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BENEŠ CONDITION FOR DISCONTINUOUS EXPONENTIAL MARTINGALE

Abstract. It is known that the Girsanov exponent $\mathfrak{z}_t$, being solution of Doléans-Dade equation $\mathfrak{z}_t = 1 + \int_0^t \mathfrak{z}_s\, \alpha(s) dB_s$ generated by Brownian motion B_t and a random process $\alpha(t)$ with $\int_0^t \alpha^2(s)\,ds < \infty$ a.s., is the martingale provided that the Beneš condition

$$|\alpha(t)|^2 \leqslant \text{const.}\big[1 + \sup_{s\in[0,t]} B_s^2\big],\ \forall\ t > 0,$$

holds true. In this paper, we show that $\int_0^t \alpha(s)dB_s$ can be replaced by a purely discontinuous square integrable martingale M_t paths from the Skorokhod space $\mathbb{D}_{[0,\infty)}$ having jumps $\alpha(s) \triangle M_t > -1$. The method of proof differs from the original Beneš proof.

§1. Introduction and main result

Let B_t be a Brownian motion process adapted to a filtration $(\mathcal{F}_t)$. The process $\alpha(t)$ is predictable relative $(\mathcal{F}_t)$ too and $\int_0^t \alpha^2(s)\,ds < \infty$, a.s., $t > 0$. It is well known that classical Girsanov's exponential process

$$\mathfrak{z}_t = \exp\bigg(\int_0^t \alpha(s)\,dB_s - \frac{1}{2}\int_0^t \alpha^2(s)\,ds\bigg)$$

is a positive local martingale (and as such a supermartingale) with $\mathsf{E}\,\mathfrak{z}_t \leqslant 1$. Also it is well known that the random process $(\mathfrak{z}_t)_{t\in[0,T]}$ is a martingale provided that

$$\mathsf{E}\,\mathfrak{z}_T = 1. \tag{1.1}$$

Sufficient conditions for (1.1) to hold represent an important question both in theory (e.g. [2, 3, 8, 9, 12]) and applications ([1] etc).

In this paper, we make an attempt to replace B_t by a purely discontinuous square integrable martingale M_t with $M_0 = 0$ and paths from Skorokhod space $\mathbb{D}_{[0,\infty)}$ of right continuous functions having limits to the left (hereafter $\langle M\rangle_t$, M_{t-}, and $\triangle M_t := M_t - M_{t-}$ denote the predictable quadratic variation, left limit, and jump process of M_t).

Key words and phrases: Girsanov's exponential martingale, uniform integrability.

Throughout this paper we fix a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,\infty)}, \mathsf{P})$ satisfying the "general conditions" (for more details, see e.g. [4] or [10]).

In order to clarify our setting notice first that $\mathfrak{z}_t$ is nothing but a multiplicative decomposition of $\exp\big(\int\limits_0^t \alpha(s)\,dB_s\big)$ up to a local martingale:

$$\mathfrak{z}_t = \exp\Big(\int\limits_0^t \alpha(s)\,dB_s\Big)\exp\Big(-\frac{1}{2}\int\limits_0^t \alpha^2(s)\,ds\Big).$$

Replacing $\int_0^t \alpha(s)\,dB_t$ by $\int_0^t \alpha(s)\,dM_t$ we have to deal with the following multiplicative decomposition: with $\alpha(s)\,\triangle M_s > -1$,

$$\begin{aligned}\mathfrak{z}_t = \exp&\Big(\int\limits_0^t \alpha(s)\,dM_s\Big)\\ &\times\exp\Big(-\sum_{s\in[0,t]}\Big\{\alpha(s)\,\triangle M_s - \log\big[1+\alpha(s)\,\triangle M_s\big]\Big\}\Big). \quad (1.2)\end{aligned}$$

Formula (1.2) is nothing but the unique solution of the Doléans-Dade equation ($\mathfrak{z}_{t-} = \lim\limits_{s\uparrow t}\mathfrak{z}_s$)

$$\mathfrak{z}_t = 1 + \int\limits_0^t \mathfrak{z}_{s-}\,\alpha(s)\,dM_s. \qquad (1.3)$$

We assume that the martingale M_t preserves main property of Brownian motion to be homogenous process with independent increments. In particular, this means $\langle M\rangle_t$ is in proportion to t, that is, $\langle M\rangle_t = \lambda t$, where λ is a positive constant. Denote $\mu(dt,dz)$ an integer-valued random measure associated with the jump process $\triangle M_t$: for any interval $[0,t]$ and any measurable set $\Gamma \in \mathbb{R}_+ \setminus \{0\}$,

$$\mu([0,t]\times\Gamma) = \sum_{s\in[0,t]} I_{\{\triangle M_s \in \Gamma\}}.$$

Aforementioned assumptions of M_t impose a structure of $\nu(dt,dz)$ the compensator of $\mu(dt,dz)$ (for more details of compensator definition see, e.g., [4, 10]):

$$\nu(dt,dz)=dt\,K(dz),$$
$$K(dz)\quad\text{is }\sigma\text{-finite measure},\quad\int\limits_{\mathbb{R}_+}z^2K(dz)=\lambda.$$

Then, in particular, M_t and $\langle M\rangle_t$ can be presented as:

$$M_t=\int\limits_0^t\int\limits_{\mathbb{R}_+}z\big[\mu(ds,dz)-K(dz)\,ds\big]\quad\text{"Itô's integral"}$$
$$\langle M\rangle_t=\int\limits_0^t\int\limits_{\mathbb{R}_+}z^2K(dz)\,ds\equiv\lambda t.$$

The aim of this paper is to find a reasonable condition providing $\mathsf{E}\,\mathfrak{z}_T=1$.

Formula (1.2) does not give any hope of a reasonable adaptation of Novikov [12], Kazamaki [7] and latest Krylov [9] conditions guaranteeing $\mathsf{E}\,\mathfrak{z}_T=1$.

However, we found that the Beneš type condition [2] (see also Karatzas and Shreve [6] and Üstünel and Zakai [13])

$$\text{“}\Big\{\alpha^2(t)\leqslant\text{const.}\big[1+\sup_{s\in[0,t]}B_s^2\big]\Big\}_{t\in[0,T]}\text{”}\Rightarrow\text{“}\mathsf{E}\,\mathfrak{z}_T=1\text{”}\tag{1.4}$$

is compatible with M_t-setting. We propose a new proof of (1.4). It is essentially different from [2], and serves the following theorem.

Theorem 1.1. *Assume* $\int_{\mathbb{R}_+}|z|^3K(dz)<\infty$. *Then for any* $T>0$ (*comp.* (1.4))

$$\text{“}\Big\{\alpha^2(t)\leqslant\text{const.}\Big[1+\sup_{s\in[0,t]}M_{s-}^2\Big]\Big\}_{t\in[0,T]}\text{”}\Rightarrow\text{“}\mathsf{E}\,\mathfrak{z}_T=1\text{”}.\tag{1.5}$$

§2. Preliminaries

We consider first a case $|\alpha(t)|<\mathbf{r}$. Formally, the idea of proof in this setting is traditional and only details are new.

Since $\mathfrak{z}_t$ is a local martingale there exists a localizing sequence of stopping times $(\tau_n)_{n\geqslant1}$, $\tau_n\uparrow\infty$ such that $\mathsf{E}\,\mathfrak{z}_{T\wedge\tau_n}\equiv1$ and $\mathfrak{z}_{T\wedge\tau_n}\xrightarrow[n\to\infty]{\text{a.s.}}\mathfrak{z}_T$,

$\mathsf{E}\,\mathfrak{z}_{T\wedge\tau_n} \equiv 1$. Hence, it is left to verify an uniform integrability of the family $\{\mathfrak{z}_{T\wedge\tau_n}\}_{n\to\infty}$ providing

$$1 \equiv \lim_{n\to\infty} \mathsf{E}\,\mathfrak{z}_{T\wedge\tau_n} = \mathsf{E}\lim_{n\to\infty} \mathfrak{z}_{T\wedge\tau_n} = \mathsf{E}\,\mathfrak{z}_T. \tag{2.1}$$

A trivial part of this proof is related to a bounded process $\alpha(t)$. Set $\tau_n = \inf\{t : \mathfrak{z}_t \geqslant n\}$, $\mathfrak{z}^n_t = \mathfrak{z}_{t\wedge\tau_n}$ and notice that $\mathfrak{z}_{(t\wedge\tau_n)-} \leqslant n$. In view of (1.3)

$$\mathfrak{z}^n_t = 1 + \int\limits_0^{t\wedge\tau_n} \mathfrak{z}_{s-}\,\alpha(s)\,dM_s = 1 + \int\limits_0^t I_{\{s\leqslant\tau_n\}}\mathfrak{z}^n_{s-}\,\alpha(s)\,dM_s.$$

Since $I_{\{s\leqslant\tau_n\}}\,\mathfrak{z}^n_{s-}\,|\alpha(s)| \leqslant n\mathbf{r}$, the process $\mathfrak{z}^n_t$ is a square integrable martingale and

$$\mathsf{E}\,(\mathfrak{z}^n_t)^2 = 1 + \mathsf{E}\int\limits_0^t \mathfrak{z}^n_{s-}\,I_{\{s\leqslant\tau_n\}}\,\alpha^2(s)\,d\langle M\rangle_s \ (\leqslant 1 + (n\mathbf{r})^2\lambda t). \tag{2.2}$$

Hence $\mathfrak{z}^n_t$ is the martingale with $\mathsf{E}\,\mathfrak{z}^n_T = 1$ for any $T > 0$. Since $\mathfrak{z}^n_T \xrightarrow[n\to\infty]{\text{a.s.}} \mathfrak{z}_T$, (2.1) holds true if $\sup\limits_n \mathsf{E}\,(\mathfrak{z}^n_T)^2 < \infty$. The latter condition is nothing but the Vallée-Poussin criteria of the uniform integrability of $\{\mathfrak{z}^n_T\}_{n\to\infty}$. Since $I_{\{s\leqslant\tau_n\}}\,\alpha^2(s)\,d\langle M\rangle_s \leqslant \mathbf{r}^2\lambda\,ds$, (2.2) can be transformed into an inequality $\mathsf{E}\,(\mathfrak{z}^n_t)^2 \leqslant 1 + \mathbf{r}\lambda\int_0^t \mathsf{E}\,(\mathfrak{z}^n_{s-})^2\,ds \equiv 1 + \mathbf{r}\lambda\int_0^t \mathsf{E}\,(\mathfrak{z}^n_s)^2\,ds$.

In other words, a bounded function $V^n_t := \mathsf{E}\,(\mathfrak{z}^n_t)^2$ solves Gronwall–Bellman inequality $V^n_t \leqslant 1 + \mathbf{r}\lambda\int\limits_0^t V^n_s\,ds$ providing $V^n_T \leqslant e^{\mathbf{r}\lambda T}$.

§3. The proof of Theorem 1.1 with unbounded $\alpha^2(t)$

1. Since $\alpha^2(t)$ is unbounded function we shall use (1.5).

Choose a stopping time $\tau_n = \{t : M^2_{t-} \geqslant n\}$ and use (1.5) with generic constant $\mathbf{r}$ [1]

$$I_{\{t\leqslant\tau_n\}}\,\alpha^2(t) \leqslant \mathbf{r}\big[1 + \sup_{s\in[0,t\wedge\tau_n]} M^2_{s-}\big].$$

Then, with $\mathfrak{z}^n_t = \mathfrak{z}_{t\wedge\tau_n}$ and $\alpha^n(s) = I_{\{s\leqslant\tau_n\}}\,\alpha(s)$, one holds $\mathfrak{z}^n_t = 1 + \int\limits_0^t \mathfrak{z}^n_{s-}\,\alpha^n(s)\,dM_s$. Since $M^2_{(t\wedge\tau_n)-} \leqslant n$, we have $(\alpha^n(s))^2 \leqslant \mathbf{r}\big[1 + n\big]$. So,

[1] Taking different values at different appearances.

according to the result of Section 2, $\mathsf{E}\,\mathfrak{z}_T^n \equiv 1$. Aforementioned property guarantees the existence of a new probability measure $\mathsf{P}_T^n \ll \mathsf{P}$ with the density $\frac{d\mathsf{P}_T^n}{d\mathsf{P}} = \mathfrak{z}_{T\wedge\tau_n}$ Now, we shall verify the uniform integrability of the family $\{\mathfrak{z}_T^n\}_{n\to\infty}$ with a help of Vallée-Poussin's criteria with a function $\psi(x)$ different from $f(x) = x^2$ as in Section 2. Following Hitsuda [5], we choose

$$\psi(x) = \begin{cases} x\log(x) + 1 - x, & x > 0, \\ 1, & x = 0, \end{cases} \qquad \lim_{x\to\infty} \frac{\psi(x)}{x} = \infty$$

and prove that

$$\sup_n \ \mathsf{E}\,\psi(\mathfrak{z}_T^n) < \infty. \tag{3.1}$$

Denote E_T^n the expectation symbol of new measure P_T^n. Then (3.1) can be transformed into

$$\sup_n \ \mathsf{E}_T^n \log\big(\mathfrak{z}_T^n\big) < \infty. \tag{3.2}$$

In view of $\alpha(s)\,\triangle M_s \geqslant -1$ and (1.2), we have

$$\begin{aligned} &\log\big(\mathfrak{z}_T^n\big) \\ &= \int_0^{T\wedge\tau_n} \alpha(s)\,dM_s - \sum_{s\in[0,T\wedge\tau_n]} I_{\{s\leqslant\tau_n\}}\Big\{\underbrace{\alpha(s)\,\triangle M_s - \log\big[1+\alpha(s)\,\triangle M_s\big]}_{\geqslant 0}\Big\} \\ &\leqslant \int_0^T I_{\{s\leqslant\tau_n\}}\,\alpha(s)\,dM_s. \end{aligned}$$

Hence, (3.2) holds true provided that

$$\sup_n \ \mathsf{E}_T^n \int_0^T I_{\{s\leqslant\tau_n\}}\,\alpha(s)\,dM_s < \infty. \tag{3.3}$$

A verification of (3.3) requires a description of $M_{t\wedge\tau_n}$ as P_T^n-semimartingale.

Lemma 3.1. *$M_{t\wedge\tau_n} = A^n_t + M^n_t$ with*

$$A^n_t = \int_0^t \int_{\mathbb{R}_+} z^2\, I_{\{s\leqslant\tau_n\}}\, \alpha(s)\, K(dz)\, ds$$

$$M^n_t = \int_0^t \int_{\mathbb{R}_+} I_{\{s\leqslant\tau_n\}}\, z\, [\mu(ds,dz) - \nu^n(ds,dz)],$$

where $\nu^n(dt,dz) = I_{\{t\leqslant\tau_n\}}[1+\alpha(t)z]\,K(dz)dt$ is P^n_T-compensator of $I_{\{t\leqslant\tau_n\}}\,\mu(dt,dz)$ such that M^n_t is a square integrable martingale having

$$\langle M^n\rangle_t = \int_0^t \int_{\mathbb{R}_+} z^2\, I_{\{s\leqslant\tau_n\}}[1+\alpha(t)z]\,K(dz)dt.$$

Proof. It is well known that $\mathsf{P}^n_T \ll \mathsf{P}$ implies that P-martingale $M_{t\wedge\tau_n}$ remains P^n_T-semimartingale having a structure $M_t = A^n_t + M^n_t$, where A^n_t is the drift and M^n_t is a local martingale:

$$M^n_t = \int_0^t \int_{\mathbb{R}_+} I_{\{s\leqslant\tau_n\}}\, z\, [\mu(ds,dz) - \nu^n(ds,dz)],$$

where $\nu^n(ds,dz)$ is P^n_T-compensator of the integer-valued measure $I_{\{s\leqslant\tau_n\}}\,\mu(ds,dz)$. In order to compute $\nu^n(ds,dz)$ one can use a standard procedure. Take predictable and bounded function $u(s,z)$ equals zero in a vicinity of $\{z=0\}$ and write

$$\begin{aligned}
\mathsf{E}^n_T \int_0^T \int_{\mathbb{R}_+} u(s,z)\, I_{\{s\leqslant\tau_n\}}\, \mu(ds,dz) &= \mathsf{E}\, \mathfrak{z}^n_T \int_0^T \int_{\mathbb{R}_+} u(s,z)\, I_{\{s\leqslant\tau_n\}}\, \mu(ds,dz)\\
&= \mathsf{E} \int_0^T \int_{\mathbb{R}_+} u(s,z)\, \mathfrak{z}^n_s\, I_{\{s\leqslant\tau_n\}}\, \mu(ds,dz)\\
&= \mathsf{E} \int_0^T \int_{\mathbb{R}_+} u(s,z)\, \frac{\mathfrak{z}^n_s}{\mathfrak{z}^n_{s-}}\, \mathfrak{z}^n_{s-}\, I_{\{s\leqslant\tau_n\}}\, \mu(ds,dz).
\end{aligned}$$

Due to Doléans-Dade's equation (1.3), $\frac{\mathfrak{z}^n_s}{\mathfrak{z}^n_{s-}} = \big[1 + I_{\{s\leqslant\tau_n\}}\,\alpha(s)\triangle M_s\big]$. Notice also that

$$\int\limits_{\mathbb{R}_+} u(s,z)\triangle M_s\, I_{\{s\leqslant\tau_n\}}\,\mu(ds,dz) = \int\limits_{\mathbb{R}_+} u(s,z)z\, I_{\{s\leqslant\tau_n\}}\,\mu(ds,dz).$$

Therefore,

$$\begin{aligned}
&\mathsf{E}^n_T \int\limits_0^T\int\limits_{\mathbb{R}_+} u(s,z)\, I_{\{s\leqslant\tau_n\}}\,\mu(ds,dz)\\
&= \mathsf{E}\int\limits_0^T\int\limits_{\mathbb{R}_+} u(s,z)\mathfrak{z}^n_{s-}[1 + I_{\{s\leqslant\tau_n\}}\,\alpha(s)\triangle M_s]\, I_{\{s\leqslant\tau_n\}}\,\mu(ds,dz)\\
&= \mathsf{E}\int\limits_0^T\int\limits_{\mathbb{R}_+} u(s,z)\mathfrak{z}^n_{s-}\, I_{\{s\leqslant\tau_n\}}[1 + \alpha(s)z]\,\mu(ds,dz)\\
&= \mathsf{E}\int\limits_0^T\int\limits_{\mathbb{R}_+} u(s,z)\mathfrak{z}^n_{s-}\, I_{\{s\leqslant\tau_n\}}[1 + \alpha(s)z]\,K(dz)\,ds\\
&= \mathsf{E}\,\mathfrak{z}^n_T\int\limits_0^T\int\limits_{\mathbb{R}_+} u(s,z)\, I_{\{s\leqslant\tau_n\}}[1 + \alpha(s)z]\,K(dz)\,ds\\
&= \mathsf{E}^n_T\int\limits_0^T\int\limits_{\mathbb{R}_+} u(s,z)\, I_{\{s\leqslant\tau_n\}}[1 + \alpha(s)z]\,K(dz)\,ds.
\end{aligned}$$

Thus, the arbitrariness of $u(s,z)$ implies

$$\nu^n(dt,dz) = I_{\{t\leqslant\tau_n\}}\,[1 + \alpha(t)z]\,K(dz)\,dt.$$

Also $\int\limits_{\mathbb{R}_+} z^2K(dz) < \infty$ and $\int\limits_{\mathbb{R}_+} |z|^3K(dz) < \infty$ imply

$$\int\limits_0^t\int\limits_{\mathbb{R}_+} z^2\nu^n(ds,dz) = \int\limits_0^t\int\limits_{\mathbb{R}_+} I_{\{s\leqslant\tau_n\}}[z^2 + |z|^3\,\alpha(s)]\,K(dz)\,ds$$

$$\leqslant \mathbf{r}\Big[t + \int_0^t I_{\{s\leqslant\tau_n\}}\,\alpha(s)\,ds\Big]$$

$$\leqslant \mathbf{r}\Big[t + \int_0^t I_{\{s\leqslant\tau_n\}}\big(1+\alpha^2(s)\big)\,ds\Big]$$

$$\leqslant \mathbf{r}\Big[t + \int_0^t I_{\{s\leqslant\tau_n\}}\,\alpha^2(s)\,ds\Big] \leqslant \text{const.}$$

This property of $\nu^n(ds,dz)$ guarantees the existence of P^n_T-square integrable martingale M^n_t with $\langle M^n\rangle_t$ announced in the lemma. The formula of A^n_t, announced in the lemma, follows from

$$A^n_t := M_{t\wedge\tau_n} - M^n_t = \int_0^t \int_{\mathbb{R}_+} z^2\, I_{\{s\leqslant\tau_n\}}\,\alpha(s)\,K(dz)\,ds.$$

□

2. Taking into account Lemma 3.1 and continue the proof of theorem we transform (3.3) into

$$\sup_n\ \mathsf{E}^n_T \int_0^T I_{\{s\leqslant\tau_n\}}\,\alpha(s)\,dA^n_s + \sup_n\ \mathsf{E}^n_T \int_0^T I_{\{s\leqslant\tau_n\}}\,\alpha(s)\,dM^n_s < \infty$$

and even into

$$\sup_n\ \mathsf{E}^n_T \int_0^T I_{\{s\leqslant\tau_n\}}\,\alpha(s)\,dA^n_s < \infty \tag{3.4}$$

since $\mathsf{E}^n_T \int_0^T I_{\{s\leqslant\tau_n\}}\,\alpha(s)\,dM^n_s = 0$. Thus, (3.4) is the sufficient condition of $\mathsf{E}\,\mathfrak{z}_T = 1$ to hold. Note that (3.4) is equivalent to

$$\sup_n\ \mathsf{E}^n_T \int_0^T \int_{\mathbb{R}_+} I_{\{s\leqslant\tau_n\}}\,\alpha^2(s)\,z^2 K(dz)\,ds < \infty.$$

Also, $\int_{\mathbb{R}_+} z^2 K(dz)\,ds = \lambda$ enables to simplify (3.4) up to

$$\sup_n\ \mathsf{E}^n_T \int_0^T I_{\{s\leqslant\tau_n\}}\,\alpha^2(s)\,ds < \infty.$$

For notational convenience set

$$V_t^n = \mathsf{E}_T^n \, I_{\{t\leqslant\tau_n\}} \, \alpha^2(t)$$

and show that V_t^n solves the Gronwall–Bellman inequality

$$V_t^n \leqslant \mathbf{r}\Big[1 + \int_0^t V_s^n \, ds\Big] \tag{3.5}$$

for any number n and some positive constant $\mathbf{r}$ independent of n. If (3.5) holds, then $\int_0^T V_s^n \, ds \leqslant e^{\mathbf{r}T} - 1$ and (3.4) holds true.

3. In this section we derive (3.5) with the help of Beneŝ' condition from (1.5). It implies $I_{\{t\leqslant\tau_n\}} \, \alpha^2(t) \leqslant \mathbf{r}\Big[1 + \sup_{s\in[0,t\wedge\tau_n]} M_{s-}^2\Big]$ and, therefore,

$$V_t^n \leqslant \mathbf{r}\bigg[1 + \mathsf{E}_T^n \sup_{s\in[0,t\wedge\tau_n]} M_{s-}^2\bigg]. \tag{3.6}$$

Due to $M_{t\wedge\tau_n} = A_t^n + M_t^n$, the following upper bound holds:

$$\begin{aligned}
\sup_{s\in[0,t\wedge\tau_n]} M_{s-}^2 \leqslant \sup_{s\in[0,t\wedge\tau_n]} M_s^2 &\leqslant 2\Big[\sup_{s\leqslant t}(A_t^n)^2 + \sup_{s\leqslant t}(M_t^n)^2\Big] \\
&= 2\bigg[\bigg(\int_0^t \int_{\mathbb{R}_+} z^2 \, I_{\{s\leqslant\tau_n\}} \, \alpha(s) \, K(dz) \, ds\bigg)^2 \\
&\quad + 2\bigg(\sup_{t'\leqslant t} \int_0^{t'} \int_{\mathbb{R}_+} I_{\{s\leqslant\tau_n\}} \, z \, [\mu(ds,dz) - \nu^n(ds,dz)]\bigg)^2\bigg].
\end{aligned}$$

By applying the Cauchy–Schwarz inequality, we obtain

$$\begin{aligned}
\mathsf{E}_T^n \bigg(\int_0^t \int_{\mathbb{R}_+} z^2 \, I_{\{s\leqslant\tau_n\}} \, \alpha(s) \, K(dz) \, ds\bigg)^2 &\leqslant \lambda^2 \mathsf{E}_T^n \int_0^t I_{\{s\leqslant\tau_n\}} \, \alpha^2(s) \, ds \\
&= \lambda^2 \int_0^t V_s^n \, ds.
\end{aligned}$$

and by the maximal Doob inequality we find that

$$\begin{aligned}
&\mathsf{E}_T^n\Big(\sup_{t'\leqslant t}\int_0^{t'}\int_{\mathbb{R}_+} I_{\{s\leqslant\tau_n\}}\, z\,[\mu(ds,dz)-\nu^n(ds,dz)]\Big)^2\\
&\leqslant 4\mathsf{E}_T^n\int_0^t\int_{\mathbb{R}_+} I_{\{s\leqslant\tau_n\}} z^2[1+\alpha(s)z]\,K(dz)dt\\
&\leqslant 4t\int_{\mathbb{R}_+} z^2K(dz)+4\int_{\mathbb{R}_+} z^3K(dz)\int_0^t I_{\{s\leqslant\tau_n\}}\,\alpha(s)\,ds\\
&\leqslant 4t\int_{\mathbb{R}_+} z^2K(dz)+4\int_{\mathbb{R}_+} z^3K(dz)\int_0^t I_{\{s\leqslant\tau_n\}}[1+\alpha^2(s)]\,ds\\
&\leqslant 4T\lambda+4T\int_{\mathbb{R}_+} z^3K(dz)+4T\int_{\mathbb{R}_+} z^3K(dz)\int_0^t V_s^n\,ds.
\end{aligned}$$

All these results provide $\mathsf{E}_T^n \sup_{s\in[0,t\wedge\tau_n]} M_{s-}^2 \leqslant \mathbf{r}\Big[1+\int_0^t V_s^n\,ds\Big]$.

Thus, (3.5) is valid with $\mathbf{r}$ independent of n. □

§4. Example

Set $M_t=\Pi_t-t$ and $\alpha(t)=h(\Pi_{t-})$, where Π_t is Poisson process with the unit parameter and $h(x)$ is a measurable function satisfying the linear grows condition:

$$|h(x)|\leqslant\mathbf{r}[1+|x|].$$

For this model of $\alpha(s)$ the Beneŝ holds, so that, the Doléans-Dade process

$$\mathfrak{z}_t=1+\int_0^t \mathfrak{z}_{s-}\Big(\int_0^s h(\Pi_{s'}-s')\,ds'\Big)\,d[\Pi_s-s]$$

is the martingale with $\mathsf{E}\,\mathfrak{z}_T\equiv 1$.

References

1. L. B. Andersen, V. V. Piterbarg, *Moment explosions in stochastic volatility models.* — Finance Stoch. **11** (2007), 29–50; DOI 10.1007/s00780-006-0011-7.
2. V. E. Benes, *Existence of optimal stochastic control laws.* — SIAM J. Control **9** (1971), 446–475.
3. I. V. Girsanov, *On transforming a certan class of stochastic processes by absolutely continuous substitution of measures.* — Theory Probab. Appl. **5** (1960), 285–301.
4. J. Jacod, A. N. Shiryaev, *Limit theorems for stochastic processes.* 2nd ed. Springer-Verlag, Berlin (2003).
5. M. Hitsuda, *Representation of Gaussian processes equivalent to Wiener process.* — Osaka J. Math. **5** (1968), 299–312.
6. I. Karatzas, S. E. Shreve, *Brownian Motion and Stochastic Calculus.* Springer-Verlag, New York–Berlin–Heidelberg (1991).
7. N. Kazamaki, *On a problem of Girsanov.* — Tôhoku Math. J. **29** (1977), 597–600.
8. N. Kazamaki, T. Sekiguchi, *On the transformation of some classes of martingales by a change of law.* — Tôhoku Math. J. **31** (1979), 261–279.
9. N. V. Krylov, *A simple proof of a result of A. Novikov* (May 8, 2009); arXiv:math/020713v2 [math.PR].
10. R. Sh. Liptser, A. N. Shiryayev, *Theory of Martingales.* Kluwer Acad. Publ. (1989).
11. R. Sh. Liptser, A. N. Shiryaev, *Statistics of Random Processes.* I. 2nd ed., Springer-Verlag, Berlin–New York (2000).
12. A. A. Novikov, *On the conditions of the uniform integrability of the continuous nonnegative martingales.* — Theory Probab. Appl. **24** (1979), No. 4, 821–825.
13. A. Üstünel, M. Zakai, *Transformation of measure on Wiener space.* Springer-Verlag, Berlin–New York (2000).

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