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APPLICATION OF A BERNSTEIN-TYPE INEQUALITY TO RATIONAL INTERPOLATION IN THE DIRICHLET SPACE

Abstract. We prove a Bernstein-type inequality involving the Bergman and Hardy norms, for rational functions in the unit disk $\mathbb{D}$ having at most n poles all outside of $\frac{1}{r}\mathbb{D}$, $0 < r < 1$. The asymptotic sharpness of this inequality is shown as $n \to \infty$ and $r \to 1^-$. We apply our Bernstein-type inequality to an efficient Nevanlinna-Pick interpolation problem in the standard Dirichlet space, constrained by the H^2-norm.

Introduction

a. Statement of the problems. Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ be the unit disk of the complex plane and let $\mathrm{Hol}\,(\mathbb{D})$ be the space of holomorphic functions on $\mathbb{D}$. Let also X and Y be two Banach spaces of holomorphic functions on $\mathbb{D}$: $X, Y \subset \mathrm{Hol}\,(\mathbb{D})$. Here and later on, H^∞ stands for the space (algebra) of bounded holomorphic functions in the unit disk $\mathbb{D}$ endowed with the norm $\|f\|_\infty = \sup_{z\in\mathbb{D}} |f(z)|$. We suppose that $n \geq 1$ is an integer, $r \in [0, 1)$, and we consider the following two problems.

Problem 1. Let $\mathcal{P}_n$ be the complex space of analytic polynomials of degree less than or equal to n, and let

$$\mathcal{R}_{n,r} = \left\{ \frac{p}{q} \,:\, q \in \mathcal{P}_n,\ \mathrm{d}^\circ p < \mathrm{d}^\circ q,\ q(\zeta) = 0 \Longrightarrow |\zeta| \geq \frac{1}{r} \right\}$$

(where $\mathrm{d}^\circ p$ means the degree of any $p \in \mathcal{P}_n$) be the set of all rational functions in $\mathbb{D}$ of degree less than or equal to $n \geq 1$ having at most n poles all outside of $\frac{1}{r}\mathbb{D}$. Notice that for $r = 0$, we get $\mathcal{R}_{n,0} = \mathcal{P}_{n-1}$. Our first problem is to search the "best possible" constant $\mathcal{C}_{n,r}(X, Y)$ such that

$$\|f'\|_X \leq \mathcal{C}_{n,r}(X, Y)\, \|f\|_Y$$

for all $f \in \mathcal{R}_{n,r}$.

Key words and phrases: Bernstein-type inequality, Bergman space, Besov space.

Problem 2. Let $\sigma = \{\lambda_1, \dots, \lambda_n\}$ be a finite subset of $\mathbb{D}$. What is the best possible interpolation by functions of the space Y for the traces $f_{|\sigma}$ of functions of the space X, in the worst case? The case where $X \subset Y$ is of no interest, and so one can suppose that either $Y \subset X$ or X and Y are incomparable. More precisely, our second problem is to compute or estimate the following interpolation constant

$$I(\sigma, X, Y) = \sup_{f \in X,\, \|f\|_X \leq 1} \inf \left\{ \|g\|_Y : g_{|\sigma} = f_{|\sigma} \right\}.$$

We also define

$$\mathcal{I}_{n,r}(X, Y) = \sup \{ I(\sigma, X, Y) : \operatorname{card} \sigma \leq n, \, |\lambda| \leq r, \, \forall \lambda \in \sigma \}.$$

b. Motivations.

Problem 1. Bernstein-type inequalities for rational functions are applied:

1.1) in matrix analysis and in operator theory (see "Kreiss Matrix Theorem" [5, 10] or [11, 14]) for resolvent estimates of power bounded matrices,

1.2) to "inverse theorems on rational approximation" by using the classical Bernstein decomposition (see [3, 8, 9]),

1.3) to effective H^∞ interpolation problems (see [13] and our Theorem B below in Subsection d), and more generally to our Problem 1.

Problem 2. We can give three main motivations for Problem 2.

2.1) It was explained in [13] (the case of $Y = H^\infty$) why the classical interpolation problems, those of Nevanlinna-Pick (1908) and Carathéodory-Schur (1916) (see [7, p. 231] for these two problems) on one hand, and Carleson's free interpolation problem (1958) (see [6, p. 158]), on the other hand, are of the nature of our interpolation problem.

2.2) It was also explained in [13] why this constrained interpolation is motivated by some applications to matrix analysis and operator theory.

2.3) It was proved in [13] that for $X = H^2$ (see Subsection c for the definition of H^2) and $Y = H^\infty$, we have

$$\frac{1}{4\sqrt{2}} \frac{\sqrt{n}}{\sqrt{1-r}} \leq \mathcal{I}_{n,r}\left(H^2, H^\infty\right) \leq \sqrt{2} \frac{\sqrt{n}}{\sqrt{1-r}}. \tag{1}$$

The above estimate (1) answers a question of L. Baratchart (private communication), which is part of a more complicated question arising in an applied situation in [1] and [2]: given a set $\sigma \subset \mathbb{D}$, how to estimate $I\left(\sigma, H^2, H^\infty\right)$ in terms of $n = \operatorname{card}(\sigma)$ and $\max_{\lambda \in \sigma} |\lambda| = r$ only?

c. The spaces X and Y considered here. Now we define some Banach spaces X and Y of holomorphic functions in $\mathbb{D}$, which we will consider throughout this paper. From now on, if $f \in \mathrm{Hol}(\mathbb{D})$ and $k \in \mathbb{N}$, $\hat{f}(k)$ stands for the kth Taylor coefficient of f.

1. The standard Hardy space $H^2 = H^2(\mathbb{D})$,

$$H^2 = \left\{ f \in \mathrm{Hol}\,(\mathbb{D}) : \, \|f\|_{H^2}^2 = \sup_{0 \le r < 1} \int_{\mathbb{T}} |f(rz)|^2 \, \mathrm{d}m(z) < \infty \right\},$$

where m stands for the normalized Lebesgue measure on $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$. An equivalent description of the space H^2 is

$$H^2 = \left\{ f = \sum_{k \ge 0} \hat{f}(k) z^k : \, \|f\|_{H^2} = \left(\sum_{k \ge 0} \left| \hat{f}(k) \right|^2 \right)^{\frac{1}{2}} < \infty \right\}.$$

2. The standard Bergman space $L_a^2 = L_a^2\,(\mathbb{D})$,

$$L_a^2 = \left\{ f \in \mathrm{Hol}\,(\mathbb{D}) : \, \|f\|_{L_a^2}^2 = \frac{1}{\pi} \int_{\mathbb{D}} |f(z)|^2 \, \mathrm{d}A(z) < \infty \right\},$$

where A is the standard area measure, also defined by

$$L_a^2 = \left\{ f = \sum_{k \ge 0} \hat{f}(k) z^k : \, \|f\|_{L_a^2} = \left(\sum_{k \ge 0} \left| \hat{f}(k) \right|^2 \frac{1}{k+1} \right)^{\frac{1}{2}} < \infty \right\}.$$

3. The analytic Besov space $B_{2,\,2}^{\frac{1}{2}}$ (also known as the standard Dirichlet space) defined by

$$B_{2,\,2}^{\frac{1}{2}} = \left\{ f = \sum_{k \ge 0} \hat{f}(k) z^k : \, \|f\|_{B_{2,\,2}^{\frac{1}{2}}} = \left(\sum_{k \ge 0} (k+1) \left| \hat{f}(k) \right|^2 \right)^{\frac{1}{2}} < \infty \right\}.$$

Then if $f \in B_{2,\,2}^{\frac{1}{2}}$, we have the following equality

$$\|f\|_{B_{2,\,2}^{\frac{1}{2}}}^2 = \|f'\|_{L_a^2}^2 + \|f\|_{H^2}^2 \,, \tag{2}$$

which establishes a link between the spaces $B_{2,\,2}^{\frac{1}{2}}$ and L_a^2.

d. The results. Here and later on, the letter c denotes a positive constant that may change from one step to the next. For two positive functions a and b, we say that a is dominated by b, denoted by $a = O(b)$, if there is a constant $c > 0$ such that $a \leq cb$; and we say that a and b are comparable, denoted by $a \asymp b$, if both $a = O(b)$ and $b = O(a)$.

Problem 1. Our first result (Theorem A below) is a partial case ($p = q = 2$, $s = \frac{1}{2}$) of the following result by Dyakonov, see [4]: if $p \in [1, \infty)$, $s \in (0, +\infty)$, $q \in [1, +\infty]$, then there exists a constant $c_{p,\,s} > 0$ such that

$$\mathcal{C}_{n,\,r}\left(B^{s-1}_{p,\,p},\, H^q\right) \leq c_{p,\,s} \sup \|B'\|^s_{H^\gamma}\,, \tag{3}$$

where γ is such that $\frac{s}{\gamma} + \frac{1}{q} = \frac{1}{p}$, and the supremum is taken over all finite Blaschke products B of order n with n zeros outside of $\frac{1}{r}\mathbb{D}$. Here $B^s_{p,\,p}$ stands for the Hardy-Besov space which consists of analytic functions f on $\mathbb{D}$ satisfying

$$\|f\|_{B^s_{p,\,p}} = \sum_{k=0}^{n-1}\left|f^{(k)}(0)\right| + \int_{\mathbb{D}} (1-|w|)^{(n-s)p-1}\left|f^{(n)}(w)\right|^p \mathrm{d}A(w) < \infty.$$

For the (tiny) partial case considered here, our proof is different and the constant $c_{2,\,\frac{1}{2}}$ is asymptotically sharp as r tends to 1^- and n tends to $+\infty$.

Theorem A. *Suppose $n \geq 1$ and $r \in [0,\, 1)$. We have*

(i)

$$\widetilde{a}(n,\,r)\sqrt{\frac{n}{1-r}} \leq \mathcal{C}_{n,\,r}\left(L^2_a,\, H^2\right) \leq \widetilde{A}(n,\,r)\sqrt{\frac{n}{1-r}}, \tag{4}$$

where

$$\widetilde{a}(n,\,r) \geq \left(1 - \frac{1-r}{n}\right)^{\frac{1}{2}} \text{ and } \widetilde{A}(n,\,r) \leq \left(1 + r + \frac{1}{\sqrt{n}}\right)^{\frac{1}{2}}.$$

(ii) *Moreover, the sequence*

$$\left(\frac{\mathcal{C}_{n,\,r}\left(L^2_a,\, H^2\right)}{\sqrt{n}}\right)_{n\geq 1}$$

is convergent for all $r \in [0, 1)$, and

$$\lim_{n\to\infty} \frac{\mathcal{C}_{n,\,r}\left(L^2_a,\, H^2\right)}{\sqrt{n}} = \sqrt{\frac{1+r}{1-r}}. \tag{5}$$

Notice that it was proved in [12] that there exists a limit

$$\lim_{n\to\infty} \frac{\mathcal{C}_{n,\,r}\left(H^2,\, H^2\right)}{n} = \frac{1+r}{1-r}, \tag{6}$$

for every $r,\ 0 \le r < 1$.

Problem 2. Looking at the motivation 2.3, we replace the algebra H^∞ by the Dirichlet space $B^{\frac{1}{2}}_{2,\,2}$. We show that the "gap" between $X = H^2$ and $Y = H^\infty$ (see (1)) is asymptotically the same as the one which exists between $X = H^2$ and $Y = B^{\frac{1}{2}}_{2,\,2}$. In other words,

$$\mathcal{I}_{n,\,r}\left(H^2,\, B^{\frac{1}{2}}_{2,\,2}\right) \asymp \mathcal{I}_{n,\,r}\left(H^2,\, H^\infty\right) \asymp \sqrt{\frac{n}{1-r}}. \tag{7}$$

More precisely, we prove the following Theorem B, in which the right inequality in (10) is a consequence of the right inequality in (4) in the above Theorem A.

Theorem B. *Suppose $n \ge 1$ and $r \in [0,\, 1)$. Then*

$$\mathcal{I}_{n,\,r}\left(H^2,\, B^{\frac{1}{2}}_{2,\,2}\right) \le \left[\left(\mathcal{C}_{n,\,r}\left(L^2_a,\, H^2\right)\right)^2 + 1\right]^{\frac{1}{2}}. \tag{8}$$

For $\lambda \in \mathbb{D}$, consider the corresponding one-point interpolation set $\sigma_{n,\,\lambda} = \underbrace{\{\lambda, \lambda, \dots, \lambda\}}_{n}$. We have

$$I\left(\sigma_{n,\,\lambda}, H^2,\, B^{\frac{1}{2}}_{2,\,2}\right) \ge \sqrt{\frac{n}{1-|\lambda|}}\left[\frac{(1+|\lambda|)^2 - \frac{2}{n} - \frac{2|\lambda|}{n}}{2(1+|\lambda|)}\right]^{\frac{1}{2}}. \tag{9}$$

In particular,

$$\left[\frac{1+r}{2}\left(1 - \frac{1}{n}\right)\right]^{\frac{1}{2}} \sqrt{\frac{n}{1-r}} \le \mathcal{I}_{n,\,r}\left(H^2,\, B^{\frac{1}{2}}_{2,\,2}\right) \le \tag{10}$$

$$\le \left(1 + r + \frac{1}{\sqrt{n}} + \frac{1-r}{n}\right)^{\frac{1}{2}} \sqrt{\frac{n}{1-r}},$$

$$\sqrt{\frac{\frac{1+r}{2}}{1-r}} \le \liminf_{n\to\infty} \frac{\mathcal{I}_{n,\,r}\left(H^2,\, B^{\frac{1}{2}}_{2,\,2}\right)}{\sqrt{n}} \le \tag{11}$$

$$\limsup_{n\to\infty} \frac{\mathcal{I}_{n,\,r}\left(H^2,\, B^{\frac{1}{2}}_{2,\,2}\right)}{\sqrt{n}} \le \sqrt{\frac{1+r}{1-r}},$$

and

$$\frac{\sqrt{2}}{2} \le \liminf_{r\to1^-} \liminf_{n\to\infty} \sqrt{\frac{1-r}{n}}\mathcal{I}_{n,\,r}\left(H^2,\, B^{\frac{1}{2}}_{2,\,2}\right) \le \tag{12}$$

$$\le \limsup_{r\to1^-} \limsup_{n\to\infty} \sqrt{\frac{1-r}{n}}\mathcal{I}_{n,\,r}\left(H^2,\, B^{\frac{1}{2}}_{2,\,2}\right) \le \sqrt{2}.$$

In the next section, we first give some definitions introducing the main tools used in the proofs of Theorem A and Theorem B. After that, we prove these theorems.

Proofs of Theorems A and B

From now on, if $\sigma = \{\lambda_1,\, \dots,\, \lambda_n\} \subset \mathbb{D}$ is a finite subset of the unit disk, then

$$B_\sigma = \prod_{j=1}^{n} b_{\lambda_j}$$

is the corresponding finite Blaschke product where $b_\lambda = \frac{\lambda - z}{1-\bar{\lambda}z}$, $\lambda \in \mathbb{D}$. In Definitions 1, 2, 3 and in Remark 1 below, $\sigma = \{\lambda_1,\, \dots,\, \lambda_n\}$ is a sequence in the unit disk $\mathbb{D}$ and B_σ is the corresponding Blaschke product.

Definition 1. *Malmquist family.* For $k \in [1,\, n]$, we set $f_k = \frac{1}{1-\bar{\lambda}_k z}$, and define the family $(e_k)_{1\le k\le n}$ (which is known as the Malmquist basis, see [6, p. 117]), by

$$e_1 = \frac{f_1}{\|f_1\|_2} \text{ and } e_k = \left(\prod_{j=1}^{k-1} b_{\lambda_j}\right)\frac{f_k}{\|f_k\|_2}, \tag{13}$$

for $k \in [2,\, n]$; we have $\|f_k\|_2 = \left(1-|\lambda_k|^2\right)^{-1/2}$.

Definition 2. *The model space K_{B_σ}.* We define K_{B_σ} to be the n-dimensional space

$$K_{B_\sigma} = \left(B_\sigma H^2\right)^\perp = H^2 \ominus B_\sigma H^2. \tag{14}$$

Definition 3. *The orthogonal projection P_{B_σ} onto K_{B_σ}.* We define P_{B_σ} to be the orthogonal projection of H^2 onto its n-dimensional subspace K_{B_σ}.

Remark 1. The Malmquist family $(e_k)_{1\leq k\leq n}$ corresponding to σ is an orthonormal basis of K_{B_σ}. In particular,

$$P_{B_\sigma} = \sum_{k=1}^{n} (\cdot,\, e_k)_{H^2}\, e_k, \tag{15}$$

where $(\cdot,\, \cdot)_{H^2}$ means the scalar product on H^2.

Proof of Theorem A. *Proof of* (i). 1) First, we prove the right inequality in (4). Using the Cauchy-Schwarz inequality and the fact that $\widehat{f'}(k) = (k+1)\widehat{f}(k+1)$ for all $k \geq 0$, we get

$$\|f'\|^2_{L^2_a} = \sum_{k\geq 0} \frac{\left|\widehat{f'}(k)\right|^2}{k+1} = \sum_{k\geq 0} \frac{(k+1)^2\left|\widehat{f}(k+1)\right|^2}{k+1} =$$

$$= \sum_{k\geq 1} k\left|\widehat{f}(k)\right|^2 \leq \left(\sum_{k\geq 1} k^2\left|\widehat{f}(k)\right|^2\right)^{\frac{1}{2}} \left(\sum_{k\geq 1}\left|\widehat{f}(k)\right|^2\right)^{\frac{1}{2}} =$$

$$= \|f'\|_{H^2}\,\|f\|_{H^2} \leq \mathcal{C}_{n,\,r}\left(H^2,\, H^2\right)\|f\|^2_{H^2},$$

and hence,

$$\|f'\|_{L^2_a} \leq \sqrt{\mathcal{C}_{n,\,r}\,(H^2,\, H^2)}\,\|f\|_{H^2},$$

which means that

$$\mathcal{C}_{n,\,r}\left(L^2_a,\, H^2\right) \leq \sqrt{\mathcal{C}_{n,\,r}\,(H^2,\, H^2)}.$$

Then it remains to use [12]:

$$\mathcal{C}_{n,\,r}\left(H^2,\, H^2\right) \leq \left(1 + r + \frac{1}{\sqrt{n}}\right)\frac{n}{1-r},$$

for all $n \geq 1$ and $r \in [0,\, 1)$.

2) The proof of the left inequality in (4) repeates that of [12, (i)] (for the left inequality) except that this time, we replace the Hardy norm $\|\cdot\|_{H^2}$ by the Bergman norm $\|\cdot\|_{L^2_a}$. Indeed, we use the same test function $e_n = \frac{(1-r^2)^{\frac{1}{2}}}{1-rz} b_r^{n-1}$ (the nth vector of the Malmquist family associated with the singleton $\sigma_{n,\,r} = \underbrace{\{r,\, r, \ldots,\, r\}}_{n}$, see Definition 1) and prove by the same

change of the variable "$\circ b_r$" (in the integral over the unit disk $\mathbb{D}$ that defines the L_a^2-norm) that

$$\|e_n'\|_{L_a^2}^2 = \frac{n}{1-r}\left(1 - \frac{1-r}{n}\right),$$

which gives

$$\mathcal{C}_{n,\,r}\left(L_a^2,\, H^2\right) \geq \sqrt{\frac{n}{1-r}}\left(1 - \frac{1-r}{n}\right)^{\frac{1}{2}}.$$

Proof of (ii). This is again the same proof as [12, (ii)] (three steps). More precisely, at Step 2, we use the same test function

$$f = \sum_{k=0}^{s+2}(-1)^k e_{n-k},$$

(where $s = (s_n)_n$ was defined in [12]), and the same change of the variable "$\circ\, b_r$" in the integral over $\mathbb{D}$. □

Proof of Theorem B. *Proofs of inequality* (8) *and of the right inequality in* (10). Let σ be a sequence in $\mathbb{D}$, and $B = B_\sigma$ the finite Blaschke product corresponding to σ. If $f \in H^2$, we use the same function g as in [13], which satisfies $g_{|\sigma} = f_{|\sigma}$. More precisely, let $g = P_B f \in K_B$ (see Definitions 2, 3 and Remark 1 above for the definitions of K_B and P_B). Then $g - f \in BH^2$ and, by the definition of $\mathcal{C}_{n,\,r}\left(L_a^2,\, H^2\right)$, we obtain

$$\|g'\|_{L_a^2}^2 \leq \left(\mathcal{C}_{n,\,r}\left(L_a^2,\, H^2\right)\right)^2 \|g\|_{H^2}^2 .$$

Now, applying identity (2) to g, we get

$$\|g\|_{B_{2,\,2}^{\frac{1}{2}}}^2 \leq \left[\left(\mathcal{C}_{n,\,r}\left(L_a^2,\, H^2\right)\right)^2 + 1\right] \|g\|_{H^2}^2 .$$

Using the fact that $\|g\|_{H^2} = \|P_B f\|_{H^2} \leq \|f\|_{H^2}$, we finally get

$$\|g\|_{B_{2,\,2}^{\frac{1}{2}}} \leq \left[\left(\mathcal{C}_{n,\,r}\left(L_a^2,\, H^2\right)\right)^2 + 1\right]^{\frac{1}{2}} \|f\|_{H^2} ,$$

and as a result,

$$I\left(\sigma,\, H^2,\, B_{2,2}^{\frac{1}{2}}\right) \leq \left[\left(\mathcal{C}_{n,\,r}\left(L_a^2,\, H^2\right)\right)^2 + 1\right]^{\frac{1}{2}} .$$

It remains to apply the right inequality in (4) in Theorem A to prove the right inequality in (10).

Proof of inequality (9). 1) We use the same test function

$$f=\sum_{k=0}^{n-1}(1-|\lambda|^2)^{\frac12}b_\lambda^k\left(1-\overline{\lambda}z\right)^{-1}$$

as that used in the proof of [13, Theorem B] (the lower bound). Since f is the sum of n elements of H^2 that are an orthonormal family known as Malmquist's basis (associated with $\sigma_{n,\lambda}=\underbrace{\{\lambda,\lambda,\dots,\lambda\}}_{n}$, see Remark 1 above or [6, p. 117]) , we have $\|f\|_{H^2}^2=n$.

2) Since the spaces H^2 and $B_{2,2}^{\frac12}$ are rotation invariant, we have

$$I\left(\sigma_{n,\lambda},H^2,\,B_{2,2}^{\frac12}\right)=I\left(\sigma_{n,\mu},H^2,\,B_{2,2}^{\frac12}\right)$$

for every λ, μ with $|\lambda|=|\mu|=r$. Let $\lambda=-r$. To get a lower estimate for $\|f\|_{B_{2,2}^{\frac12}/b_\lambda^n B_{2,2}^{\frac12}}$, consider g such that $f-g\in b_\lambda^n\mathrm{Hol}(\mathbb{D})$, i.e., such that $f\circ b_\lambda-g\circ b_\lambda\in z^n\mathrm{Hol}(\mathbb{D})$.

3) First, we observe that

$$\|g\circ b_\lambda\|^2_{B_{2,2}^{\frac12}}=\left\|(g\circ b_\lambda)'\right\|^2_{L_a^2}+\|g\circ b_\lambda\|^2_{H^2}=\|b_\lambda'\cdot(g'\circ b_\lambda)\|^2_{L_a^2}+\|g\circ b_\lambda\|^2_{H^2}$$

$$=\int_{\mathbb{D}}|b_\lambda'(u)|^2\,|g'(b_\lambda(u))|^2\,du+\|g\circ b_\lambda\|^2_{H^2}=\int_{\mathbb{D}}|g'(w)|^2\,dw+\|g\circ b_\lambda\|^2_{H^2}$$

(we have used the change of variables $w=b_\lambda(u)$). We get

$$\|g\circ b_\lambda\|^2_{B_{2,2}^{\frac12}}=\|g'\|^2_{L_a^2}+\|g\circ b_\lambda\|^2_{H^2}=\|g\|^2_{B_{2,2}^{\frac12}}+\|g\circ b_\lambda\|^2_{H^2}-\|g\|^2_{H^2}\,,$$

and

$$\begin{aligned}\|g\|^2_{B_{2,2}^{\frac12}}&=\|g\|^2_{H^2}+\|g\circ b_\lambda\|^2_{B_{2,2}^{\frac12}}-\|g\circ b_\lambda\|^2_{H^2}\\&\geq\|g\circ b_\lambda\|^2_{B_{2,2}^{\frac12}}-\|g\circ b_\lambda\|^2_{H^2}\,.\end{aligned}$$

Now, we have

$$\begin{aligned}f\circ b_\lambda&=\sum_{k=0}^{n-1}z^k\frac{(1-|\lambda|^2)^{\frac12}}{1-\overline{\lambda}b_\lambda(z)}=\left(1-|\lambda|^2\right)^{-\frac12}\left(1+(1-\overline{\lambda})\sum_{k=1}^{n-1}z^k-\overline{\lambda}z^n\right)\\&=(1-r^2)^{-\frac12}\left(1+(1+r)\sum_{k=1}^{n-1}z^k+rz^n\right).\end{aligned}$$

4) Next,

$$\|g\circ b_\lambda\|^2_{B^{\frac12}_{2,\,2}} - \|g\circ b_\lambda\|^2_{H^2} = \sum_{k\geq 1} k\left|\widehat{g\circ b_\lambda}(k)\right|^2$$
$$\geq \sum_{k=1}^{n-1} k\left|\widehat{g\circ b_\lambda}(k)\right|^2 = \sum_{k=1}^{n-1} k\left|\widehat{f\circ b_\lambda}(k)\right|^2,$$

because $\widehat{g\circ b_\lambda}(k) = \widehat{f\circ b_\lambda}(k)$, for every $k\in[0,\,n-1]$. This gives

$$\|g\circ b_\lambda\|^2_{B^{\frac12}_{2,\,2}} - \|g\circ b_\lambda\|^2_{H^2} \geq \frac{1}{1-r^2}\left((1+r)^2\sum_{k=1}^{n-1} k\right)$$
$$= \frac{(1+r)^2}{1-r^2}\frac{n(n-1)}{2} = \frac{1+r}{1-r}\frac{n(n-1)}{2} = \frac{1+r}{1-r}\frac{(n-1)}{2}\|f\|^2_{H^2}$$

for all $n\geq 2$ because $\|f\|^2_{H^2} = n$. Finally,

$$\|g\|^2_{B^{\frac12}_{2,\,2}} \geq \frac{n}{1-r}\frac{1+r}{2}\left(1-\frac1n\right)\|f\|^2_{H^2}.$$

In particular,

$$\mathcal{I}_{n,\,r}\left(H^2,\,B^{\frac12}_{2,\,2}\right) \geq \sqrt{\frac{n}{1-r}}\left[\frac{1+r}{2}\left(1-\frac1n\right)\right]^{\frac12}.$$

□

Some comments.

a. Extension of Theorem A to the spaces $B^s_{2,\,2}$, $s\geq 0$. Using the techniques developed in the proof of our Theorem A (combined with complex interpolation between Banach spaces and induction argument), it is possible to specify the sharp numerical constant $c_{2,\,s}$ in Dyakonov's result (3) (see item d in the Introduction) and to prove the asymptotic sharpness (at least for $s\in\mathbb{N}\cup\frac12\mathbb{N}$) of (3). In the same spirit, it can be shown that

$$\lim_{n\to\infty}\frac{\mathcal{C}_{n,\,r}\left(B^{s-1}_{2,\,2},\,H^2\right)}{n^s} = \left(\frac{1+r}{1-r}\right)^s. \tag{16}$$

Our Theorem A corresponds to the case of $s=\frac12$.

b. Extension of Theorem B to the spaces $B^s_{2,2}$, $s \geq 0$. The proof of the upper bound in our Theorem B can be extended so as to give an upper (asymptotic) estimate of the interpolation constant $\mathcal{I}_{n,r}\left(H^2, B^s_{2,2}\right)$, $s \geq 0$. More precisely, applying Dyakonov's result (3) (see item d in the Introduction), we get

$$\mathcal{I}_{n,r}\left(H^2, B^s_{2,2}\right) \leq \tilde{c}_s \left(\frac{n}{1-r}\right)^s, \quad \text{with} \quad \tilde{c_s} \asymp c_{2,s}, \tag{17}$$

where $c_{2,s}$ is defined in (3) and specified in (16). Looking at the above comment 1, we see that $\tilde{c}_s \asymp (1+r)^s$ for sufficiently large values of n. Our Theorem B corresponds again to the case of $s = \frac{1}{2}$. In this Theorem B, we prove the sharpness of (17) for $s = \frac{1}{2}$. However, for the general case of $s \geq 0$, the asymptotic sharpness of $\left(\frac{n}{1-r}\right)^s$ as $r \to 1^-$ and $n \to \infty$ is less obvious. Indeed, the key of the proof (for the sharpness) is based on the property that the Dirichlet norm (that of $B^{1/2}_{2,2}$) is "nearly" invariant under composition with an elementary Blaschke factor b_λ, as this is the case for the H^∞ norm. N. K. Nikolski conjectured that

$$\mathcal{I}_{n,r}\left(H^2, B^s_{2,2}\right) \asymp \begin{cases} \frac{n^s}{\sqrt{1-r}} & \text{if } s \geq \frac{1}{2} \\ \left(\frac{n}{1-r}\right)^s & \text{if } 0 \leq s \leq \frac{1}{2}, \end{cases} \tag{18}$$

the motivation was the position of the spaces $B^s_{2,2}$, $s \geq 0$, with respect to the algebra H^∞.

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