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ON THE BEHAVIOR OF THE PERIODIC COXETER LAPLACIAN IN SOME REPRESENTATIONS RELATED TO THE ANTIFERROMAGNETIC ASYMPTOTIC MODE AND CONTINUAL LIMITS

Abstract. We consider some problems related to the asymptotic behavior of the so-called periodic Coxeter Laplacian (a distinguished operator in the group algebra of the symmetric group essentially coinciding with the Hamiltonian of the XXX Heisenberg model of spins) in some representations corresponding to the antiferromagnetic asymptotic mode, as well as in some related continual limits.

§1. Introduction

In this note, which is a direct continuation of [5], we consider some problems related to the asymptotic behavior of the so-called periodic Coxeter Laplacian (a distinguished operator in the group algebra of the symmetric group essentially coinciding with the Hamiltonian of the XXX Heisenberg model of spins) in some representations corresponding to the *antiferromagnetic* asymptotic mode, as well as in some related continual limits.

Recall that the *periodic Coxeter Laplacian* is the operator $L_N = Ne - (s_1 + \ldots + s_N)$ in the group algebra of the symmetric group $\mathfrak{S}_N$, where s_k is the Coxeter transposition $(k, k+1)$ (hereafter we always adopt the convention that $N+1 \equiv 1$). Consider the representation π of $\mathfrak{S}_N$ in the tensor product $(\mathbb{C}^2)^{\otimes N}$ by permutations of factors. Then the operator $L = \pi(L_N)$ is related to the Hamiltonian of the XXX Heisenberg model on the periodic one-dimensional lattice with N sites (see, e.g., [2, 4]) by the formula $H = \frac{J}{4}(2L - N)$, where $J > 0$ corresponds to the ferromagnetic case, and $J < 0$ to the antiferromagnetic one.

See the first part [5] for a discussion of motivation, background, notation, etc. In this note, we show that the ground state of the XXX antiferromagnet, i.e., the eigenvector of L with the greatest eigenvalue, lies in the

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irreducible representation corresponding to the Young diagram with the greatest possible second row (Theorem 1); show that the limiting density of eigenvalues of the normalized Coxeter Laplacian in the antiferromagnetic mode is a δ-measure (Propositions 1 and 2); show that the normalized Coxeter Laplacian has scalar weak limits in several natural representations of the infinite symmetric group (Propositions 3, 4, and 5), the corresponding constants being equal to 1 for the irreducible "two-block" induced representation, $\frac{1}{2}$ for the factor representation with Thoma parameters $\alpha = (\frac{1}{2}, \frac{1}{2}), \beta = 0$, and $\frac{5}{4}$ for the discrete elementary representation with the tableau $t_0 = \begin{pmatrix} 1 & 3 & 5 & \dots \\ 2 & 4 & 6 & \dots \end{pmatrix}$; consider a continual limit of the Bethe algebras (Sec. 3.1); show that a family of operators essentially coinciding with the Fourier transform of the Coxeter Laplacian in this continual limit yields a representation of the Witt algebra (Sec. 3.2).

Most problems considered in this note were posed by and discussed with A. M. Vershik within our joint project aimed to combine the known results on the representation theory of finite and infinite symmetric groups and a circle of results related to the quantum inverse scattering method and Bethe ansatz. The author is also grateful to P. P. Kulish for fruitful discussions.

§2. Antiferromagnetic mode

In [5], we considered the asymptotic behavior of the periodic Coxeter Laplacian in the ferromagnetic mode, which corresponds to considering it in the representations ϱ_r of $\mathfrak{S}_N$ induced from the identity representations of the Young subgroups $\mathfrak{S}_r \times \mathfrak{S}_{N-r}$ when r is fixed and $N \to \infty$. Now we are interested in another asymptotic mode, namely, assuming for simplicity that $N = 2n$ is even (the case of odd N can be treated in a similar way), we consider the operator L_N in the representation ϱ_{2n}^n of $\mathfrak{S}_{2n}$ induced from the identity representation of $\mathfrak{S}_n \times \mathfrak{S}_n$ as $n \to \infty$ and in the irreducible representation π_{μ_N} with diagram $\mu_N = (n, n)$. This mode corresponds to considering the antiferromagnetic XXX chain. In particular, the ground state of this model corresponds to the eigenvector of L_N in ϱ_N^n with the greatest eigenvalue.

2.1. The ground state.

Theorem 1. *The ground state of the antiferromagnetic XXX chain lies in the irreducible representation π_{μ_N} of $\mathfrak{S}_N$ with diagram $\mu_N = (n, n)$.*

Let T be the image in the representation π of the periodic shift in $\mathfrak{S}_N$, i.e., the one-cycle permutation $T_N = (1\,2\ldots N)$. Recall that it commutes with the Hamiltonian H and the Laplacian L.

Lemma 1. *The ground state lies in the eigenspace of T corresponding to the eigenvalue $\alpha = 1$ for n even and $\alpha = -1$ for n odd.*

Proof. The induced representation ϱ_{2n}^n can be realized in the linear space with basis (e_I) parametrized by n-tuples $I \subset \{1,\ldots,2n\}$, $|I| = n$. Consider the graph $G = (V, E)$ whose vertices are exactly such n-tuples I and two vertices I_1 and I_2 are joined by an edge if and only if the tuples I_1 and I_2 are obtained from each other by changing one element by 1 (mod$2n$). Then the periodic Coxeter Laplacian is exactly the graph-theoretic Laplacian of G. Thus the largest eigenvalue λ of L can be found by the well-known formula

$$\lambda = \max_x \frac{\sum_{I_1\sim I_2}(x_{I_1} - x_{I_2})^2}{\sum_{I\in V}(x_I)^2}, \qquad (1)$$

where $I_1 \sim I_2$ means that I_1 and I_2 are joined by an edge and the maximum is taken over all nonzero vectors $(x_I)_{I\in V}$ orthogonal to the subspaces of constant vectors.

Now observe that the graph G is obviously bipartite: V is the disjoint union of V_e and V_o, where V_e (respectively, V_o) consists of n-tuples I such that the sum of its elements is even (respectively, odd), and each edge joins a vertex from V_e and a vertex from V_o. Let x be a unit largest eigenvector of L, which is simultaneously an eigenvector of T with eigenvalue α. We want to prove that $\alpha = 1$ if n is even and $\alpha = -1$ if n is odd. But observe that if we replace x_I by $|x_I|$ for $I \in V_e$ and by $-|x_I|$ for $I \in V_o$, then the norm of the vector will not change and the sum in the numerator of (1) will not decrease. It easily follows that for a largest eigenvector we have $x_I \geq 0$ for $I \in V_e$ and $x_I \leq 0$ for $x \in V_o$ (or vice versa). Now let n be even. Then the orbits of T are contained in V_e or V_o, which implies that $\alpha = 1$. If n is odd, in a similar way we obtain that $\alpha = -1$. □

Proof of the theorem. Since $\rho_N^n = \oplus_{k=0}^n \pi_{N-k,k}$ (where $\pi_{N-k,k}$ is the irreducible representation of $\mathfrak{S}_N$ corresponding to the two-row Young diagram $(N-k, k)$), and for the representation ρ_N^{n-1} induced from the identity representation of $\mathfrak{S}_{n+1} \times \mathfrak{S}_{n-1}$ we have $\rho_N^{n-1} = \oplus_{k=0}^{n-1} \pi_{N-k,k}$, it suffices to show that the largest eigenvector of L in the space of ρ_N^n does not lie in the subspace of ρ_N^{n-1}. But this follows from the lemma. Indeed, if, say, n is even, then the largest eigenvector of L in ρ_N^n lies in the $\alpha = 1$ eigenspace of the shift T, but exactly the same argument as in the proof of the lemma shows that the largest eigenvector of L in ρ_N^{n-1} must lie in the $\alpha = -1$ eigenspace of T. $\square$

2.2. The limiting density of eigenvalues.

Proposition 1. *The limiting distribution of the eigenvalues of the operator* $\frac{1}{N}L_N$ *in the representation* π_{μ_N} *is the* δ*-measure at* $1/2$.

Proof. In the proof of this proposition we use the following result. Denote $\tilde{L}_N = Ne - L_N = (1,2) + (2,3) + \ldots + (N-1,N) + (N,1)$. Let π be a representation of the symmetric group $\mathfrak{S}_N$ (and its group algebra), χ be its character, and $M = \dim \pi$ be its dimension. Let $\tilde{\lambda}_1, \ldots, \tilde{\lambda}_M$ be all eigenvalues of $\tilde{L}_N$ in the representation π. It follows from the results of [1] that for every positive integer k,

$$\sum_{j=1}^{M} \tilde{\lambda}_j^k = \sum_{j_1=1}^{N} \cdots \sum_{j_k=1}^{N} \chi(\sigma_{j_1} \ldots \sigma_{j_k}). \tag{2}$$

Now let $\pi = \pi_{\mu_N}$. Then

$$\frac{1}{M} \sum_{j=1}^{M} \left(\frac{\tilde{\lambda}_j}{N} \right)^k = \frac{1}{N^k} \sum_{j_1=1}^{N} \cdots \sum_{j_k=1}^{N} \frac{\chi_{\mu_N}(\sigma_{j_1} \ldots \sigma_{j_k})}{\dim \chi_{\mu_N}}.$$

It is not difficult to see that the leading term of the last expression is

$$\frac{\chi_{\mu_N}(\sigma_{k,N})}{\dim \chi_{\mu_N}},$$

where $\sigma_{k,N}$ is a permutation with cycle type $(2^k, 1^{N-2k})$. Now it follows from the asymptotic theory of characters of the symmetric group (see [8]) that this limit equals the value of the character $\chi_{\alpha,0}$ of the infinite symmetric group $\mathfrak{S}_\infty$ with Thoma parameters $\alpha = (1/2, 1/2)$, $\beta = 0$ at a permutation of cycle type 2^k, and the same theory says that this value is equal to $(\frac{1}{2})^k$. Thus we see that

$$\frac{1}{M} \sum_{j=1}^{M} \left(\frac{\tilde{\lambda}_j}{N} \right)^k \to \left(\frac{1}{2} \right)^k,$$

which implies that the limiting distribution of the eigenvalues $\frac{1}{N}\tilde{\lambda}_j$ of $\frac{1}{N}\tilde{L}_N$ is $\delta_{1/2}$, so that the limiting distribution of the eigenvalues $\frac{1}{N}\lambda_j = 1 - \frac{1}{N}\tilde{\lambda}_j$ is also $\delta_{1/2}$. $\square$

Proposition 2. *The limiting distribution of the eigenvalues of the operator* $\frac{1}{N}L_N$ *in the induced representation* $\varrho_N^{N/2}$ *is also the* δ*-measure at* $1/2$.

Proof. An easy corollary of the previous proposition. □

2.3. Limiting operators in various representations. The inductive limit of the representations $\varrho_N^{N/2}$ as $N \to \infty$ is the irreducible "two-block" induced representation $\varrho^{\infty,\infty}$ of the infinite symmetric group $\mathfrak{S}_\infty$ of type ∞^2 (see [6]), namely, the representation of $\mathfrak{S}_\infty$ induced from the identity representation of the Young subgroup $\mathfrak{S}_{\{1,3,5,\dots\}} \times \mathfrak{S}_{\{2,4,6,\dots\}}$.

Proposition 3. *The weak limit of the operators* $\frac{1}{N}L_N$ *in the representations* $\varrho_N^{N/2}$ *as* $N \to \infty$ *is identity operator* E *in the space of* $\varrho^{\infty,\infty}$.

Proof. In this case, $\varrho^{\infty,\infty}$ can be realized in the space $L^2(\Pi)$ where Π is the set of $0-1$ sequences tail-equivalent to $\xi = 010101\dots$. Let H_m be the subspace of functions supported by sequences m-equivalent to ξ (i.e., coinciding with ξ starting from the mth position). Then $\cup_{m=1}^{\infty} H_m$ is dense in $L^2(\Pi)$. But for $f, g \in H_m$, obviously, $((k,k+1)f, g) = 0$ for $k > m$. Hence $\lim(\frac{1}{N}\tilde{L}_N f, g) = 0$, since only finitely many terms survive. The proposition follows. □

We can also consider the primary representation $\dim \mu_N \cdot \pi_{\mu_N}$. Obviously, the spectrum of the operator L_N in this representation is just a multiple of its spectrum in the irreducible representation π_{μ_N}. But the limit of such primary representations is the factor representation $\rho_{(\frac{1}{2},\frac{1}{2};0)}$ of the infinite symmetric group $\mathfrak{S}_\infty$ with Thoma parameters $\alpha = (\frac{1}{2}, \frac{1}{2})$, $\beta = 0$.

Proposition 4. *The weak limit of the operators* $\frac{1}{N}L_N$ *in the representations* $\dim \mu_N \cdot \pi_{\mu_N}$ *as* $N \to \infty$ *is the scalar operator* $\frac{1}{2}E$ *in the space of the factor representation* $\rho_{(\frac{1}{2},\frac{1}{2};0)}$.

Proof. Let us consider the "dynamic" realization of the factor representation $(\frac{1}{2}, \frac{1}{2}; 0)$ (see [9]). Consider the space of sequences $\mathcal{X} = \prod_{k=1}^{\infty}\{0,1\}$ with the $(\frac{1}{2}, \frac{1}{2})$-Bernoulli measure m. The infinite symmetric group $\mathfrak{S}_\infty$ acts on $\mathcal{X}$ by substitutions of coordinates, and this action preserves the measure m. Define an equivalence relation $\sim$ on $\mathcal{X}$ as follows: $x \sim y$ if there exists $\sigma \in \mathfrak{S}_\infty$ such that $y = \sigma x$. Let $\widetilde{\mathcal{X}} = \{(x,y) : x, y \in \mathcal{X}, x \sim y\}$

be the principal groupoid with diagonal $\mathcal{X}$ constructed from this equivalence relation. Consider the measure $\widetilde{m}$ on the groupoid $\widetilde{\mathcal{X}}$ induced by the measure m on the diagonal $\mathcal{X}$, and set $\mathcal{K} = L^2(\mathcal{X}, \widetilde{m})$. Thus the scalar product in $\mathcal{K}$ is given by $(h_1, h_2) = \int_{\mathcal{X}} \sum_{y\sim x} h_1(x,y)\overline{h_2(x,y)}dm(x)$. The infinite symmetric group acts in the space $\mathcal{K}$ according to the formula $(V_g h)(x,y) = h(g^{-1}x, y)$. Let $\Phi \in \mathcal{K}$ be the characteristic function of the diagonal: $\Phi(x,y) = 1$ if $x = y$ and 0 otherwise. The restriction of the representation V to the cyclic hull $\mathcal{K}_0$ of the vector Φ is just the factor representation $\rho_{(\frac{1}{2},\frac{1}{2};0)}$.

Now let $H_m \subset \mathcal{K}_0$ be the set of functions supported by pairs (x,y) coinciding from the mth position. Again, $\cup_{m=1}^{\infty} H_m$ is dense in $\mathcal{K}_0$. But for $f, g \in H_m$ and $k > m$ we have $((k, k+1)f, g) = \frac{1}{2}(f, g)$. The proposition follows. □

Remark. It is easy to see that the weak limit of the operators $\frac{1}{N}L_N$ in the "two-row" factor representation with Thoma parameters $\alpha = (p, 1-p)$, $\beta = 0$ is the scalar operator with constant $1 - p^2 - (1-p)^2$.

It is interesting to look at this result using the "tableaux realization" of the factor representation, which acts in the space $L^2(\mathcal{B}, \tilde{M}^{1/2,1/2})$, where $\mathcal{B}$ is the groupoid of infinite Young bitableaux and $\tilde{M}^{1/2,1/2}$ is the measure on this groupoid induced by the central measure $M^{1/2,1/2}$ on its diagonal (the space $\mathcal{T}$ of infinite Young tableaux) associated with the Thoma parameters $\alpha = (\frac{1}{2}, \frac{1}{2})$, $\beta = 0$. The action of the Coxeter generators in this model is given by Young's orthogonal form, and it is not very difficult to see that the result of the previous proposition in this setting is equivalent to the following assertion: for every $n = 1, 2, \ldots$, for every tableau s of length n,

$$\lim_{N\to\infty} \mathbb{E}\left(\frac{1}{N}\sum_{k=n+1}^{n+N} \frac{1}{r_k(t)} \,\middle|\, [t]_n = s\right) = \frac{1}{2}, \tag{3}$$

where $r_k(t)$ is the kth axial distance of the tableau t (equal to $c_{k+1}(t) - c_k(t)$ where c_l is the content of the lth cell in t) and the conditional expectation is taken with respect to the central measure $M^{1/2,1/2}$. The direct proof of this assertion proceeds through symmetric functions approach, using the expression for the cylinder distributions of $M^{1/2,1/2}$ in terms of Schur functions and Pieri-type identities.

In a similar way, if, instead of the factor representation, we consider the *concomitant* representation corresponding to the Thoma parameters

$\alpha = (\frac{1}{2}, \frac{1}{2})$, $\beta = 0$ (see [7]), then the limit of the operators $\frac{1}{N}L_N$ will be the scalar operator with the (obviously, smaller) constant

$$1 - \lim_{N\to\infty} \mathbb{E}\left(\frac{1}{N}\sum_{k=n+1}^{n+N}\left(\frac{1}{r_k(t)} + \sqrt{1-\frac{1}{r_k^2(t)}}\right)\,\middle|\,[t]_n = s\right).$$

Another representation it is natural to consider is the *elementary* representation equal to the inductive limit of the irreducible representations π_{μ_N}.

Proposition 5. *The weak limit of the operators $\frac{1}{N}L_N$ in the representations π_{μ_N} as $N \to \infty$ is the scalar operator $\frac{5}{4}E$ in the space of the corresponding elementary representation.*

Proof. The elementary representation acts in the discrete l^2 space spanned by the tableaux tail-equivalent to the "main" tableau $t_0 = \begin{pmatrix} 1 & 3 & 5 & \dots \\ 2 & 4 & 6 & \dots \end{pmatrix}$. Assume for simplicity that $N = 2n$ is even. As follows from Young's orthogonal form, $s_k t_0 = -t_0$ for odd k, and $s_k t_0 = \frac{1}{2}t_0 - \frac{\sqrt{3}}{2}t_0^k$ for even k, where t_0^k is the tableau obtained from t_0 by swapping k and $k+1$. It easily follows that

$$\lim_{N\to\infty}(\frac{1}{N}L_N t_0, t) = \frac{5}{4}\delta_{t_0 t}.$$

It is also obvious that the same formula holds for every tableau s tail-equivalent to t_0, and the proposition follows. □

§3. Continual limits

3.1. Continual limit of the Bethe algebras in the ferromagnetic case. The *Bethe algebra* $\mathfrak{T}_N$ is a commutative subalgebra of the group algebra $\mathbb{C}[\mathfrak{S}_N]$ generated by operators naturally arising in the quantum inverse scattering method for the XXX Heisenberg model (for more details, see [5]). A system of generators of this algebra is

$$R_{N,k} = \sum_{i_1<\dots<i_k} s_{i_1}\dots s_{i_k}, \quad k = 0, 1, \dots, N-1,$$

where the inequalities are understood with respect to the cyclic order (which does not lead to any ambiguity, since nonneighbor Coxeter generators commute).

Now, following [3], consider a new system of generators $I_k = I_k^{(N)}$, $k = 0, \ldots, N-2$, of the Bethe algebra $\mathfrak{T}_N$. They are defined as follows. Consider the generating function

$$\log\left(1 + \sum_{k=1}^{N-2} u^k R_{N,k}\right) = \sum_{k=1}^{\infty} u^k I_k. \tag{4}$$

This formula defines $I_k^{(N)}$ for $k \geq 1$. Also, put $I_0^{(N)} = T_N$ (the periodic shift). Then, as shown in [3], $I_0^{(N)}, I_1^{(N)}, \ldots, I_{N-2}^{(N)}$ is a system of generators of $\mathfrak{T}_N$ enjoying the following nice property: for each $k = 1, 2, \ldots$ there exists an element $\theta_k \in \mathbb{C}[\mathfrak{S}_{k+1}]$ such that

$$I_k^{(N)} = \sum_{l=0}^{N-1} T_N^l \theta_k T_N^{-l} \qquad \text{for all } N, \tag{5}$$

that is, $I_k^{(N)}$ for an arbitrary N is obtained as the sum of all elements of the orbit of some fixed element of $\mathfrak{S}_{k+1}$ (which does not depend on N) under the action of the periodic shift by conjugation.

In particular, it is not difficult to see that for $k = 1$ we have $\theta_1 = (12)$ and $I_1^{(N)} = s_1 + \ldots + s_N$ is essentially the periodic Coxeter Laplacian L_N (more exactly, $I_1^{(N)} = Ne - L_N$).

Now consider first the case of the representation ρ^1 induced from the identity representation of $\mathfrak{S}_1 \times \mathfrak{S}_{N-1}$. As long as we are interested only in the representation of the Bethe algebra $\mathfrak{T}_N$, which is a commutative algebra containing the periodic shift T_N, the space of this representation can be regarded as the space of functions $f : \{1/N, 2/N, \ldots, N/N\} \to \mathbb{C}$ on the discrete circle $C_N = \{1/N, 2/N, \ldots, N/N\}$. Denote by $S = S_N$ the (periodic) shift on this space, i.e., $(Sf)(x) = f(x + 1/N)$, with $N/N + 1 \equiv 1/N$.

Lemma 2. *In this representation, $\rho^1(I_0^{(N)}) = S_N$ and*

$$\rho^1(I_k^{(N)}) = \frac{1}{k}\left[(-1)^{k-1}N + (S_N - 1)^k - (1 - S_N^{-1})^k\right], \quad k = 1, \ldots, N-2. \tag{6}$$

Proof. It is not difficult to compute that

$$\rho^1(R_{N,k}) = \binom{N-1}{k-1} S^{-1} + \sum_{l=0}^{k} \binom{N-2-l}{k-l} S^l, \quad k = 1, \ldots, N-2. \tag{7}$$

Then

$$\begin{aligned} G(u) &:= \rho^1(\sum_{k=1}^{N-2} u^k R_{N,k}) \\ &= \sum_{k=1}^{N-2} u^k \binom{N-1}{k-1} S^{-1} + \sum_{k=1}^{N-2} u^k \sum_{l=0}^{k} u^k \binom{N-2-l}{k-l} S^l \\ &= G_1(u) + G_2(u). \end{aligned}$$

The second sum equals

$$\sum_{l=0}^{N-2} S^l \sum_{k=l}^{N-2} u^k \binom{N-2-l}{k-l}.$$

Now

$$\sum_{k=l}^{N-2} u^k \binom{N-2-l}{k-l} = \sum_{m=0}^{N-2-l} u^{m+l} \binom{N-2-l}{m} = u^l (1+u)^{N-2-l},$$

so that

$$G_2(u) = \sum_{l=0}^{N-2} u^l (1+u)^{N-2-l} S^l.$$

Note that since we are interested only in the first $N-2$ terms of $\log(1+G(u))$, we can add to $G(u)$ arbitrary terms of degree $k > N-2$. Thus let

$$\tilde{G}_2(u) = \sum_{l=0}^{\infty} u^l (1+u)^{N-2-l} S^l = (1+u)^{N-2} \frac{1}{1 - \frac{Su}{1+u}}$$

and

$$\tilde{G}_1(u) = S^{-1} \sum_{k=1}^{\infty} u^k \binom{N-1}{k-1} = u(1+u)^{N-1}.$$

Let $\tilde{G}(u) = \tilde{G}_1(u) + \tilde{G}_2(u)$. Now it is a simple calculation to check that

$$\log \tilde{G}(u) = \log \left(\sum_{k=0}^{\infty} u^k \tilde{A}_k \right),$$

where $\tilde{A}_k$ are given by the right-hand side of (6) for $k = 0, 1, \ldots$. □

It is convenient to get rid of the identity terms and consider the generators

$$A_k^{(N)} = \rho^1(I_k^{(N)}) - \frac{(-1)^k N}{k} = \frac{1}{k}\left[(S_N - 1)^k - (1 - S_N^{-1})^k\right], \quad k = 1, \dots, N-2.$$

In particular, $A_1^{(N)} = -\rho^1(L_N)$. Put also $A_0^{(N)} = S_N - 1$.

Now let $N \to \infty$, so that the discrete circle turns into the ordinary continual circle.

Lemma 3. *As $N \to \infty$,*

$$N^{k+1} A_k^{(N)} f(x) \to f^{(k+1)}(x) \quad \textit{for every } k = 0, 1, \dots, \tag{8}$$

that is, the limit of the (scaled) generators $A_k^{(N)}$ in this representation of the Bethe algebra is the operator of taking the $(k+1)$th derivative.

Proof. Follows from the above formulas for the generators and the well-known theorem from the theory of finite differences that

$$\sum_{j=1}^{n} (-1)^j \binom{n}{j} P(j) = 0$$

for any polynomial P of degree less than n. □

Thus we see that *the continual limit of the Bethe algebra in the representation ρ^1 is the algebra of difference operators with constant coefficients on the circle.* Of course, the image of the Bethe algebra in the representation ρ^1 is generated by only one operator $\rho^1(T_N)$, and the limit algebra is generated by the operator of taking the first derivative.

Now consider the representation ρ^2 induced from the identity representation of $\mathfrak{S}_2 \times \mathfrak{S}_{N-2}$. In this case, the space of the representation can be realized as $\{f : C_N \times C_N \to \mathbb{C} \mid f(x,y) = f(y,x),\ f(x,x) = 0\}$. In view of (5), the image of $I_k^{(N)}$ in this representation involves the terms $T_N^l \theta_k T_N^{-l}$, each involving at most $k+1$ neighboring elements. It follows that if the distance between i and j (on the circle) is greater than $k+1$, then

$$\rho^2(\tilde{I}_k^{(N)}) f(i/N, j/N) = A_{k,1}^{(N)} f(i/N, j/N) + A_{k,2}^{(N)} f(i/N, j/N),$$

where $\tilde{I}_k^{(N)} = I_k^{(N)} - \frac{(-1)^k N}{k}$ $A_{k,l}^{(N)}$ acts as $A_k^{(N)}$ on the lth argument of f. On the other hand, if k is fixed and $N \to \infty$, and we fix x and y, then for sufficiently large N we have $|x-y| > \frac{k+1}{N}$, so that this condition is satisfied, which implies that

$$N^{k+1}\rho^2(\tilde{I}_k^{(N)})f(x,y) \to \frac{\partial^{k+1} f}{\partial x^{k+1}} + \frac{\partial^{k+1} f}{\partial y^{k+1}}, \quad k=0,1,\dots. \tag{9}$$

Obviously, analogous reasoning works for the representation ρ^m with arbitrary $m=1,2,\dots$, so that we obtain the following result.

Proposition 6. *As* $N \to \infty$,

$$N^{k+1}\rho^m(\tilde{I}_k^{(N)})f(x_1,\dots,x_m) \to \sum_{j=1}^{m} \frac{\partial^{k+1} f}{\partial x_j^{k+1}}, \quad k=0,1,\dots. \tag{10}$$

In particular, in this limit, the periodic Coxeter Laplacian L_N turns into the ordinary Laplacian:

$$N^2\rho^m(L_N)f = N^2\rho^m(-\tilde{I}_1^{(N)})f \to -\left(\sum_{j=1}^{m} \frac{\partial^2}{\partial x_j^2}\right) f.$$

3.2. A representation of the Witt algebra.

Proposition 7. *In the continual limit of the representation* ρ^l, *the following limits exist:*

$$-\lim_{N\to\infty} N \sum_{j=1}^{N} j e^{\frac{2\pi i j n}{N}}(e - s_j) = W_n, \quad n \in \mathbb{Z};$$

the operators W_n *form a representation of the Witt algebra, i.e., satisfy the commutation relations*

$$[W_n, W_m] = (m-n)W_{m+n}.$$

Proof. Let us first consider the case $l=1$. Denote

$$V_n^{(N)} = -N\rho^1\Big(\sum_{j=1}^{N} j e^{\frac{2\pi i j n}{N}}(e - s_j)\Big).$$

Then, realizing ρ^1 in the space of functions $f : \{1/N, 2/N, \ldots, N/N\} \to \mathbb{C}$ on the discrete circle $C_N = \{1/N, 2/N, \ldots, N/N\}$ (as in the previous section), we have

$$V_n^{(N)} f(k/N) = -N \left(k e^{\frac{2\pi i k n}{N}} \left(f(\frac{k}{N}) - f(\frac{k+1}{N}) \right) \right.$$
$$\left. + (k-1) e^{\frac{2\pi i (k-1) n}{N}} \left(f(\frac{k}{N}) - f(\frac{k-1}{N}) \right) \right).$$

Now if $k/N \to \theta$, then the right-hand side can be written as

$$-N \left(k e^{2\pi i n\theta} \left(f(\theta) - f(\theta + \frac{1}{N}) \right) \right.$$
$$\left. + (k-1) e^{2\pi i n\theta} e^{-\frac{2\pi i n}{N}} \left(f(\frac{k}{N}) - f(\theta - \frac{1}{N}) \right) \right),$$

and elementary calculus shows that it tends to

$$-\left(e^{2\pi i n\theta} \frac{\partial}{\partial\theta} \right) f;$$

these are the standard generators of the Witt algebra.

For $l > 1$, arguing as in the previous section, we see that in the $N \to \infty$ limit, the operators under consideration are the sums of the operators W_n acting separately on each variable, so that we obtain the lth tensor power of the standard representation of the Witt algebra. □

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