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G. Carlier, F. Santambrogio

A CONTINUOUS THEORY OF TRAFFIC CONGESTION AND WARDROP EQUILIBRIA

Abstract. In the classical Monge–Kantorovich problem, the transportation cost only depends on the amount of mass sent from sources to destinations and not on the paths followed by each particle forming this mass. Thus, it does not allow for congestion effects, which depend instead on the proportion of mass passing through a same point or on a same path. Usually the travelling cost (or time) of a path depends on "how crowded" this path is. Starting from a simple network model, we shall define equilibria in the presence of congestion. We will then extend this theory to the continuous setting mainly following the recent papers [8,10]. After an introduction with almost no mathematical details, we will give a survey of the main features of this theory.

§1. Introduction

The understanding of traffic congestion and its effects on the performances of a road network has always been an intriguing issue, for the questions it brings both in modelization and in real life behavior. Problems like "if I choose this secondary uncongested road it would take less time, but if everybody does the same it would be much worse" are classical, and naturally lead to challenging game-theory and optimization issues.

In the 50's (see [22]) Wardrop formalized the main rule that should lead the congestion of a network through two principles: first, all the paths connecting the same two locations which are actually followed by some vehicles must provide the same travelling time (a time which depends on their length as well as on congestion); second, all the other possible paths must provide a larger travelling time. This may be mathematically translated into the fact that only paths which are *geodesics* for a certain metric on the network are used, but this metric is exactly induced by the the way vehicles use the networks. This gives an equilibrium problem that one can see as a fixed point (which are the ways of choosing some paths on

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the network, so that, looking at the geodesics induced by this choice, we find again the same paths?). This concept of equilibrium, called *Wardrop equilibrium*, is just a particular case of *Nash equilibria*, in a game where the players are the vehicles and the goal of each of them is to minimize its travelling time.

Soon after Wardrop formulated his principles, Beckmann et al. (see [2]) discovered that Wardrop equilibria have a variational characterization. Actually, a traffic configuration (i.e. the set of choices of all paths) is an equilibrium if and only if it optimizes a global criterion, taking into account the total congestion. When the quantity of traffic on each path on the network is considered to be a real number (instead of an integer number of vehicles), the functions H involved in the optimization problem are primitives of increasing functions, which gives convexity. The good point, from the mathematical point of view, is that such an optimization problem being convex, it allows for powerful duality and numerical methods to approach it. A discretized version for integer vehicles exists as well. On the other hand, one could be disappointed that, except in very special cases, the total cost which is optimized does not correspond to the total travelling time of all the vehicles, which means that looking for the equilibrium and for the social optimum is in general not the same, and that the equilibrium is in general not efficient. The literature on *price of anarchy* and *selfish routing* (see the book [16] and the references therein) precisely addresses the relations between equilibria and social optima, focusing in particular in the lack of efficiency of equilibria.

Similar considerations may be extended to the more recent framework of *continuous traffic congestion*, where the network is replaced by a domain in $\mathbb{R}^2$ and vehicles are allowed to move in every direction, thus giving rise to a density of traffic congestion, that we will call *traffic intensity*. In such a case, one can write a convex optimization problem on the space of these densities, and then prove, again, that at the optimum the paths which are actually followed are geodesics for the congested metric induced by the traffic intensity. It is just for the sake of clarity that in this framework one usually starts from the optimization and then gets the equilibrium as an optimality condition (mathematically, the reason is the fact that, when optimizing, the functional will force the traffic intensity, which will be defined as a measure, to be absolutely continuous, and its density will appear in the metric it induces, while geodesics in the case of possibly singular measures are not well-defined).

A natural question is whether this continuous counterpart is meaningful in terms of modeling, since car traffic actually occurs on one-dimensional networks. And the answer is yes, at least for two reasons: first it fits the situation of *pedestrian congestion* (which is usually considered in the literature as a two-dimensional problem); second, it may be useful to look at *large scale* traffic problems, when one only wants to detect average values of the traffic intensity in different zones of a large congested area. Yet, we stress that the continuous model that we present in this paper is not exactly the homogenized limit of the discrete ones on grid networks whose step goes to zero (see [3] for such limit issues). It is anyway the most natural model that may play the role of Wardrop's one in a continuous setting and it shares its main qualitative features. It will be described through the formalism of measures on the set of paths (which exactly accounts for the statistical properties of the set of choices of all the vehicles), which is a classical tool in transport theory, in connection with *optimal transport* (see [20, 21] for a recent account of optimal transport theory).

A key element in transport theory is the concept of *transport plan*, i.e. a probability measure γ on the product space of pairs origin-destination, which is the main unknown in Monge-Kantorovitch theory. Here, as it is the case in other path-dependent problems (branched transport, see [**?**], fluid mechanics, see [9],...), γ may also be fixed a priori, since the main unknown is the way the traffic distributes over the multiple paths connecting the same pair origin-destination. This is what we call the *short-term* problem, since for immediate applications one usually knows the proportion of commuters moving every day between two given points. On the other hand, one can also consider the problem where γ is allowed to vary, and only its marginals μ_0 and μ_1 are prescribed (which means that we know the total number of paths leaving every origin, and the total arriving at each destination, without any information on the coupling between them). This may be interpreted as *long-term*[a]problem: think for instance of a urban area where people move from home to work; it is quite clear that the addresses of those who work in a certain spot may change from year to year, but that, globally, the population density of all the neighborhoods and the distribution of offices and working places will stay the same for much more time. The problem where we optimize also over γ is not at all specific to the continuous framework, it may also be considered in networks, and leads to an extra equilibrium optimality condition. Actually one gets that the optimal configuration must realize a coupling γ

which optimizes a Monge-Kantorovitch transport cost, computed according to the metric induced by the traffic intensity itself. It is once more an equilibrium problem!

On the other hand, a peculiar feature of this long-term problem which is very specific to the continuous formulation is its tight connection with a *minimal flow problem.* This problem (minimizing a total integral cost $\int H(|v(x)|)dx$ among vector fields v with prescribed divergence $\nabla \cdot v = \mu_0 - \mu_1$) is also due to Beckmann [1]. It is strongly related to the Monge-Kantorovitch transport cost (in the case $H(t) = t$) with the possible additional effect that, due to congestion, "where the flow is stronger the cost is proportionally higher". It is clear, from the fact that the data on origins and destinations appear only through $\mu_0 - \mu_1$, that this problem may not be linked to the short-term one. On the contrary, we will explain that in the long-term case this problem is actually equivalent to the traffic optimization giving Wardrop equilibria. Indeed, the optimal traffic intensity turns out to be equal to $|v|$ and the paths actually followed by the commuters are integral curves of a vector field obtained from the optimal v.

The fact that one can equivalently look, at least in the long-term case, at Beckmann's minimization leads to a more classical *calculus of variations* problem. In particular, it is possible to write down the optimality conditions for such a minimization as a PDE. This PDE, of the form $v = G(\nabla u)$, with $\nabla \cdot G(\nabla u) = \mu_0 - \mu_1$, may be strongly degenerate, depending on the function $G = \nabla H^*$ that one chooses. Even if simple choices lead to the Laplace or to the $p-$Laplace equation, it turns out the cases that are realistic in congestion modeling are exactly those leading to much more degenerate PDEs (for instance one can find a G that vanishes on a whole ball around the origin). This has motivated the study of the regularity properties of the solutions of these equations, since, by the way, some regularity is needed so as to properly define the integral curves of the optimal vector field (see [8, 19]).

Notice anyway that the equations which are involved in this formulation are elliptic PDEs, and no variable playing the role of time appears in them. This is due to the fact that our model is stationary: it only accounts for sort of a *cyclical, neverending movement,* where every path is constantly occupied by the same density of vehicles, since those who arrive are immediately replaced by others. This point is in common with the models on networks, where one can think that the traffic intensity stands for an average occupation ratio of each road during a period. This is a difference with respect to other recent mathematical models involving congestion effects, like what one can see in *Mean Field Games*. In these continuous non-atomic differential games, introduced by J.-M. Lasry and P.-L. Lions (see [14]), a continuum of agents moves in a domain, minimizing some criteria taking into account lengths and travelling times as well as congestion ratios they meet at every time. Due to the explicit presence of time, the PDE describing the optimal evolutions (or the equilibria, since here as well some equivalences are available) are given by a system coupling a transport equation and an Hamilton-Jacobi equation, which is very different from the framework we are going to describe in the next sections.

The last point that we want to stress in this introduction concerns the methods for numerical approximation. Exactly as in the discrete network case, these methods are mainly based on the dual problem of the convex optimization giving the equilibrium as an optimum. This is for dimensional reasons: indeed, the primal problem has as many variables as possible paths, while the dual has only one variable per edge in the network, standing for the metric at every edge. In the continuous case, this introduces a dual variable ξ, which is a positive function on the domain and is used as a metric on it. One needs to optimize a convex criterion on ξ (like an L^p norm), perturbed by a combination $\int c_\xi(x,y)d\gamma$ of the distances c_ξ, defined as the Riemannian distances with the conformal metric given by ξ times the identity matrix, computed on the pairs origin-destinations. Obviously this is much easier in the short-term problem, since the transport plan γ is fixed, and requires instead an optimization over γ for the long-term one.

In order to do numerics, it is hence necessary (see [5, 6]) to be able to compute the distances c_ξ on a discretization grid and to differentiate the results with respect to ξ (the problem being convex, a simple gradient descent can be used to approximate the optimum). This is done thanks to the so-called *Fast Marching Method* (FMM), a numerical discretization, endowed with a very efficient way of computing the discretized solutions, which is suitable for some Hamilton-Jacobi equations. It is somehow the way how Hamilton-Jacobi strikes back in the problem, and here the equation we have to deal with is the *Eikonal equation* $|\nabla u| = \xi$, which is solved by $u = c_\xi(x,\cdot)$. The numerical method, based on a variation of the FMM, which allows to differentiate c_ξ with respect to ξ is one of the new contributions in this subject and has been studied in [6]. It is likely to be interesting in itself and has also been applied to other problems, different from traffic congestion.

Numerics will shortly be addressed at the end of Section 4, which is devoted to the duality in the short-term case (since, as we underlined above, the duality formulation is easier when γ is fixed). This section follows two general sections on the models and the relations between equilibria and optimization in the network (Section 2) and in the continuous (Section 3) cases, respectively. Section 5, on the contrary, is specific to the long-term problem and presents the equivalences with Beckmann's minimal flow optimization as well as the PDE issues which arise from this formulation.

§2. Wardrop equilibria in a simple congested network model

The main data of the model are a finite oriented connected graph $G = (N, E)$ modelling the network, and edge travel times functions $g_e : w \in \mathbb{R}_+ \mapsto g_e(w)$ giving, for each edge $e \in E$, the travel time on arc e when the flow on this edge is w. The functions g_e are all nonnegative, continuous, nondecreasing and they are meant to capture the congestion effects (which may be different on the different edges, since some roads may be longer or wider and may have different responses to congestion). The last ingredient of the problem is a transport plan on pairs of nodes $(x, y) \in N^2$ interpreted as pairs of sources/destinations. We denote by $(\gamma_{x,y})_{(x,y)\in N^2}$ this transport plan: $\gamma_{x,y}$ represents the "mass" to be sent from x to y. We denote by $C_{x,y}$ the set of simple paths connecting x to y, so that $C := \cup_{(x,y)\in N^2} C_{x,y}$ is the set of all simple paths. A generic path will be denoted by σ and we will use the notation $e \in \sigma$ to indicate that the path σ uses the edge e.

The unknown of the problem is the flow configuration. The edge flows are denoted by $w = (w_e)_{e\in E}$ and the path flows are denoted by $q = (q_\sigma)_{\sigma\in C}$: this means that w_e is the total flow on edge e and q_σ is the mass traveling on the path σ. Of course the w_e's and q_σ's are nonnegative and constrained by the mass conservation conditions:

$$\gamma_{x,y} = \sum_{\sigma \in C_{x,y}} q_\sigma, \quad \forall (x, y) \in N^2 \tag{1}$$

and

$$w_e = \sum_{\sigma \in C \,:\, e\in\sigma} q_\sigma, \quad \forall e \in E. \tag{2}$$

Given the edge flows $w = (w_e)_{e\in E}$, the total travel-time of the path $\sigma \in C$ is

$$T_w(\sigma) = \sum_{e\in\sigma} g_e(w_e). \tag{3}$$

In [22], Wardrop defined a notion of noncooperative equilibrium that has been very popular since among engineers working in the field of congested transport and that may be described as follows. Roughly speaking, a Wardrop equilibrium is a flow configuration such that every actually used path should be a shortest path taking into account the congestion effect i.e. formula (3). This leads to

Definition 1. *A Wardrop equlibrium is a flow configuration* $w = (w_e)_{e\in E}$, $q = (q_\sigma)_{\sigma\in C}$ *(all nonnegative of course), satisfying the mass conservation constraints* (1) *and* (2)*, such that, in addition, for every* $(x, y) \in N^2$ *and every* $\sigma \in C_{x,y}$*, if* $q_\sigma > 0$ *then*

$$T_w(\sigma) = \min_{\sigma'\in C_{x,y}} T_w(\sigma').$$

A few years after Wardrop introduced his equilibrium concept, Beckmann, McGuire and Winsten [2] realized that Wardrop equilibria can be characterized by the following variational principle:

Theorem 1. *The flow configuration* $w = (w_e)_{e\in E}$, $q = (q_\sigma)_{\sigma\in C}$ *is a Wardrop equilibrium if and only if it solves the convex minimization problem*

$$\inf_{(w,q)} \sum_{e\in E} H_e(w_e) \text{ s.t. nonnegativity constraints and (1)–(2) hold} \tag{4}$$

where, for each e*, we take* H_e *to be the primitive of* g_e*, i.e.* $H_e(w) = \int_0^w g_e(s)ds$.

Proof. Note that due to (2), one can deduce w from q so that (4) is an optimization problem on $q = (q_\sigma)_{\sigma\in C}$ only. Assume that $q = (q_\sigma)_{\sigma\in C}$ (with associated edge flows $(w_e)_{e\in E}$) is optimal for (4) then for every admissible $\eta = (\eta_\sigma)_{\sigma\in C}$ with associated (through (2)) edge-flows $(u_e)_{e\in E}$, one has

$$\begin{aligned} 0 &\le \sum_{e\in E} H'_e(w_e)(u_e - w_e) = \sum_{e\in E} g_e(w_e) \sum_{\sigma\in C\,:\,e\in\sigma} (\eta_\sigma - q_\sigma) \\ &= \sum_{\sigma\in C} (\eta_\sigma - q_\sigma) \sum_{e\in\sigma} g_e(w_e) \end{aligned}$$

so that

$$\sum_{\sigma\in C} q_\sigma T_w(\sigma) \le \sum_{\sigma\in C} \eta_\sigma T_w(\sigma)$$

minimizing the right-hand side thus yields

$$\sum_{(x,y)\in N^2} \sum_{\sigma\in C_{x,y}} q_\sigma T_w(\sigma) = \sum_{(x,y)\in N^2} \gamma_{x,y} \min_{\sigma'\in C_{x,y}} T_w(\sigma')$$

which exactly says that (q, w) is a Wardrop equilibrium. To prove the converse, it is enough to see that problem (4) is convex so that the inequality above is indeed sufficient for a global minimum. □

The previous characterization actually is the reason why Wardrop equilibria became so popular. Not only, one deduces for free existence results, but also uniqueness for w (not for q) as soon as the functions g_e are increasing (so that H_e is strictly convex). The variational formulation (4) also admits a dual formulation. Another major advantage of (4) is that the techniques of numerical convex optimization can be used to compute Wardrop equilibria, however there are as many variables as the number of paths which obviously restricts computations to small networks, the dual formulation has much less variables but involves nonsmooth terms. Let us also mention an interesting extension of the model to a stochastic setting by Baillon and Cominetti [4].

Remark 1. It would be very tempting to deduce from theorem 1 that equilibria are efficient since they are minimizers of (4). One has to be cautious with this quick interpretation since the quantity $\sum_{e\in E} H_e(w_e)$ does not represent the natural total social cost measured by the total time lost in commuting which reads as

$$\sum_{e\in E} w_e g_e(w_e). \tag{5}$$

The efficient transport patterns are minimizers of (5) and thus are different from equilibria in general. Efficient and equilibria configurations coincide in the special case of power functions where $g_e(w) = a_e w^\alpha$, but this case is not realistic since it implies that traveling times vanish if there is no traffic... Moreover, a famous counter-example due to Braess shows that it may be the case that adding an extra road on which the travelling time is aways zero leads to an equilibrium where the total commuting time is increased! This illustrates the striking difference between efficiency and equilibrium, a topic which is very well-documented in the finite-dimensional network setting where it is frequently associated to the literature on the so-called *price of anarchy* (see [16]).

Remark 2. In the problem presented in this paragraph, the transport plan γ is fixed, this may be interpreted as a *short-term problem.* Instead, we could consider the *long-term* problem where only the distribution of sources μ_0 and the distribution of destinations μ_1 are fixed. In this case, one requires in addition, in the definition of an equilibrium that γ is efficient in the sense that it minimizes among transport plans between μ_0 and μ_1 the total cost

$$\sum \gamma_{x,y} d_w(x,y) \quad \text{with} \quad d_w(x,y) := \min_{\sigma \in C_{x,y}} T_w(\sigma).$$

In the long-term problem where one is allowed to change the assignment as well, equilibria still are characterized by a convex minimization problem where one also optimizes over γ.

§3. Optimal transport with congestion and equilibria in a continuous framework

The aim of this paragraph is to generalize the previous analysis to a continuous framework. In the continuous setting, there will be no network, all paths in a certain given region will therefore be admissible. The first idea is to formulate the whole path-dependent transport pattern in terms of a probability measure Q on the set of paths (this is the continuous analogue of the path flows $(q_\sigma)_\sigma$ of the previous paragraph). The second one is to measure the intensity traffic generated by Q in a similar way as one defines transport density in the Monge's problem (this is the continuous analogue of the arc flows $(w_e)_e$ of the previous paragraph). The last and main idea will be in modelling the congestion effect through a metric that is monotone increasing in the traffic intensity (the analogue of $g_e(w_e)$).

We will deliberately avoid to enter into technicalities so the following description will be pretty informal (see [10] for details). From now on, Ω denotes an open bounded connected subset of $\mathbb{R}^2$ (a city say), and we are also given :

- either two probability measures μ_0 and μ_1 (distribution of sources and destinations) on $\overline{\Omega}$ in the case of the long-term problem,
- or a transport plan γ (joint distribution of sources and destinations) that is a joint probability on $\overline{\Omega} \times \overline{\Omega}$) in the short-term case.

Given an absolutely continuous curve σ; $[0,1] \mapsto \overline{\Omega}$ and a continuous function φ, let us set

$$L_\varphi(\sigma) := \int_0^1 \varphi(\sigma(t))|\dot{\sigma}(t)|dt. \tag{6}$$

A transport pattern is by definition a probability measure Q on $C := C([0,1], \overline{\Omega})$ concentrated on absolutely continuous curves that is compatible with mass conservation, i.e. such that either

$$e_{0\#}Q = \mu_0,\ e_{1\#}Q = \mu_1$$

(where, as usual, $f_\# \mu$ denotes the push forward of the measure μ through the map f) in the case of the long-term problem, or

$$(e_0, e_1)_\# Q = \gamma, \text{ with } e_t(\sigma) := \sigma(t),\ \forall t \in [0,1]$$

in the case of the short-term problem. We shall denote by $\mathcal{Q}(\mu_0, \mu_1)$ and $\mathcal{Q}(\gamma)$ the set of admissible transport patterns respectively for the long-term and for the short-term problem:

$$\mathcal{Q}(\mu_0, \mu_1) := \{Q\ :\ e_{0\#}Q = \mu_0,\ e_{1\#}Q = \mu_1\}$$

and

$$\mathcal{Q}(\gamma) := \{Q\ :\ (e_0, e_1)_\# Q = \gamma\}.$$

In the remainder of this paragraph, we will focus for simplicity on the long-term problem. We are interested in finding an equilibrium i.e. a $Q \in \mathcal{Q}(\mu_0, \mu_1)$ that is supported by geodesics for a metric ξ_Q depending on Q itself (congestion).

The intensity of traffic associated to $Q \in \mathcal{Q}(\mu_0, \mu_1)$ is by definition the measure $i_Q \in \mathcal{M}(\overline{\Omega})$, defined by

$$\int \varphi di_Q := \int_{C([0,1],\overline{\Omega})} \Big(\int_0^1 \varphi(\gamma(t))|\dot{\gamma}(t)|dt \Big) dQ(\gamma) = \int_C L_\varphi(\sigma) dQ(\sigma).$$

for all $\varphi \in C(\overline{\Omega}, \mathbb{R}_+)$. This definition is a generalization of the notion of transport density and the interpretation is the following: for a subregion A, $i_Q(A)$ represents the total cumulated traffic in A induced by Q, it is indeed the average over all paths of the length of this path intersected with A.

The congestion effect is then captured by the *metric* associated to Q:

$$\xi_Q(x) := g(x, i_Q(x)), \quad \text{for} \quad i_Q \ll \mathcal{L}^2 \ (+\infty \text{ otherwise})$$

for a given increasing function $g(x,.) : \mathbb{R}_+ \to \mathbb{R}_+$. The fact that there exists at least one $Q \in \mathcal{Q}(\mu_0, \mu_1)$ such that $i_Q \ll \mathcal{L}^2$ is not always true and depends on μ_0 and μ_1 but again we do not wish to enter the details, let us only indicate that this condition is satisfied when μ_0 and μ_1 are "well behaved" (this is a nontrivial fact for which we refer to the regularity results of De Pascale and Pratelli [12] and to the more recent paper [18]). Let us now describe what a reasonable definition of an equilbrium should look like. If the overall transport pattern is Q, an agent commuting from x to y choosing a path $\sigma \in C_{x,y}$ (i.e. an absolutely continuous curve σ such that $\sigma(0) = x$ and $\sigma(1) = y$) spends time

$$L_{\xi_Q}(\sigma) = \int_0^1 g(\sigma(t), i_Q(\sigma(t))|\dot{\sigma}(t)|dt.$$

She will then try to minimize this time i.e. to achieve the corresponding geodesic distance.

$$c_{\xi_Q}(x,y) := \inf_{\sigma \in C_{x,y}} L_{\xi_Q}(\sigma)$$

Paths in $C_{x,y}$ such that $c_{\xi_Q}(x,y) = L_{\xi_Q}(\sigma)$ are called geodesics (for the metric induced by the congestion effect generated by Q). A first requirement, in the definition of an equilibrium therefore is that Q-a.e. path σ is a geodesic between its endpoints $\sigma(0)$ and $\sigma(1)$. The transportation pattern may be disintegrated with respect to $\gamma_Q := (e_0, e_1)_\# Q$:

$$Q = \gamma_Q \otimes (p^{x,y})$$

i.e.

$$\int_C \Phi(\sigma) dQ(\sigma) = \int_{\overline{\Omega} \times \overline{\Omega}} \Big(\int_{C_{x,y}} \Phi(\sigma) dp^{x,y}(\sigma) \Big) d\gamma_Q(x,y), \quad \forall \Phi.$$

In other words, $\gamma_Q(A \times B)$ is the probability that a path has starting point in A and a terminal point in B. Denoting by $\Pi(\mu_0, \mu_1)$ the set of transport plans between μ_0 and μ_1 (that is the set of probability measures on $\overline{\Omega} \times \overline{\Omega}$ having μ_0 and μ_1 as marginals), the requirement that $Q \in \mathcal{Q}(\mu_0, \mu_1)$ obviously translates into $\gamma_Q \in \Pi(\mu_0, \mu_1)$. Given starting and terminal points (x,y), $p^{x,y}$ is a probability on $C_{x,y}$ that represents the probability over paths conditional on (x,y). The requirement that Q gives full mass to geodesics says that for γ_Q-a.e. (x,y), $p^{x,y}$ is supported on the set of geodesics between x and y but this does require any particular property on the coupling γ_Q. We thus supplement the definition of an equlbrium by the additional requirement that γ_Q should solve the optimal transportation problem:

$$\inf_{\gamma\in\Pi(\mu_0,\mu_1)} \int_{\overline{\Omega}\times\overline{\Omega}} c_{\xi_Q}(x,y)d\gamma(x,y). \tag{7}$$

This yields:

Definition 2. *A Wardrop equilibrium (for the long-term problem) is a $Q \in \mathcal{Q}(\mu_0,\mu_1)$ such that*

$$Q(\{\sigma \ : \ L_{\xi_Q}(\sigma) = c_{\xi_Q}(\sigma(0),\sigma(1)) = 1 \tag{8}$$

and $\gamma_Q := (e_0,e_1)_{\#}Q$ solves the optimal transport problem (7).

Of course in the short-term case, γ_Q is fixed equal to γ so that Wardrop equilibria are defined by condition (8) only.

Let us then consider the (convex) variational problem

$$\inf_{Q\in\mathcal{Q}(\mu_0,\mu_1)} \int_{\Omega} H(x,i_Q(x))dx \tag{9}$$

where $H'(x,.) = g(x,.)$, $H(x,0) = 0$. We shall refer to (9) as the congested optimal mass transportation problem for reasons that will be clarified later. Under some technical assumptions that we do not reproduce here, the main results of [10] can be summarized by:

Theorem 2. *Problem* (9) *admits at least one minimizer. Moreover $\overline{Q} \in \mathcal{Q}(\mu_0,\mu_1)$ solves* (9) *if and only if it is a Wardrop equilibrium. In particular there exist Wardrop equilibria.*

The full proof is quite involved since it requires to take care of some regularity issues in details. But the intuition of why solutions of (9) are Wardrop equilibria can be understood easily from the following formal manipulations. By convexity arguments, it is easily seen that $\overline{Q} = \overline{\gamma}\otimes\overline{p}^{x,y} \in \mathcal{Q}(\mu_0,\mu_1)$ solves (9) if and only if it satifies the variational inequalities

$$\int_{\Omega} \overline{\xi} i_{\overline{Q}} = \inf\left\{\int_{\Omega} \overline{\xi} i_Q \ : \ Q \in \mathcal{Q}(\mu_0,\mu_1)\right\} \text{ with } \overline{\xi}(x) := H'(x, i_{\overline{Q}}(x)), \tag{10}$$

which we may rewrite as

$$
\begin{aligned}
\int_{\Omega} \overline{\xi} i_{\overline{Q}} &= \int_{C} L_{\overline{\xi}}(\sigma) d\overline{Q}(\sigma) \\
&= \int_{\overline{\Omega}\times\overline{\Omega}} \left(\int_{C^{x,y}} L_{\overline{\xi}}(\sigma) d\overline{p}^{x,y}(\sigma) \right) d\overline{\gamma}(x,y) \\
&= \inf_{(\gamma,p)} \int_{\overline{\Omega}\times\overline{\Omega}} \left(\int_{C^{x,y}} L_{\overline{\xi}}(\sigma) dp^{x,y}(\sigma) \right) d\gamma(x,y) \\
&= \inf_{\gamma\in\Pi(\mu_0,\mu_1)} \int_{\overline{\Omega}\times\overline{\Omega}} \left(\inf_{p\in\mathcal{M}^1_+(C^{x,y})} \int_{C^{x,y}} L_{\overline{\xi}}(\sigma) dp(\sigma) \right) d\gamma(x,y) \\
&= \inf_{\gamma\in\Pi(\mu_0,\mu_1)} \int_{\overline{\Omega}\times\overline{\Omega}} \left(\inf_{\sigma\in C^{x,y}} L_{\overline{\xi}}(\sigma) \right) d\gamma(x,y)
\end{aligned}
$$

Let us then define the geodesic distance $c_{\overline{\xi}}$ by

$$
c_{\overline{\xi}}(x,y) := \inf_{\sigma\in C^{x,y}} L_{\overline{\xi}}(\sigma),
$$

we firstly get

$$
\begin{aligned}
\int_{\overline{\Omega}\times\overline{\Omega}} c_{\overline{\xi}}(x,y) d\overline{\gamma}(x,y) &\le \int_{C} L_{\overline{\xi}} d\overline{Q} \\
&= \inf_{\gamma\in\Pi(\mu_0,\mu_1)} \int_{\overline{\Omega}\times\overline{\Omega}} c_{\overline{\xi}}(x,y) d\gamma(x,y)
\end{aligned}
$$

so that $\overline{\gamma}$ solves the Monge-Kantorovich problem:

$$
\inf_{\gamma\in\Pi(\mu_0,\mu_1)} \int_{\overline{\Omega}\times\overline{\Omega}} c_{\overline{\xi}}(x,y) d\gamma(x,y).
$$

Secondly, we obtain

$$\int_C L_{\overline{\xi}}(\sigma)d\overline{Q}(\sigma) = \int_{\overline{\Omega}\times\overline{\Omega}} c_{\overline{\xi}}(x,y)d\overline{\gamma}(x,y)$$
$$= \int_C c_{\overline{\xi}}(\sigma(0),\sigma(1))d\overline{Q}(\sigma)$$

and since $L_{\overline{\xi}}(\sigma) \geq c_{\overline{\xi}}(\sigma(0),\sigma(1))$, we get

$$L_{\overline{\xi}}(\sigma) = c_{\overline{\xi}}(\sigma(0),\sigma(1)) \quad \text{for } \overline{Q}\text{-a.e. } \sigma.$$

or, in an equivalent way, for $\overline{\gamma}$-a.e. (x,y) one has:

$$L_{\overline{\xi}}(\sigma) = c_{\overline{\xi}}(x,y) \quad \text{for } \overline{p}^{x,y}\text{-a.e. } \sigma$$

which exactly proves that $\overline{Q}$ is a Wardrop equilibrium.

Remark 3. The use of the weighted length functional $L_{\overline{\xi}}$ and thus also the geodesic distance $c_{\overline{\xi}}$ above is purely formal since defining these quantities actually makes sense only if $\overline{\xi}$ is continuous or at least l.s.c.. We therefore refer the interested reader to [10] for details on how to define these objects when $\overline{\xi}$ is just an L^q function. Let us also mention that a recent regularity result (see [19]) actually proves that $\overline{\xi}$ is in fact a continuous function (in dimension 2 and under reasonable assumptions on the data).

Remark 4. For the short-term problem, a similar variational characterization holds, namely that $\overline{Q} \in \mathcal{Q}(\gamma)$ is a (short-term) Wardrop equilibrium if and only if it solves

$$\inf_{Q\in\mathcal{Q}(\gamma)} \int_\Omega H(x, i_Q(x))dx. \tag{11}$$

We have proved that, as in the finite-dimensional network case, Wardrop equilibria have a variational characterization which is in principle easier to deal with than the definition. Unfortunately, the convex problems (9) and (11) may be difficult to solve since they involve measures on sets of curves that is two layers of infinite dimensions! The next two paragraphs are precisely intended to consider different formulations that turn out to be much more tractable:

- for the short-term problem (11), we will see that the equilibrium metrics solve a kind of dual problem that can be solved numerically,

- for the long-term problem (9), we will deduce optimal Q's from a minimal flow problem à la Beckmann and a construction à la Moser, in other words, the problem will amount to solve a certain nonlinear elliptic PDE (which turns out to be quite degenerate in realistic congestion models).

§4. Duality for the short-term problem

The purpose of this Section is to give a dual and tractable formulation of the variational problem for the short-term problem (11). For every $x \in \Omega$ and $\xi \geq 0$, let us define

$$H^*(x,\xi) := \sup\{\xi i - H(x,i),\ i \geq 0\}, \quad \xi_0(x) := g(x,0).$$

By our assumptions on g, one has $H^*(x,\xi) = 0$ for every $x \in \Omega$ and $\xi \leq \xi_0(x)$. Let us recall Young's inequality:

$$H(x,i) + H^*(x,\xi) \geq \xi i, \quad \forall i \geq 0,\ \forall \xi \geq \xi_0(x). \tag{12}$$

Notice that the inequality (12) is strict unless $\xi = g(x,i) \geq \xi_0(x)$. In particular, for $Q \in \mathcal{Q}(\gamma)$, we have the identity

$$H(x, i_Q(x)) + H^*(x, \xi_Q(x)) = \xi_Q(x) i_Q(x) \tag{13}$$

and

$$H(x, i_Q(x)) + H^*(x,\xi) > \xi i_Q(x),\ \forall \xi \geq \xi_0(x),\ \xi \neq \xi_Q(x) \tag{14}$$

(for $\xi_Q(x) := g(x, i_Q(x))$). Let us now define the functional

$$J(\xi) = \int_{\Omega} H^*(x,\xi(x))dx - \int_{\overline{\Omega}\times\overline{\Omega}} c_\xi(x,y)d\gamma(x,y) \tag{15}$$

where, as usual, c_ξ is the geodesic distance associated to the metric ξ i.e.

$$c_\xi(x,y) := \inf_{\sigma \in C_{x,y}} L_\xi(\sigma).$$

Consider now:

$$\sup\{-J(\xi) : \xi \geq \xi_0\} \tag{16}$$

Theorem 3. *The following duality formula holds*

$$\min(11) = \max(16) \tag{17}$$

and ξ solves (16) if and only if $\xi = \xi_Q$ for some $Q \in \mathcal{Q}(\gamma)$ solving (11).

Proof. Let $Q \in \mathcal{Q}(\gamma)$ (so that $\xi_Q \geq \xi_0$) and let $\xi \geq \xi_0$; from (12) and

$$\int_\Omega \xi(x) i_Q(x)\, dx = \int_C L_\xi(\sigma)\, dQ(\sigma). \tag{18}$$

we first get:

$$\begin{aligned}\int_\Omega H(x, i_Q(x))dx &\geq \int_\Omega \xi i_Q - \int_\Omega H^*(x, \xi(x))dx \\ &= \int_C L_\xi(\sigma) dQ(\sigma) - \int_\Omega H^*(x, \xi(x))dx.\end{aligned}$$

Using the fact that

$$L_\xi(\sigma) \geq c_\xi(\sigma(0), \sigma(1)) \tag{19}$$

and $Q \in \mathcal{Q}(\gamma)$ we then have

$$\int_C L_\xi(\sigma) dQ(\sigma) \geq \int_C c_\xi(\sigma(0), \sigma(1)) dQ(\sigma) = \int_{\overline{\Omega}\times\overline{\Omega}} c_\xi(x, y) d\gamma(x, y).$$

Since $Q \in \mathcal{Q}(\gamma)$ and $\xi \geq \xi_0$ are arbitrary and since we already know that the infimum of (11) is attained we thus deduce

$$\min(11) \geq \sup(16). \tag{20}$$

Now let $Q \in \mathcal{Q}(\gamma)$ solve (11) and set $\xi := \xi_Q$ (recall that ξ_Q does not depend on the choice of the minimizer Q). From the equivalence between Wardrop equilibria and solutions of (11), we know that

$$L_\xi(\sigma) = c_\xi(\sigma(0), \sigma(1)) \text{ for } Q\text{-a.e. } \sigma \in C.$$

With (18), integrating the previous identity and using $Q \in \mathcal{Q}(\gamma)$ we then get:

$$\int_\Omega \xi i_Q = \int_C L_\xi(\sigma) dQ(\sigma) = \int_{\overline{\Omega}\times\overline{\Omega}} c_\xi(x, y) d\gamma(x, y).$$

Using (13), (20) and the fact that $Q \in \mathcal{Q}(\gamma)$ solves (11) yields:

$$\begin{aligned}\sup(16) \leq \min(11) = \int_\Omega H(x, i_Q(x))dx &= \int_\Omega \xi i_Q - \int_\Omega H^*(x, \xi(x))dx \\ &= \int_{\overline{\Omega}\times\overline{\Omega}} c_\xi(x, y) d\gamma(x, y) - \int_\Omega H^*(x, \xi(x))dx\end{aligned}$$

so that ξ solves (16) and (17) is satisfied. Finally if ξ solves (16) and $Q \in \mathcal{Q}(\gamma)$ solves (11), then with (18) and (19), one has

$$\int_{\Omega} \xi i_Q - \int_{\Omega} H^*(x, \xi(x))dx \geq \int_{\overline{\Omega}\times\overline{\Omega}} c_\xi(x,y)d\gamma(x,y) - \int_{\Omega} H^*(x,\xi(x))dx$$
$$= \max(16) = \min(11) = \int_{\Omega} H(x, i_Q(x))dx$$

and thus we deduce from (12) and (14) that $\xi = \xi_Q$. □

Remark 5. Under reasonable continuity and strict monotonicity assumptions on the congestion function g, the dual problem (16) has a unique solution so that the equilibrium metric ξ_Q and the equilibrium intensity of traffic i_Q are unique although Wardrop equilibria Q might not be unique.

Numerics. In [5, 6], we designed a consistent numerical scheme to approximate the equilibrium metric ξ_Q by a descent method on the dual which can be done in an efficient way by the *Fast Marching Algorithm.* One can recover the corresponding equilibrium intensity i_Q by inverting the relation $\xi(x) = g(x, i_Q(x))$.

The goal is to find a method to approximate the minimizers of functional J in (15). This is done by means of a discretization grid, and the values of ξ are considered as defined at the nodes of the grid. The first integral becomes a sum on all the points of the grid, while, for the second, one needs to replace the transport plan γ with a discretized one defined on pairs of points (x, y) on the same grid, and to define $c_\xi(x, y)$ consequently.

To define such a distance $c_\xi(x_0, \cdot)$, for a fixed source x_0, as a function of the second variable, one uses the fact that it is the unique viscosity solution of the Eikonal non-linear PDE

$$\begin{cases} \|\nabla \mathcal{U}^\xi(x)\| = \xi, \\ \mathcal{U}^\xi(x)(x_0) = 0, \end{cases} \tag{21}$$

The computation of $\mathcal{U}^\xi(x)$ thus requires the discretization of (21) so that a numerical scheme captures the viscosity solution of the equation. By dropping the dependence on ξ and x_0 of the distance map $\mathcal{U}^\xi = \mathcal{U}$ to ease the notations, the geodesic distance map $\mathcal{U}^\xi$ is discretized on a grid of $n \times n$ points, so that $\mathcal{U}_{i,j}$ for $0 \leq i, j < n$ is an approximation of $\mathcal{U}^\xi(ih, jh)$ where the grid step is $h = 1/n$. The metric ξ is also discretized so that $\xi_{i,j} = \xi(ih, jh)$.

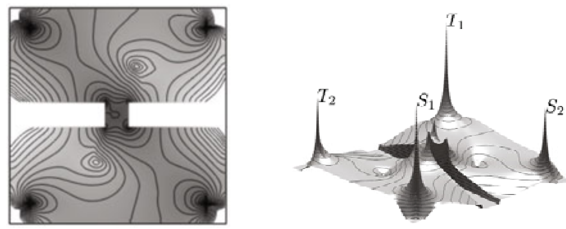

Figure 1. Traffic intensity at equilbrium in a city with a river and a bridge.

Classical finite difference schemes do not capture the viscosity solution of (21); upwind derivative should be used instead

$$D_1\mathcal{U}_{i,j} := \max\{(\mathcal{U}_{i,j} - \mathcal{U}_{i-1,j}), (\mathcal{U}_{i,j} - \mathcal{U}_{i+1,j}), 0\}/h,$$
$$D_2\mathcal{U}_{i,j} := \max\{(\mathcal{U}_{i,j} - \mathcal{U}_{i,j-1}), (\mathcal{U}_{i,j} - \mathcal{U}_{i,j+1}), 0\}/h.$$

As proposed by Rouy and Tourin [17], the discrete geodesic distance map $\mathcal{U} = (\mathcal{U}_{i,j})_{i,j}$ is found as the solution of the following discrete non-linear equation that discretizes (21)

$$D\mathcal{U} = \xi \quad \text{where} \quad D\mathcal{U}_{i,j} = \sqrt{D_1\mathcal{U}_{i,j}^2 + D_2\mathcal{U}_{i,j}^2}. \tag{22}$$

Rouy and Tourin [17] showed that this discrete geodesic distance $\mathcal{U}$ converges to $\mathcal{U}^\xi$ when h tends to 0. The Fast Marching Algorithm exactly uses a clever way of ordering the points of the grid so as to solve recursively all the equations in (22).

Once we are able to compute the value of $J(\xi)$ for every discrete metric ξ on the grid, we want to differentiate it w.r.t. ξ, so as to to take advantage of a gradient descent algorithm. Actually, one can see that J is not always differentiable in ξ, but, since all the terms $c_\xi(x, y)$ may be proven to be concave in ξ, we face a convex function, and we can look for its sub-differential. Differentiating the equations in (22) (see [6]) one gets a new set of equations on the gradient $\nabla_\xi c_\xi(x, y)$. The same loop of the Fast Marching Algorithm allows to solve them in a quite efficient way, thus giving an element of the sub-differential. Afterwards, usual subgradient algorithms allow to approximate the optimal solution $\bar{\xi}$.

An example is given in the following figure:

In a symmetric configuration of two sources S_1 and S_2, and two targets T_1 and T_2; we consider a river where there is no traffic and a bridge linking the two sides of the river (see the map on the left in Figure 1, where the grey scale and the level lines are meant to show the equilibrium traffic intensity). We chose the traffic weights such that $\gamma_{1,1}+\gamma_{1,2} = 2(\gamma_{2,1}+\gamma_{2,2})$ and $\frac{\gamma_{2,2}}{\gamma_{2,1}} = \frac{\gamma_{1,1}}{\gamma_{1,2}} = 2$. The traffic intensity going out from S_1 is twice S_2's. One can note, in the picture at the right in Figure 1, the two hollows on each side of the river appearing because of the inter-sides and intra-sides crossed traffics, together with the traffic peaks close to the bridge ant do the points S_i and T_j.

§5. Beckmann-like reformulation of the long-term problem

In the long-term problem (9), we have one more degree of freedom since the transport plan is not fixed. This will enable us to reformulate the problem as a variational divergence constrained problem à la Beckmann and ultimately to reduce the equilibrium problem to solving some nonlinear PDE. For $Q \in \mathcal{Q}(\mu_0, \mu_1)$, let us define the vector-field σ_Q through

$$\forall X \in C(\overline{\Omega}, \mathbb{R}^d)$$
$$\int_{\overline{\Omega}} X(x)\sigma_Q(x)dx := \int_{C([0,1],\overline{\Omega})} \left(\int_0^1 X(\gamma(t)) \cdot \dot{\gamma}(t)dt \right) dQ(\gamma)$$

which is a kind of vectorial traffic intensity. Taking a gradient field $X = \nabla u$ in the previous definition yields

$$\begin{aligned} \int_{\overline{\Omega}} \nabla u \sigma_Q &= \int_{C([0,1],\overline{\Omega})} [u(\sigma(1)) - u(\sigma(0))] dQ(\gamma) \\ &= \int_{\Omega} u(\mu_1 - \mu_0) \end{aligned}$$

which means that

$$\nabla \cdot \sigma_Q = \mu_0 - \mu_1,$$

moreover it is easy to check that

$$|\sigma_Q| \leq i_Q.$$

Since H is increasing, it proves that the value of the scalar problem (9) is larger than that of the minimal flow problem à la Beckmann:

$$\inf_{\sigma\,:\,\nabla\cdot\sigma=\mu_0-\mu_1}\int_\Omega \mathcal{H}(\sigma(x))dx \tag{23}$$

where $\mathcal{H}(\sigma)=H(|\sigma|)$ and H is taken independent of x only for simplicity. Conversely, if σ is a minimizer of (23) and $Q\in\mathcal{Q}(\mu_0,\mu_1)$ is such that $i_Q=|\sigma|$ then Q solves the scalar problem (9) (i.e. is an equilibrium).

To build such a Q, we can formally use the following construction *à la* Moser (see Moser [15] and Dacorogna and Moser [11]). Assuming σ smooth and μ_0 and μ_1 absolutely continuous, with nice densities bounded away from 0, let us consider the nonautonmous ODE

$$\dot X(t,x)=\frac{\sigma(X(t,x))}{(1-t)\mu_0(X(t,x))+t\mu_1(X(t,x))},\ X(0,x)=x,$$

and define $\overline{Q}$ by

$$\overline{Q}=\ \delta_{X(.,x)}\otimes\mu_0.$$

Set $\mu_t=(1-t)\mu_0+t\mu_1$ and

$$v(t,x)=\frac{\sigma(x)}{\mu_t(x)}$$

then by construction μ_t solves the continuity equation:

$$\partial_t\mu_t+\nabla\cdot(\mu_t v)=0$$

By construction we also have $e_{0\#}\overline{Q}=\mu_0$ and, because of the uniqueness in the continuity equation, $X(t,.)_\#\mu_0=\mu_t=(1-t)\mu_0+t\mu_1$. In particular the image of μ_0 by the flow at time 1, $X(1,.)$ is μ_1, which proves that $e_{1\#}\overline{Q}=\mu_1$ hence $\overline{Q}\in\mathcal{Q}(\mu_0,\mu_1)$. Moreover for every test-function φ:

$$\begin{aligned}\int_\Omega \varphi\, di_{\overline{Q}} &= \int_\Omega\int_0^1 \varphi(X(t,x))|v(t,X(t,x))|dtd\mu_0(x)\\ &= \int_0^1\int_\Omega \varphi(x)|v(t,x)|\mu_t(x)dxdt\\ &= \int_\Omega \varphi(x)|\sigma(x)|dx\end{aligned}$$

so that $i_{\overline{Q}}=|\sigma|$ and then $\overline{Q}$ is optimal.

The previous argument works as soon as σ is regular enough (say, Lipschitz continuous). To get regularity, one needs to look at the optimality conditions satisfied by σ as a minimizer of (23). By duality, the solution of (23) is $\sigma = \nabla\mathcal{H}^*(\nabla u)$ where $\mathcal{H}^*$ is the Legendre transform of $\mathcal{H}$ and u solves the PDE:

$$\begin{cases} \nabla\cdot(\nabla\mathcal{H}^*(\nabla u)) = \mu_0 - \mu_1, & \text{in } \Omega, \\ \nabla\mathcal{H}^*(\nabla u)\cdot\nu = 0, & \text{on } \partial\Omega, \end{cases} \tag{24}$$

This equation turns out to be a standard Laplace equation if $\mathcal{H}$ is quadratic, or it becomes a $p-$Laplace equation for other power functions. In these cases, regularity results are well-known, under regularity assumptions on μ_0 and μ_1. Yet, let us recall that $H' = g$ where g is the congestion function, so it is natural to have $g(0) > 0$: the metric is positive even if there is no traffic! This means that the radial function $\mathcal{H}$ is not differentiable at 0 and then its subdifferential at 0 contains a ball. By duality, this implies $\nabla\mathcal{H}^* = 0$ on this ball which makes (24) very degenerate, even worse than the $p-$Laplacian. For instance, a reasonable model of congestion is $g(t) = 1 + t^{p-1}$ for $t \geq 0$, with $p > 1$, so that

$$\mathcal{H}(\sigma) = \frac{1}{p}|\sigma|^p + |\sigma|,\ \mathcal{H}^*(z) = \frac{1}{q}(|z|-1)_+^q,\ \text{with } q = \frac{p}{p-1} \tag{25}$$

so that the optimal σ is

$$\sigma = \Big(|\nabla u| - 1\Big)_+^{q-1}\frac{\nabla u}{|\nabla u|},$$

where u solves the very degenerate PDE:

$$\nabla\cdot\left(\Big(|\nabla u| - 1\Big)_+^{q-1}\frac{\nabla u}{|\nabla u|}\right) = \mu_0 - \mu_1, \tag{26}$$

with Neumann boundary condition

$$\Big(|\nabla u| - 1\Big)_+^{q-1}\frac{\nabla u}{|\nabla u|}\cdot\nu = 0.$$

Note that there is no uniqueness for u but there is for σ.

For this degenerate equation (more degenerate than the p-laplacian since the diffusion coefficient identically vanishes in the zone where $|\nabla u| \leq 1$), getting Lipschitz continuity on σ is not reasonable. Yet, Sobolev regularity of σ and Lipschitz regularity results for solutions of this PDE can be found in [8]. This enables one to build a flow *à la* DiPerna-Lions [13] and then to justify rigorously the construction above, even without a Cauchy-Lipschitz flow. Interestingly, in two dimensions it is also available (see [19]) a continuity result on the optimal σ, obtained as a consequence of a fine analysis of this degenerate elliptic PDE. Besides the interest for this regularity result in itself, we also stress that continuity for σ implies continuity for the optimal i_Q, and this exactly gives the regularity which is required in the proof of Theorem 2 (the main difficulty being defining $c_{\bar{\xi}}$ for a non-continuous $\bar{\xi}$, and this is the reason why our proof in Section 3 is only formal).

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CEREMADE, UMR CNRS 7534,
Université Paris-Dauphine,
Pl. de Lattre de Tassigny,
75775 Paris Cedex 16,
France
E-mail: carlier@ceremade.dauphine.fr

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Laboratoire de Mathématiques,
UMR CNRS 8628, Faculté des Sciences,
Université Paris-Sud XI,
91405 Orsay Cedex,
France
E-mail: filippo.santambrogio@math.u-psud.fr