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M. S. Nikulin, R. Tahir

APPLICATION OF SEDYAKIN'S MODEL AND BIRNBAUM–SAUNDERS FAMILY FOR STATISTICAL ANALYSIS OF REDUNDANT SYSTEMS WITH ONE WARM STAND-BY UNIT

Abstract. The purpose of this paper is to determine the improved system reliability by introducing the redundancy of the components. This paper considers one warm stand-by unit with one main unit. We study the reliability of such systems using the probability models in terms of the Sedyakin's Model. Parametric family of Birnbaum-Saunders distributions is considered as the distribution of failure times of both main and stand-by units. Parametric point and interval estimation is obtained from censored data.

§1. Introduction

Recently the series of papers on the redundant system was published by Bagdonavičius *et al.* [4–7]. We give the reliability of redundant system based on parametric Birnbaum–Saunders family of distributions. Redundancy which means the duplication of critical components of a system is a common and useful approach to increase the reliability of the system. The redundant systems contain more than one subcomponents and all must fail before the system fails. The functioning component or unit is called the operating component or main unit and the other subcomponents are called stand-by units. If the main unit fails then the first stand-by unit is commuted automatically and if this unit fails the second stand-by unit is commuted and so on. So the redundant system fails only if all units fails. A system with two, three or many replications of each element is respectively termed as dual modular redundant, triple modular redundant, and multi-modular redundant. Redundant system with one main unit and $m-1$ stand-by units is denoted by $S(1, m-1)$, $m \geqslant 2$. Redundant system is different from the backup system system in a way that with redundant

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system data is continuously passed from the primary to secondary component while backup systems may loos data and may take many hours to become operational. Redundancy increases the cost and complexity of the system design and sometimes it is the unique way to provide high reliability to modern electrical, biotechnical and mechanical systems. Many low-risk industries do not need redundancy in order to be successful. However, in high-risk industries such as aerospace and nuclear, where the cost of failure is high enough, the redundancy becomes essential to have the high reliability.

Here we consider a system $S(1,1)$ with one main unit operating in hot conditions and one stand-by unit operating in warm conditions. In the terminology of accelerated life testing main unit is working under accelerated stress with respect to stand by unit. We suppose that the switching on from the warm to hot state does not do any damage to the unit and the switching time is stochastic. The model with fluent switching, important for practice, is considered. On the base of this supposition Bagdonavičius *et al.* [4] proposed the test for *general fluent switching hypothesis* H_0 and for *particular fluent switching hypothesis* H_0^*. The hypothesis H_0 is formulated using *Sedyakin's reliability principle* [16] and the hypothesis H_0^* is formulated using AFT model for the reliability of redundant systems. Sedyakin's model explains that after the failure of main unit the stand-by unit follow the same reliability curve as that of the main unit. Following precedent results of Bagdonavičius *et al.* [4] we study the asymptotic properties of the test statistics. After testing, we construct parametric estimators of the cumulative distribution function $K_2(t)$ of redundant system, using censored reliability data of components under different stresses. Then confidence intervals for the commutative distribution function of redundant system $S(1,1)$ are constructed under the assumption that the distribution of failure times of both units follow the Birnbaum–Saunders family.

§2. Redundant system with warm stand-by unit

Redundancy is mostly used technique to increase the reliability of the technical systems. We consider the redundant system with one main and one stand-by unit in parallel and we denote the system by $S(1,1)$. For configuring the redundant system the functioning state of the stand-by unit is very important according to the criticality of the system. In the last century several people study the system with hot and cold stand-by units but recently there are many papers where the intermediate warm conditions are used for stand-by units. If the stand-by unit is working in hot conditions as the main unit, the failing probability of stand-by unit is equal to the main unit i.e. 0.5, because both are functioning under the same stress. And if the stand-by unit is in cold state, it requires commuting time to come in hot conditions after the failure of main unit which can interrupt the system's continuity and also can cause the increase in failure rate due to the sudden change in stress that is due to the burn-in period (failure in early life). So the warm reserving can be the better choice where the stand-by unit is functioning under lower stress than the main one. In this way the probability of failure of the stand-by unit is smaller than that of the main unit and commuting is fluent but we don't discuss here the problem of the choice of optimality of the warm state which depends on the practical problem. Recently a lot of work has been done for statistical analysis of redundant system by Bagdonavicius *et al.* [4–7] and with warm stand-by units. They used many parametric models such as exponential, Weibull, lognormal, loglogistic, generalized Weibull, and inverse Gaussian for determining the reliability of redundant system (see Nikulin *et al.* [15]). Here we follow their results and use the famous Birnbaum–Saunders (BS) family of distributions as the distribution of failure times.

§3. Birnbaum–Saunders (BS) family of life distributions

The family of Birnbaum–Saunders (BS) distributions is widely used for failure time data especially when the failures are due to crack. This family was proposed by Birnbaum and Saunders [10] with two parameters, named as shape and scale parameters. Fatigue failure is due to repeated applications of a common cyclic stress pattern. The PDF and hazard function of this distribution is unimodal and is very popular in modeling fatigue failures in industry as an alternative to other unimodal distributions such as the lognormal and inverse Gaussian. Desmond [11] worked on the relationship between Birnbaum–Saunders distribution and the family of inverse Gaussian distributions. The hazard functions for both of these distributions are very similar. The cumulative distribution function of two-parameter Birnbaum-Saunders distribution is

$$F(t;\alpha,\beta) = \Phi\left[\frac{1}{\alpha}\left\{\left(\frac{t}{\beta}\right)^{\frac{1}{2}} - \left(\frac{\beta}{t}\right)^{\frac{1}{2}}\right\}\right], \quad 0 < t < \infty, \quad \alpha,\beta > 0, \quad (1)$$

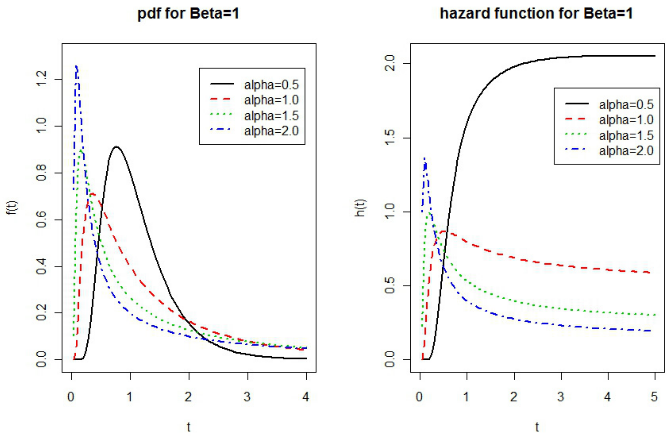

Fig. 1. Pdf and hazard function for BS distribution for $\beta = 1$.

where α is the shape parameter, β is the scale parameter and $\Phi(x)$ is the standard normal distribution function. The probability density function can be written as

$$f(t;\alpha,\beta) = \frac{1}{2\sqrt{2\pi}\ \alpha\beta}\left\{\left(\frac{\beta}{t}\right)^{\frac{1}{2}} + \left(\frac{\beta}{t}\right)^{\frac{3}{2}}\right\}\exp\left[-\frac{1}{2\alpha^2}\left(\frac{t}{\beta}+\frac{\beta}{t}-2\right)\right],$$
$$t > 0, \quad \alpha,\beta > 0.$$

The survival function and the hazard function can be obtained from the following relationships:

$$S(t;\alpha,\beta) = 1 - F(t;\alpha,\beta), \quad \lambda(t;\alpha,\beta) = \frac{f(t;\alpha,\beta)}{S(t;\alpha,\beta)}.$$

Extensive work has been done on the Birnbaum–Saunders distribution and its application in the failure time data but not a lot of work is done on its uses in the redundant system to enhance the reliability (see Balakrishnan *et al.* [8, 9], Kundu *et al.* [12], Lemonte *et al.* [13], Volodin and Dzhungurova [17]). The mean, variance and coefficient of variation (CV) of the BS distribution are given as

$$\mu = \frac{\beta}{2}(\alpha^2+2), \quad \sigma^2 = \frac{\beta^2}{4}(5\alpha^4+4\alpha^2), \quad CV = \frac{\sqrt{5\alpha^4+4\alpha^2}}{(\alpha^2+2)}.$$

§4. Accelerated Life Testing (ALT) in Reliability

Failure times data of high reliable units may take many years for reliability analysis. One way of obtaining quick reliability information is to use accelerated life testing (ALT) method, where higher level of experimental factors, stresses or covariates (temperature, voltage or speed) are applied on the units to increase the number of failures in less time (see Bagdonavičius and Nikulin [3], Meeker and Escobar [14]).

In ALT, it is supposed that a stress (or explanatory variable) is a deterministic time function, may be multidimensional:

$$x(\cdot) = (x_1(\cdot), \ldots, x_m(\cdot))^T : [0, \infty[\to R^m, \tag{2}$$

which is a vector of covariates itself or a realization of a stochastic process $X(\cdot)$. This process is also called the covariate process, $X(\cdot) = (X_1(\cdot), \ldots, X_m(\cdot))^T$. We denote E a set of all possible (admissible) stresses, and by E_1 a set of all constant over time covariates, $E_1 \subset E$. By tradition if $x(\cdot)$ is constant in time, we write x instead of $x(\cdot)$. There are many types of stresses (e.g. step stresses, continuous cyclic stress, cyclic stress of type switch-on-switch-off, degradation stress) but step stresses are commonly used in ALT where the units are placed initially at low stress and if they do not fail till certain moment then stress is increased continuously in steps. Step-stress can be increasing or decreasing. For our problem it is interesting to introduce the next class of step-stresses. We denote $E_2 \subset E$ a set of step-stresses of the form

$$x(t) = x_1 1_{\{0 \leqslant t < t_1\}} + x_2 1_{\{t_1 \leqslant t\}}, \quad x_1, x_2 \in E_1. \tag{3}$$

Let $T_{x(\cdot)}$ be the failure time under the stress $x(\cdot)$ and

$$S_{x(\cdot)}(t) = \mathbf{P}\{T_{x(\cdot)} > t\}, \quad F_{x(\cdot)}(t) = \mathbf{P}\{T_{x(\cdot)} \leqslant t\}, \quad f_{x(\cdot)}(t) = -S'_{x(\cdot)}(t),$$

$$\lambda_{x(\cdot)}(t) = \lim_{h \to 0} \frac{1}{h} \mathbf{P}\{T_{x(\cdot)} \in [t, t+h) | T_{x(\cdot)} \geqslant t\} = -\frac{S'_{x(\cdot)}(t)}{S_{x(\cdot)}(t)},$$

be the survival function, cumulative distribution function, probability density function, and hazard function respectively.

Let $x(\cdot), y(\cdot) \in E$ be the two stresses. A stress $y(\cdot)$ is accelerated with respect to the stress $x(\cdot)$ if $S_{y(\cdot)}(t) \leqslant S_{x(\cdot)}(t)$, or equivalently we can write $F_{y(\cdot)}(t) \geqslant F_{x(\cdot)}(t)$ for $t \geqslant 0$. If the data are censored then we have to consider the influence of $x(\cdot)$ on the distribution of censoring time C, i.e. we write $C = C_{x(\cdot)}$, and we observe

$$X_{x(\cdot)} = \min(T_{x(\cdot)}, C_{x(\cdot)}).$$

§5. Sedyakin's physical principle in reliability

Accelerated stresses are used to reduce the time on test. So a transfer functional is needed to interpolate the accelerated failure times to the failure times under usual stress (see Bagdonavičius [1], Bagdonavičius and Nikulin [2]). The physical principle in reliability proposed by N. Sedyakin [16] states that for two identical populations of units functioning under different stresses $x_1 \neq x_2$, two moments t_1 and t_1^* are equivalent if their survival probabilities are equal until these moments, i.e.

$$S_{x_1}(t_1) = S_{x_2}(t_1^*), \quad t_1^* = g(t_1), \quad \text{where} \quad g(t) = S_{x_1}^{-1}(S_{x_2}(t)). \qquad (4)$$

This principle gives an interesting way to prolong any class of survival functions $\{S_x(\cdot), x \in E_1\}$ indexed by constant in time stresses to a class of survival functions indexed by step-stresses, for example from E_2 given by (2). Figure 2 shows the increasing step-stress. x_1 and x_2 are the stresses

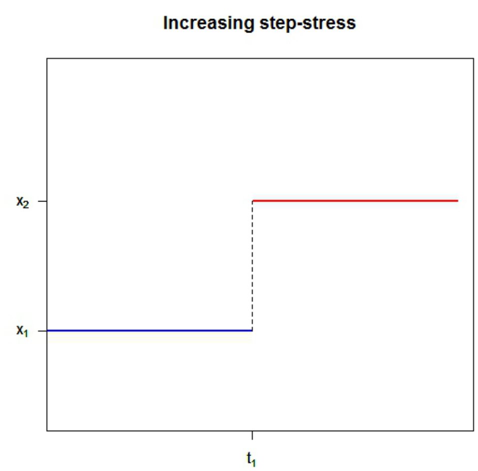

Fig. 2. Increasing step-stress for the warm stand-by unit.

corresponding to warm and hot conditions respectively. Stress x_2 is accelerated with respect to stress x_1. The moment t_1 is random in our case.

According to Sedyakin we may consider the model on E_2 for all $s \geqslant 0$

$$\lambda_{x(\cdot)}(t_1 + s) = \lambda_{x_2}(t_1^* + s). \qquad (5)$$

In terms of the survival function $S_{x(\cdot)}(t), x(\cdot) \in E_2$ that satisfies the same rule of time-shift

$$S_{x(\cdot)} = \begin{cases} S_{x_1}(t), & 0 \leqslant t < t_1, \\ S_{x_2}(t - t_1 + t_1^*), & t \geqslant t_1, \end{cases} \tag{6}$$

where t_1^* is determined by the equation (4). The model given by (5) and (6) is called the Sedyakin model on E_2.

The generalized Sedyakin (GS) model on a set of stresses E can be written by supposing that the hazard rate $\lambda_{x(\cdot)}(t)$ at any moment t is a function of the value of the stress at this moment and of the probability of survival until this moment $\lambda_{x(\cdot)}(t) = h(x(t), S_{x(\cdot)}(t)), \quad x(\cdot) \in E$, where h is a positive function. Note that the AFT model verifies this rule.

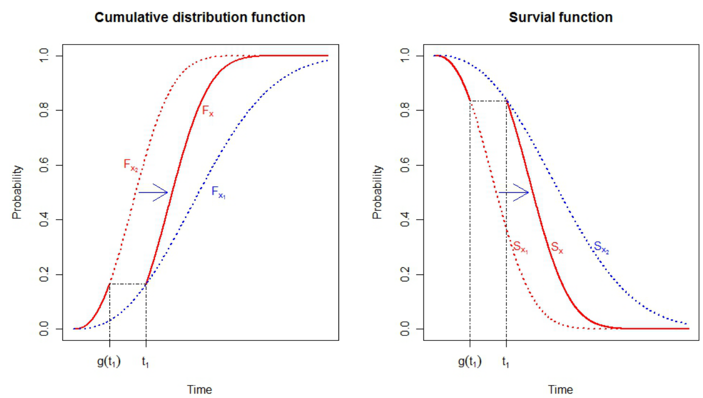

Fig. 3. Cumulative distribution function and survival function of the system under Sedyakin's principal.

§6. Sedyakin model and its application in Redundant system

Let denote by T_1, F_1 and f_1 the failure time, the cumulative distribution function and the probability density function of the main unit. Suppose the failure time of the stand-by unit be T_2. If it is working in hot conditions its distribution function is also F_1. In warm conditions the distribution function of T_2 is F_2 and the p.d.f. is f_2. After the failure of main unit the stand-by unit is switched to hot conditions and its distribution function is different from F_1 and F_2. The system fails if both units fail i.e. the failure time of the system is $T = \max(T_1, T_2)$. Let the conditional density function of T_2 given that the main unit fails at moment y is denoted by $f_2^{(y)}(x) = f_{T_2|T_1=y}(x)$ and let denote by K_2 the distribution function of the system's failure time T. The cumulative distribution function K_2 can be written as

$$K_2(t) = \mathbf{P}(T \leqslant t) = \mathbf{P}(T_1 \leqslant t, T_2 \leqslant t) = \int_0^t \mathbf{P}(T_2 \leqslant t \mid T_1 = y) dF_1(y)$$

$$= \int_0^t \left\{ \int_0^y f_2(x)dx + \int_y^t f_2^{(y)}(x)dx \right\} f_1(y)dy. \tag{7}$$

When the stand-by unit is in **cold** state then

$$f_2(x) = 0 \quad \text{if} \quad x \leqslant y, \quad \text{and} \quad f_2^{(y)}(x) = f_1(x-y) \quad \text{if} \quad x > y,$$

so from equation (7), it follows that

$$K_2(t) = \int_0^t \left\{ \int_y^t f_1(x-y)dx \right\} f_1(y)dy = \int_0^t F_1(t-y)dF_1(y).$$

When the stand-by unit is in **hot** state then

$$f_2^{(y)}(x) = f_2(x) = f_1(x),$$

so using equation (7) we can write

$$K_2(t) = \int_0^t \left\{ \int_0^y f_2(x)dx + \int_y^t f_2(x)dx \right\} f_1(y)dy$$

$$= \int_0^t \left\{ \int_0^t f_2(x)dx \right\} dF_1(y) = \int_0^t \left\{ \int_0^t f_1(x)dx \right\} dF_1(y) = [F_1(t)]^2.$$

When the stand-by unit is in **warm** conditions the following hypothesis is assumed:

$$H_0 : f_2^{(y)}(x) = f_1(x + g(y) - y), \quad \forall \quad x \geqslant y \geqslant 0, \tag{8}$$

where $g(y)$ is the moment which in hot conditions corresponds to the moment y in warm conditions in the sense that

$$F_1(g(y)) = \mathbf{P}(T_1 \leqslant g(y)) = \mathbf{P}(T_2 \leqslant y) = F_2(y),$$

so

$$g(y) = F_1^{-1}(F_2(y)).$$

Conditionally (given $T_1 = y$) the hypothesis (8) corresponds to the Sedyakin's model. In the situation considered here the switch off moments are random. The equation (7) implies that under hypothesis H_0 the distribution function of the system $S(1,1)$ is

$$K_2(t) = \int_0^t F_1(t + g(y) - y) dF_1(y). \tag{9}$$

In particular, if we suppose that the distribution of the units functioning in warm and hot conditions differ only in scale, i.e.

$$F_2(t) = F_1(rt),$$

for some unknown $r > 0$, then $g(y) = ry$. This make the sense of AFT model. In such case the following hypothesis is to be verified:

$$H_0^* : f_2^{(y)}(x) = f_1(x + ry - y), \quad \forall \quad x \geqslant y \geqslant 0, \tag{10}$$

and we can write the cumulative distribution function of the system as

$$K_2(t) = \int_0^t F_1(t + ry - y) dF_1(y). \tag{11}$$

The hypothesis H_0^* can also be considered as the generalization of the accelerated failure time (AFT) model to the case of stress with random switch-on (see Bagdonavičius [1]). In this paper we suppose that the cumulative distribution function of failure times for both units belong to the Birnbaum-Saunders family. So we need to estimate the parameters of the models and other reliability characteristics, for example, distribution function of the system from the data which can be noncensored or censored. But before going to the estimation problem, it is necessary to test the model given by the Hypotheses H_0^*.

§7. Goodness-of-fit test for hypotheses H_0^*

For testing the hypotheses H_0^* we suppose the following plan of experiment for data:

i) the failure times $T_{11}, \cdots, T_{1n_1}$ of n_1 units tested in hot conditions,

ii) the failure times $T_{21}, \cdots, T_{2n_2}$ of n_2 units tested in warm conditions,

iii) the failure times $T_1, \cdots, T_n$ of n redundant systems (with warm stand-by units).

The test is based on the difference of two estimators of the cumulative distribution function $F(\cdot)$ of the system failure time T. The first estimator is the empirical distribution function

$$\widehat{F}^{(1)}(t) = \frac{1}{n}\sum_{i=1}^{n} 1_{\{T_i \leqslant t\}}, \quad t \geqslant 0,$$

based on the third sample. The second estimator

$$\widehat{F}^{(2)}(t) = \int_0^t \widehat{F}_1(t + \widehat{r}y - y)d\widehat{F}_1(y),$$

is based on the equation (11), where

$$\widehat{r} = \frac{\widehat{\mu}_1}{\widehat{\mu}_2}, \quad \widehat{\mu}_j = \frac{1}{n_j}\sum_{i=1}^{n_j} T_{ji}, \quad j = 1, 2,$$

where $\widehat{\mu}_1$ and $\widehat{\mu}_2$ are the means of two empirical distributions based on the first and second samples respectively.

The test is based on the statistic

$$X = \sqrt{n}\int_0^\infty [\widehat{F}^{(1)}(t) - \widehat{F}^{(2)}(t)]dt. \tag{12}$$

It is natural generalization of Student's type t-test for comparing the means of two populations. Indeed, the mean failure time of the system with distribution function F is

$$\mu = \int_0^\infty [1 - F(s)]ds,$$

so the statistic (12) is the normed difference of two estimators (the second being not the empirical mean) of the mean μ. Student's type t-test is based on the difference of empirical means of two populations. Bagdonavičius *et al.* [4] shown the following theorem.

Theorem. *Suppose that* $n_i/n \to l_i \in (0,1), n \to \infty$ *and the densities* $f_i(x), i = 1, 2$ *are continuous and positive on* $(0, \infty)$. *Then under* H_0^* *the* *statistic X converges in distribution to the normal law $N(0,\sigma^2)$, where*

$$\sigma^2 = \mathrm{Var}(\mathrm{T_i}) + \frac{1}{\mathrm{l_1}}\mathrm{Var}(\mathrm{H}(\mathrm{T_{1i}})) + \frac{\mathrm{c^2 r^2}}{\mathrm{l_2^2}}\mathrm{Var}(\mathrm{T_{2i}}),$$

where

$$H(x) = x[c + r - 1 - F_1(x/r) - rF_2(x)] + r\mathbf{E}(1_{\{T_{1i} \leqslant x/r\}}T_{1i}) + r\mathbf{E}(1_{\{T_{2i} \leqslant x\}}T_{2i}),$$

$$c = \frac{1}{\mu^2}\int_0^\infty y[1 - F_2(y)]dF_1(y).$$

The test statistic is

$$Y_n = \frac{X}{\widehat{\sigma}}, \tag{13}$$

where $\widehat{\sigma}$ is a consistent estimator of σ and is estimated as

$$\widehat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n}(T_i - \widehat{\mu})^2 + \frac{n}{n_1^2}\sum_{i=1}^{n_1}[\widehat{H}(T_{1i}) - \widehat{\bar{H}}]^2 + \frac{\widehat{c}^2\widehat{r}^2 n}{n_2^2}\sum_{i=1}^{n_2}(T_{2i} - \widehat{\mu}_2)^2,$$

where

$$\widehat{\mu} = \frac{1}{n}\sum_{i=1}^{n}T_i,$$

$$\widehat{c} = \frac{1}{\widehat{\mu}_2}\int_0^\infty y[1 - \widehat{F}_2(y)]d\widehat{F}_1(y) = \frac{1}{\widehat{\mu}_2}\sum_{i=1}^{n_1}T_{1i}[1 - \widehat{F}_2(T_{1i})],$$

$$\widehat{H}(x) = x[\widehat{c} + \widehat{r} - 1 - \widehat{F}_1(x/\widehat{r}) - \widehat{r}\widehat{F}_2(x)] + \frac{\widehat{r}}{n_1}\sum_{i=1}^{n_1}1_{\{T_{1i} \leqslant x/\widehat{r}\}}T_{1i} + \frac{\widehat{r}}{n_2}\sum_{i=1}^{n_2}1_{\{T_{2i} \leqslant x\}}T_{2i},$$

$$\widehat{\bar{H}} = \frac{1}{n}\sum_{i=1}^{n_1}\widehat{H}(T_{1i}).$$

The distribution of the statistic Y_n is approximated by the standard normal distribution and the hypothesis H_0^* is rejected with approximative significance value α if $|Y_n| > z_{\alpha/2}$, where $z_{\alpha/2}$ is the $(\alpha/2)$ critical value of the standard normal distribution.

§8. Parametric point estimation of the distribution function of redundant system $S(1,1)$ using BS family of life distributions

Bagdonavičius *et al.* [4–7] studied asymptotic properties of point and interval estimators of reliability characteristics of redundant system for uncensored and censored data based on various parametric models like Weibull, lognormal, loglogistic, and generalized Weibull. They also give the nonparametric estimation of the cumulative distribution function but in this paper we study only the parametric estimation. We use their results to apply on the BS family of life distributions. From Figure 4 one can see the effect of additional stand-by unit on the cumulative distribution function K_2 of the system based on BS family.

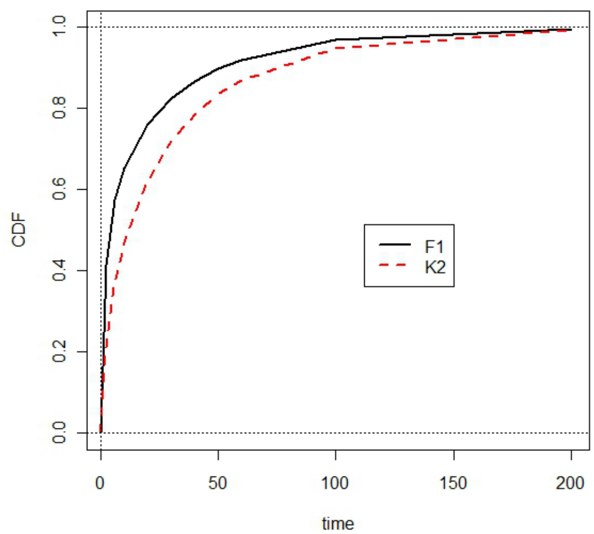

Fig. 4. Trajectories of the parametric estimators $\widehat{F_1}$ and $\widehat{K_2}$ (BS distribution).

Suppose that we have following censored data:

a) right censored sample $(X_{11}, \delta_{11}), \dots, (X_{1n_1}, \delta_{1n_1})$, of size n_1, where $X_{1i} = T_{1i} \wedge C_{1i}$, $\quad \delta_{1i} = \mathbf{1}_{\{T_{1i} \leqslant C_{1i}\}}$, T_{1i} are the failure time of units tested in hot conditions, C_{1i} are the censored times;

b) right censored sample $(X_{21}, \delta_{21}), \ldots, (X_{2n_1}, \delta_{2n_1})$, of size n_2, where $X_{2i} = T_{2j} \wedge C_{2j}$, $\quad \delta_{2j} = \mathbf{1}_{\{T_{2j} \leqslant C_{2j}\}}$, T_{2j} are the failure time of units tested in warm conditions, C_{2j} are the censored times.

Let denote $m_1 = \sum_{i=1}^{n_1} \delta_{1i}$, $m_2 = \sum_{j=1}^{n_2} \delta_{2j}$ and $m = m_1 + m_2$.

By using the data from above plan of experiment, the maximum likelihood estimator $\widehat{\gamma} = (\widehat{r}, \widehat{\boldsymbol{\theta}}^T)^T$ of the parameter $\gamma = (r, \boldsymbol{\theta}^T)^T = (r, \alpha, \beta)^T$ can be estimated from the following loglikelihood function

$$\begin{aligned}\ell(\gamma) = {} & \sum_{i=1}^{n_1} \delta_{1i} \ln f_1(T_{1i}; \boldsymbol{\theta}) + \sum_{i=1}^{n_1} (1 - \delta_{1i}) \ln S_1(T_{1i}; \boldsymbol{\theta}) + m_2 \ln r \\ & + \sum_{j=1}^{n_2} \delta_{2j} \ln f_2(rT_{2j}; \boldsymbol{\theta}) + \sum_{j=1}^{n_2} (1 - \delta_{2j}) \ln S_2(rT_{2j}; \boldsymbol{\theta}). \end{aligned} \tag{14}$$

The loglikelihood function in terms of Birnbaum–Saunders distribution can be written as

$$\begin{aligned}&\ell(\gamma) = -m(\ln \alpha + \ln \beta) + m_2 \ln r \\ &+ \sum_{i=1}^{n_1} \delta_{1i} \ln \left\{ \left(\frac{\beta}{T_{1i}} \right)^{\frac{1}{2}} + \left(\frac{\beta}{T_{1i}} \right)^{\frac{3}{2}} \right\} + \sum_{j=1}^{n_2} \delta_{2j} \ln \left\{ \left(\frac{\beta}{rT_{2j}} \right)^{\frac{1}{2}} + \left(\frac{\beta}{rT_{2j}} \right)^{\frac{3}{2}} \right\} \\ &- \frac{1}{2\alpha^2} \left[\sum_{i=1}^{n_1} \delta_{1i} \left(\frac{T_{1i}}{\beta} + \frac{\beta}{T_{1i}} - 2 \right) + \sum_{j=1}^{n_2} \delta_{2j} \left(\frac{rT_{2j}}{\beta} + \frac{\beta}{rT_{2j}} - 2 \right) \right] \\ &+ \sum_{i=1}^{n_1} (1 - \delta_{1i}) \ln S_1(T_{1i}; \alpha, \beta) + \sum_{j=1}^{n_2} (1 - \delta_{2j}) \ln S_2(rT_{2j}; \alpha, \beta). \end{aligned}$$

Maximum likelihood estimator $\widehat{\gamma}$ can be found by equating the score vector to zero, i.e. $\dot{\ell}(\gamma) = 0$.

From here we can obtain the maximum likelihood estimator for $K_2(t)$:

$$\begin{aligned}\widehat{K}_2(t) = {} & \frac{1}{2\sqrt{2\pi}\widehat{\alpha}\widehat{\beta}} \int_0^t \Phi \left\{ \frac{1}{\widehat{\alpha}} \left[\left(\frac{t + \widehat{r}y - y}{\widehat{\beta}} \right)^{1/2} - \left(\frac{\widehat{\beta}}{t + \widehat{r}y - y} \right)^{1/2} \right] \right\} \\ & \times \left\{ \left(\frac{\widehat{\beta}}{y} \right)^{1/2} + \left(\frac{\widehat{\beta}}{y} \right)^{3/2} \right\} \exp \left\{ -\frac{1}{2\widehat{\alpha}^2} \left(\frac{y}{\widehat{\beta}} + \frac{\widehat{\beta}}{y} - 2 \right) \right\} dy, \end{aligned} \tag{15}$$

where $\widehat{\alpha}, \widehat{\beta}, \widehat{r}$ are the maximum likelihood estimators.

The Fisher's information matrix is

$$I(\gamma) = -E\ddot{\ell}(\gamma),$$

and it may be replaced by $-\ddot{\ell}(\widehat{\gamma})$. For second partial derivative of the loglikelihood function we use

$$A_i = \frac{1}{\alpha}\left\{\left(\frac{T_{1i}}{\beta}\right)^{\frac{1}{2}} - \left(\frac{\beta}{T_{1i}}\right)^{\frac{1}{2}}\right\}, \quad B_j = \frac{1}{\alpha}\left\{\left(\frac{rT_{2j}}{\beta}\right)^{\frac{1}{2}} - \left(\frac{\beta}{rT_{2j}}\right)^{\frac{1}{2}}\right\}.$$

Second partial derivatives are

$$\frac{\partial^2 \ell}{\partial r^2} = -\frac{m_2}{r^2} + \frac{1}{2r^2}\sum_{j=1}^{n_2}\delta_{2j}\frac{1 + 3\frac{\beta}{rT_{2j}} + 3(\frac{\beta}{rT_{2j}})^2}{(1+\frac{\beta}{rT_{2j}})^2} - \frac{\beta}{\alpha^2 r^2}\sum_{j=1}^{n_2}\delta_{2j}\frac{1}{rT_{2j}}$$

$$+ \frac{1}{4r^2}\sum_{j=1}^{n_2}(1-\delta_{2j})\varphi(B_j)\frac{\begin{array}{c} S_2(rT_{2j};\alpha,\beta)\left[\frac{1}{\alpha}\left\{\left(\frac{rT_{2j}}{\beta}\right)^{\frac{1}{2}} + 3\left(\frac{\beta}{rT_{2j}}\right)^{\frac{1}{2}}\right\} + B_j^3 + \frac{4}{\alpha^2}B_j\right] \\ -\left\{B_j^2 + \frac{4}{\alpha^2}\right\}\varphi(B_j)\end{array}}{[S_2(rT_{2j};\alpha,\beta)]^2},$$

$$\frac{\partial^2 \ell}{\partial \alpha^2} = \frac{m}{\alpha^2} - \frac{3}{\alpha^2}\left[\sum_{i=1}^{n_1}\delta_{1i}A_i^2 + \sum_{j=1}^{n_2}\delta_{2j}B_j^2\right]$$

$$+ \frac{1}{\alpha^2}\sum_{i=1}^{n_1}(1-\delta_{1i})\frac{A_i\varphi(A_i)\left[\begin{array}{c} S_1(T_{1i};\alpha,\beta)\left[A_i^2 - 2\right] \\ -A_i\varphi(A_i)\end{array}\right]}{[S_1(T_{1i};\alpha,\beta)]^2}$$

$$+ \frac{1}{\alpha^2}\sum_{j=1}^{n_2}(1-\delta_{2j})\frac{B_j\varphi(B_j)\left[S_2(rT_{2j};\alpha,\beta)\left[B_j^2 - 2\right] - B_j\varphi(B_j)\right]}{[S_2(rT_{2j};\alpha,\beta)]^2},$$

$$\frac{\partial^2 \ell}{\partial \beta^2} = \frac{m}{\beta^2} - \frac{1}{2\beta^2}\sum_{i=1}^{n_1}\delta_{1i}\frac{1 + 2\frac{\beta}{T_{1i}} + 3(\frac{\beta}{T_{1i}})^2}{(1+\frac{\beta}{T_{1i}})^2}$$

$$+ \frac{1}{2\beta^2}\sum_{j=1}^{n_2}\delta_{2j}\frac{1 + 2\frac{\beta}{rT_{2j}} + 3(\frac{\beta}{rT_{2j}})^2}{(1+\frac{\beta}{rT_{2j}})^2} - \frac{1}{\alpha^2\beta^3}\left[\sum_{i=1}^{n_1}T_{1i} + r\sum_{j=1}^{n_2}T_{2j}\right]$$

$$- \frac{1}{4\beta^2}\sum_{i=1}^{n_1}(1-\delta_{1i})\varphi(A_i)\frac{\begin{array}{c} S_1(T_{1i};\alpha,\beta)\left[\frac{1}{\alpha}\left\{3\left(\frac{T_{1i}}{\beta}\right)^{\frac{1}{2}} + \left(\frac{\beta}{T_{1i}}\right)^{\frac{1}{2}}\right\} - \left\{A_i^2 + \frac{4}{\alpha^2}\right\}A_i\right] \\ +\left\{A_i^2 + \frac{4}{\alpha^2}\right\}\varphi(A_i)\end{array}}{(S_1(T_{1i};\alpha,\beta))^2}$$

$$- \frac{1}{4\beta^2} \sum_{j=1}^{n_2} (1-\delta_{2j}) \varphi(B_j) \frac{S_2(rT_{2j};\alpha,\beta)\left[\frac{1}{\alpha}\left\{3\left(\frac{rT_{2j}}{\beta}\right)^{\frac{1}{2}} + \left(\frac{\beta}{rT_{2j}}\right)^{\frac{1}{2}}\right\} - \left\{B_j^2 + \frac{4}{\alpha^2}\right\} B_j\right] + \left\{B_j^2 + \frac{4}{\alpha^2}\right\}\varphi(B_j)}{(S_2(rT_{2j};\alpha,\beta))^2},$$

$$\frac{\partial^2 \ell}{\partial\beta\partial r} = -\frac{1}{\beta r} \sum_{j=1}^{n_2} \delta_{2j} \frac{\left(\frac{\beta}{rT_{2j}}\right)}{\left(1+\frac{\beta}{rT_{2j}}\right)^2} + \frac{1}{2\alpha^2\beta} \sum_{j=1}^{n_2} \left(\frac{T_{2j}}{\beta} + \frac{\beta}{r^2 T_{2j}}\right)$$

$$+ \frac{1}{4\beta r} \sum_{j=1}^{n_2} (1-\delta_{2j}) \varphi(B_j) \frac{S_2(rT_{2j};\alpha,\beta) B_j \left[9 + 2\alpha^2 B_j^2\right] + \left\{B_j^2 + \frac{4}{\alpha^2}\right\}\varphi(B_j)}{(S_2(rT_{2j};\alpha,\beta))^2},$$

$$\frac{\partial^2 \ell}{\partial\alpha\partial\beta} = -\frac{1}{\alpha^3\beta}\left[\sum_{i=1}^{n_1}\left(\frac{T_{1i}}{\beta} - \frac{\beta}{T_{1i}}\right) + \sum_{j=1}^{n_2}\left(\frac{rT_{2j}}{\beta} - \frac{\beta}{rT_{2j}}\right)\right]$$

$$+ \frac{1}{2\alpha^2\beta} \sum_{j=1}^{n_2} (1-\delta_{1i}) \left\{\left(\frac{T_{1i}}{\beta}\right)^{\frac{1}{2}} + \left(\frac{\beta}{T_{1i}}\right)^{\frac{1}{2}}\right\} \varphi(A_i) \frac{S_1(T_{1i};\alpha,\beta)\left[A_i^2 - 1\right] - \varphi(A_i) A_i}{(S_1(T_{1i};\alpha,\beta))^2}$$

$$+ \frac{1}{2\alpha^2\beta} \sum_{j=1}^{n_2} (1-\delta_{2j}) \left\{\left(\frac{rT_{2j}}{\beta}\right)^{\frac{1}{2}} + \left(\frac{\beta}{rT_{2j}}\right)^{\frac{1}{2}}\right\} \varphi(B_j) \frac{S_2(rT_{2j};\alpha,\beta)\left[B_j^2 - 1\right] - \varphi(B_j) B_j}{(S_2(rT_{2j};\alpha,\beta))^2},$$

$$\frac{\partial^2 \ell}{\partial\alpha\partial r} = \frac{1}{\alpha^3} \sum_{j=1}^{n_2} \left(\frac{T_{2j}}{\beta} - \frac{\beta}{r^2 T_{2j}}\right)$$

$$+ \frac{1}{2r\alpha^2} \sum_{j=1}^{n_2} (1-\delta_{2j}) \varphi(B_j) \left\{\left(\frac{rT_{2j}}{\beta}\right)^{\frac{1}{2}} + \left(\frac{\beta}{rT_{2j}}\right)^{\frac{1}{2}}\right\}$$

$$\times \frac{S_2(rT_{2j};\alpha,\beta)\left[1 - B_j^2\right] + B_j \varphi(B_j)}{(S_2(rT_{2j};\alpha,\beta))^2}.$$

§9. Asymptotic confidence interval for $K_2(t)$

Using the results from Bagdonavičius *et al.* [7] we can construct the asymptotic $(1-\alpha)$ confidence interval $(\underline{K}_2(t), \overline{K}_2(t))$ for $K_2(t)$, where

$$\underline{K}_2(t) = \left(1 + \frac{1-\widehat{K}_2(t)}{\widehat{K}_2(t)} \exp\left\{\frac{\widehat{\sigma}_{\widehat{K}_2} z_{1-\alpha/2}}{\sqrt{\widehat{K}_2(t)(1-\widehat{K}_2(t))}}\right\}\right)^{-1},$$

$$\overline{K}_2(t) = \left(1 + \frac{1-\widehat{K}_2(t)}{\widehat{K}_2(t)} \exp\left\{-\frac{\widehat{\sigma}_{\widehat{K}_2} z_{1-\alpha/2}}{\sqrt{\widehat{K}_2(t)(1-\widehat{K}_2(t))}}\right\}\right)^{-1},$$

where

$$\widehat{\sigma}^2_{\widehat{K}_2(t)} = C_2^T(t,\widehat{\gamma})I^{-1}(\widehat{\gamma}_1)C_2(t,\widehat{\gamma}),$$

$$C_2(t,\gamma) = (C_{21}(t,\gamma), C_{22}(t,\gamma), C_{23}(t,\gamma))^T,$$

$$C_{21}(t,\gamma) = \int_0^t \frac{\partial F_1}{\partial r}(t+ry-y;\alpha,\beta)dF_1(y;\alpha,\beta),$$

$$C_{22}(t,\gamma) = \int_0^t \frac{\partial F_1}{\partial \alpha}(t+ry-y;\alpha,\beta)dF_1(y,\alpha,\beta) + F_1(t+ry-y;\alpha,\beta)d(\frac{\partial F_1}{\partial \alpha}(y;\alpha,\beta)),$$

$$C_{23}(t,\gamma) = \int_0^t \frac{\partial F_1}{\partial \beta}(t+ry-y;\alpha,\beta)dF_1(y,\alpha,\beta) + F_1(t+ry-y;\alpha,\beta)d(\frac{\partial F_1}{\partial \beta}(y;\alpha,\beta)).$$

Bagdonavičius *et al.* [4] study the reliability characteristics of redundant system $S(1, m-1)$, where they consider $(m-1)$ warm stand-by units. We can also generalize this with multiple stand-by units whose failure times follow BS family of life distributions to increase the reliability. But increase in number of stand-by units increases the cost of the system. So, the number of stand-by units are optimized in any given practical situation with respect to the reliability measures.

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University of Bordeaux, IMB,
UMR CNRS 5251, France
E-mail: `mikhail.nikouline@u-bordeaux2.fr`

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