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INVERSE SOURCE PROBLEM FOR THE 1-D SCHRÖDINGER EQUATION

Abstract. We consider the inverse problem of determining a source in the dynamical Schrödinger equation $iu_t - u_{xx} + q(x)u = w(t)a(x)$, $0 < x < 1$, with Dirichlet boundary conditions and zero initial condition. From the measurement $u_x(0,t)$, $0 < t < T$, we recover unknown $a(x)$ provided $q(x)$ and $w(t)$ are given. We describe also how to recover $a(x)$ and $q(x)$ from the measurements at the both boundary points.

§1. Introduction

We consider the initial boundary value problem (IBVP)

$$iu_t(x,t) - u_{xx}(x,t) + q(x)u(x,t) = w(t)a(x), \qquad 0 < x < 1, \quad 0 < t < T, \tag{1.1}$$

$$u(0,t) = u(1,t) = 0, \tag{1.2}$$

$$u(x,0) = 0, \tag{1.3}$$

where T is an arbitrary positive number, the (real-valued) potential q belongs to $L_1(0,1)$, $w \in H^1(0,T)$ and $w(0) \neq 0$. The function $a \in H_0^1(0,1)$ is unknown and has to be found from the trace of the derivative of the solution u to the IBVP (1.1)–(1.3) at a boundary point: $\mu(t) := u_x(0,t)$, $t \in [0,T]$.

The solution of this inverse problem is presented in Sec. 2. It is based on the version of the Boundary Control (BC) method proposed in [3]. (The review of the history and achievements of the BC method can be found in [9].) We derive the integral equation, which eventually allows us to recover the Fourier coefficients of the unknown function a.

This approach is especially efficient in the case of zero potential and a special choice of the intensity $w(t)$. Another approach to such a kind

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inverse problem was discussed in [15]) for the wave equation. In papers [12] and [13], the authors used observers techniques and developed an efficient algorithm recovering initial data in the wave and Euler–Bernoulli plate equations.

Control theoretic ideas in the inverse source problem for hyperbolic equations were used by Yamamoto in [16]. His result was based on the exact controllability of the corresponding dynamical system. Notice that our approach is, in general, less restrictive and requires only the spectral controllability.

The proposed method can be applied to the inverse source problem for parabolic, wave and beam equations (cf. [3]). The generalization of our approach to the case of abstract differential equation in Hilbert space will be discussed in a forthcoming publication.

In Sec. 3, we assume that both the potential q and source a are unknown and the derivative u_x is measured at the both boundary points $x = 0, 1$; $t \in [0, T]$. We demonstrate that the spectral function of the corresponding spectral problem can be recovered from these data. Then one can recover the potential and source using either classical Gelfand–Levitan–Krein approach or the Boundary Control method (see [7]). In more detail the algorithm solving this problem and stability estimates are discussed in [8].

§2. Observation at one end of the interval

We consider the Sturm–Liouville problem associated with (1.1), (1.2):

$$-\phi''(x) + q(x)\phi(x) = \lambda\phi(x), \quad 0 < x < 1, \tag{2.1}$$

$$\phi(0) = \phi(1) = 0. \tag{2.2}$$

The following facts are well-known.
(a) *Its spectrum* $\{\lambda_k\}_{k=1}^{\infty}$ *of the problem* (2.1), (2.2) *is pure discrete, simple, and real with the only point of accumulation at* $+\infty$.
(b) *The asymptotic representation holds,*

$$\sqrt{\lambda_k} = \pi k + o(1) \quad as \quad k \to \infty. \tag{2.3}$$

(c) *The corresponding eigenfunctions* $\{\phi_k\}_{k=1}^{\infty}$ *form an orthogonal basis in* $L^2(0, 1)$ *(which we assume to be orthonormal).*
(d) *The estimates hold*

$$|\phi_k'(0)| \asymp k. \tag{2.4}$$

This relation means that

$$0 < \inf_{k\in\mathbb{N}} |\phi_k'(0)/k| \le \sup_{k\in\mathbb{N}} |\phi_k'(0)/k| < \infty.$$

We look for the solution to (1.1)–(1.3) in the form

$$u(x,t) = \sum_{k=1}^{\infty} c_k(t)\phi_k(x). \tag{2.5}$$

Plugging (2.5) into (1.1), multiplying by ϕ_k, we derive the equations on $c_k(t)$:

$$\dot{c}_k(t) - i\lambda_k c_k(t) = -ia_k w(t), \quad c_k(0) = 0,$$

where a_k are Fourier coefficients of the function a:

$$a(x) = \sum_{k=1}^{\infty} a_k\phi_k(x), \quad \|a\|^2_{H_0^1(0,1)} \asymp \sum_{k=1}^{\infty} |a_k/k|^2. \tag{2.6}$$

It follows that

$$c_k(t) = -i\int_0^t e^{i\lambda_k(t-s)}w(s)a_k\,ds.$$

Therefore, the function $\mu(t) := u_x(0,t)$ can be presented in the form

$$\mu(t) = \sum_{k=1}^{\infty} \phi_k'(0)\int_0^t (-i)e^{i\lambda_k(t-s)}a_k w(s)\,ds = -i\int_0^t r(t-s)w(s)\,ds, \tag{2.7}$$

where

$$r(t) = \sum_{k=1}^{\infty} e^{i\lambda_k t}\phi_k'(0)a_k. \tag{2.8}$$

Statements (a) and (b) above imply that the family $\{e^{i\lambda_k t}\}_{k=1}^{\infty}$ forms a Riesz basis in closure of its linear span in $L_2(0,T)$ for any $T > 0$ (see [4, Sec. II.4]), [5, 6]). From (2.4) and (2.6) it follows that $\{\phi_k'(0)a_k\} \in \ell^2$. Therefore, the series in the RHS of (2.8) converges in $L^2(0,T)$.

Differentiating (2.7) we obtain

$$\mu'(t) = -iw(0)r(t) - i\int_0^t w'(t-s)r(s)\,ds, \quad 0 < t < T. \tag{2.9}$$

We see that, if $w \in H^1(0,T)$ and $a \in H_0^1(0,1)$, then $\mu \in H^1(0,T)$. Since $w(0) \neq 0$, the equation (2.9) is the second kind Volterra equation for $r(t)$. Given w and μ, it has a unique solution $r \in L^2(0,T)$.

When $r(t)$ is recovered, the coefficients a_k can be found from (2.8). In our case it can be done in different ways because λ_k and $\phi_k'(0)$ are known. We demonstrate the method proposed in [3] (see also [1, 2]) since it works also if λ_k are unknown — we discuss this case in the next section.

We consider the following integral eigenvalue equation

$$\int_0^T \left[r(2T-t-\tau) - \lambda R(2T-t-\tau)\right] h(\tau)\, d\tau = 0, \quad 0 \leq \tau \leq T, \tag{2.10}$$

where $R(t) = \int_0^t r(\tau)\, d\tau$. It was proved in [3] that the eigenvalues of this problem coincide with eigenvalues λ_k, $k \in \mathbb{N}$ of the problem (2.1), (2.2). More of that, if $h_k(t)$ denote the corresponding eigenfunctions of the problem (2.10), then the family $\{f_k(t)\}_{k=1}^{\infty}$, $f_k(t) = \int_0^t h_k(\tau)\, d\tau$, is biorthogonal to $\{e^{i\lambda_k(T-t)}\}_{k=1}^{\infty}$ in $L^2(0,T)$.

We consider also the equation of the form (2.10) with $r(t)$ replaced by $\overline{r(t)}$. This equation yields the sequence of eigenvalues $\overline{\lambda}_k$ and eigenfunctions $j_k(t)$. We put $g_k(t) = \int_0^t j_k(\tau)\, d\tau$ and normalize functions f_k, g_k by the rule:

$$\delta_{nk} = \int_0^T \int_0^T r(2T-t-\tau) f_n(\tau) \overline{g_k(t)}\, d\tau\, dt.$$

Then we introduce constants α_k and β_k:

$$\alpha_k = \int_0^T r(T-\tau) f_k(\tau)\, d\tau, \quad \beta_k = \int_0^T r(T-\tau) \overline{g_k(\tau)}\, d\tau.$$

It was proved in [3] that $a_k \phi_x^k(0) = \alpha_k \beta_k$. Therefore, the function $a(x)$ can be presented in the form

$$a(x) = \sum_{k=1}^{\infty} \frac{\alpha_k \beta_k}{\phi_k'(0)}\, \phi_k(x)$$

Remark 1. Our method can be applied to inverse source problems for the wave, heat and beam equations. The proposed procedure is especially efficient when $q = 0$ and, hence, the eigenfunctions and eigenvalues of (2.1), (2.2) can be presented in the explicit form (see, e.g. [15] where a special choice of the intensity $w(t)$ was considered).

§3. Observation at the both ends

We consider the dynamical system (1.1)–(1.3) and suppose that the potential $q \in L_1(0,1)$ and the source $a \in H_0^1(0,1)$ are unknown. We suppose also that for some fixed $T > 0$, two functions

$$\mu_0(t) = u_x(0,t), \quad \mu_1(t) = u_x(1,t), \quad t \in [0,T],$$

are measured. Similarly to (2.7), one obtains the representation $\mu_j = -i \int_0^t r_j(t-s) w(s)\, ds, \ j = 0,1$, where

$$r_0(t) = \sum_{k=1}^{\infty} e^{i\lambda_k t} \phi_k'(0) a_k, \quad r_1(t) = \sum_{k=1}^{\infty} e^{i\lambda_k t} \phi_k'(1) a_k.$$

Using the method described in Section 2 we can recover the eigenvalues λ_k and the products $\phi_k'(0)a_k$ and $\phi_k'(1)a_k$.

We call the source $a(x)$ *generic* if $a_k \neq 0, \ k \in \mathbb{N}$. In the case of generic source our method allows to recover the spectral data consisting of pairs

$$\left\{ \lambda_k, \frac{\phi_k'(1)}{\phi_k'(0)} \right\}, \ k \in \mathbb{N}. \tag{3.1}$$

In [14] the authors proved that this data determined the potential $q(x)$ uniquely and provided the method of its reconstruction. Here we describe another method recovering the potential.

Let $y = y(x,\lambda)$ be the solution to the Cauchy problem

$$-y'' + q(x)y = \lambda y, \quad 0 < x < 1, \quad y(0,\lambda) = 0, \quad y'(0,\lambda) = 1.$$

The eigenvalues of the Dirichlet problem (2.1), (2.2) are zeroes of the function $y(1,\lambda)$, the normalized eigenfunctions are $\phi_k(x) = \frac{y(x,\lambda_k)}{\|y(\cdot,\lambda_k)\|}$. Therefore,

$$\frac{\phi_k'(1)}{\phi_k'(0)} = \frac{y'(1,\lambda_k)}{y'(0,\lambda_k)} = y'(1,\lambda_k) =: A_k.$$

The following formulas can be found in [14])

$$\|y(\cdot,\lambda_k)\| = y'(1,\lambda_k)\dot{y}(1,\lambda_k)$$

$$y(1,\lambda)=\prod_{k\geq 1}\frac{\lambda_k-\lambda}{k^2\pi^2},\ \dot{y}(1,\lambda_k)=-\frac{1}{k^2\pi^2}\prod_{n\geq 1, n\neq k}\frac{\lambda_n-\lambda_k}{n^2\pi^2}=:B_k,$$

where dot denotes the derivative with respect to λ.

We see that the data (3.1) allows us to find the norms $\|y(\cdot,\lambda_k)\| = A_k B_k$. The set of pairs $\{\lambda_k, \|y_2(\cdot,\lambda_k)\|\}_{k=1}^{\infty}$ is the "classical" spectral data. The potential can be recovered from this data using the Gelfand-Levitan, Krein or Boundary Control method (see, e.g. [7] for more details). When the potential is found, the source $a(x)$ can be recovered via its Fourier series (see Section 2) or using the methods of observers [12, 13].

Remark 2. The inverse problem of the recovery the potential by one measurement for the dynamical Schrödinger equation on an interval was studied in [8]. The authors proposed an algorithm for the reconstruction of the potential and obtained stability estimates.

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