estimate the last two terms we use (33)

$$C(r)F_3^2(r) \leqslant c\left(\mathcal{L}^{\frac{5}{6}}(r)M^{\frac{4}{3}} + \mathcal{L}^{\frac{1}{3}}M^{\frac{7}{3}}\right)$$

and Young inequality. The second term can be estimated in the same maner.

As the result we obtain (31). Next by standard iteration procedure we finish the prove of the theorem. □

§5. Proof of the main results

As a first step we obtain theorem 2.1 without smallness condition on a pressure.

Lemma 5.1. *For arbitrary $M > 0$ there is $\varepsilon_1(M) > 0$, such that if v, p, H are the suitable weak solution near the boundary to the MHD equations in Q^+,*

$$A(R)+E(R)+A_*(R)+E_*(R)+F_3(R)+D_{\frac{12}{11}}(R) < M, \quad \forall\, 0 < R \leqslant 1 \tag{34}$$

and

$$C(1) + F_3(1) < \varepsilon_1, \tag{35}$$

then the functions v and H are Hölder continuous on $\bar{Q}^+(r_*)$ for some $0 < r_* < 1$.

Proof. Assume that the statement of the lemma is false. Then there are sequences of v_n, p_n, H_n of suitable weak solutions in Q^+, such that

$$C(v_n, 1) + F_3(H_n, 1) = \varepsilon_n \to 0, \text{ as } n \to \infty \tag{36}$$

and 0 is a singular point. Then by theorem 2.1

$$C(v_n, r) + D(p_n, r) + F_3(H_n, r) > \varepsilon_* \tag{37}$$

for all $0 < r < 1$.

On the other hand from (26), (34), (36) and the embedding theorem we have

$$D(p_n, r) \leqslant cD_{\frac{12}{11}}(p_n, r) \leqslant cr^{\alpha} M + c(r) M^{\frac{3}{2}} \varepsilon_n^{\frac{1}{2}}. \tag{38}$$

We fix $0 < r \leqslant 1$ and pass to the limit by n in (37) and (38)

$$\begin{aligned} \varepsilon_* &\leqslant \limsup_{n\to\infty} \left(C(v_n, r) + D(p_n, r) + F_3(H_n, r)\right) \\ &= \limsup_{n\to\infty} D(p_n, r) \leqslant cr^{\alpha} M. \end{aligned}$$

As a result we obtain that the inequality

$$\varepsilon_* \leqslant cr^{\alpha} M$$

must be true for arbitrary $0 < r < 1$. Hence, we have a contradiction. □

Theorem 5.1. *For arbitrary $M > 0$ there is $\varepsilon_2(M) > 0$, such that if v, p, H are the suitable weak solution near the boundary to the MHD equations in Q^+, satisfying to* (34) *and one of the following conditions holds*

$$E(1) < \varepsilon_2, \tag{39}$$

$$A(1) < \varepsilon_2, \tag{40}$$

$$C(1) < \varepsilon_2, \tag{41}$$

then the functions v and H are Hölder continuous on $\bar{Q}^+(r_*)$ for some $0 < r_* < 1$.

Proof. The proof of this theorem is similar to the proof of the previous lemma. We begin from the case (39). Let v_n, p_n, H_n are the sequences of suitable weak solutions to the MHD system, such that (34) holds,

$$E(v_n, 1) = \varepsilon_n \to 0$$

as $n \to \infty$, and $z_0 = 0$ is a singular point. Then from lemma 5.1 we have

$$C(v_n, r) + F_3(v_n, r) > \varepsilon_1 \tag{42}$$

for arbitrary $0 < r < 1$.

On the other hand

$$C(v_n, r) \leqslant \frac{c}{r^{\frac{2}{3}}} C(v_n, 1) \leqslant \frac{c}{r^{\frac{2}{3}}} A^{\frac{1}{2}}(v_n, 1) E^{\frac{1}{2}}(v_n, 1) \to 0 \tag{43}$$

as $n \to \infty$ and for any fixed $0 < r \leqslant 1$. From (23) we obtain

$$\limsup F_2(H_n, r) \leqslant c r^{\alpha} M. \tag{44}$$

Next we use interpolation inequality

$$F_3(H_n, r) \leqslant F_2^{\frac{1}{6}}(H_n, r) F_{\frac{10}{3}}^{\frac{5}{6}}(H_n, r). \tag{45}$$

To estimate the second factor in the right hand side of (45) we use (24). So from (42)-(45) we obtain

$$\varepsilon_1 \leqslant \limsup_{n \to \infty} \left(C(v_n, r) + F_3(v_n, r) \right) \leqslant c r^{\alpha_1} M^k \quad \forall\, 0 < r \leqslant \frac{1}{2},$$

and, if we choose r sufficiently small, we will have a contradiction.

Observe, that $E(r)$ and $A(r)$ take part in (23) symmetrically, so the proof of this theorem in the case (40) is similar to the previous one. In the case of (41) for obtaining (44) is sufficient to use (14). □

References

1. O. V. Besov, V .P. Il'in, S. M. Nikolskii, *Integral representations of functions and imbedding theorems.* Moscow, Nauka (1975). Translation: Wiley&Sons, 1978.
2. A. Mikhaylov, *On local regularity for suitable weak solutions of the Navier–Stokes equations near the boundary.* — Zap. Nauchn. Semin. POMI **370** (2009), 73–93.
3. G. A. Seregin, *Local regularity of suitable weak solutions to the Navier-Stokes equations near the boundary.* — J. Math. Fluid Mech. **4**, no. 1 (2002), 1–29.
4. G. A. Seregin, *Local regularity for suitible weak solutions to the Navier–Stokes equations.* — Uspekhi Mat. Nauk **62**, no. 3, 149-Ц168
5. G. A. Seregin, *Some estimates near the boundary for solutions to the non-stationary linearized Navier-Stokes equations.* — Zap. Nauchn. Semin. POMI **271** (2000), 204–223.
6. G. A. Seregin, T. N. Shilkin, V. A. Solonnikov, *Boundary patial regularity for the Navier–Stokes equations.* — Zap. Nauchn. Semin. POMI **310** (2004), 158–190.
7. V. A. Solonnikov, *Estimates of solutions of the Stokes equations in Sobolev spaces with a mixed norm.* — Zap. Nauchn. Semin. POMI **288** (2002), 204–231.
8. V. A. Solonnikov, *On the estimates of solutions of nonstationary Stokes problem in anisotropic S. L. Sobolev spaces and on the estimate of resolvent of the Stokes problem.* — Uspekhi Mat. Nauk **58**, no. 2 (350) (2003), 123–156.
9. V. Vyalov, T. Shilkin, *On the boundary regularity of weak solution to the MHD system.* — Zap. Nauchn. Semin. POMI **385** (2010), 18–53.

Chebyshev Laboratory,
St.Petersburg State University,
14th Line, 29b,
Saint-Petersburg, 199178 Russia
E-mail: vyalov@gmail.com

Поступило 7 ноября 2011 г.