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CORRECT AND SELF–ADJOINT PROBLEMS FOR BIQUADRATIC OPERATORS

Abstract. In this paper we continue the theme which has been investigated in [11, 12] and [13] and we present a simple method to prove correctness and self-adjointness of the operators of the form B^4 corresponding to some boundary value problems. We also give representations for the unique solutions for these problems. The algorithm is easy to implement via computer algebra systems. In our examples, Derive and Mathematica were used.

1. Introduction

An important tool in creating correct operators and solving boundary value problems containing differential or integro-differential equations is a theory of the correct extensions of minimal operators. Correct extensions of densely defined minimal operators in Banach and Hilbert spaces have been investigated by M. I. Vishik [2], A. A. Dezin [7], M. Otelbaev [8], R. Oinarov [9] and many others. Self-adjoint extensions of a densely defined minimal symmetric operator A_0 have been studied by a number of authors, such as M. G. Krein [1], E. A. Coddington, A. Dijksma [3, 4], V. I. Gorbachuk and M. L. Gorbachuk [5], A. N. Kochubei [6] and many others.Correct self-adjoint and positive extensions of nondensely defined minimal symmetric operators A_0 have been considered in [10]. Correct self-adjoint problems for quadratic and cubic operators have been investigated in [11, 12] and [13].

In this paper using the operator B, defined by

$$Bx = \widehat{A}x - (\widehat{A}F)C\langle \widehat{A}x, F^t\rangle_{H^m} = f, \quad D(B) = D(\widehat{A}),$$

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A comment from the first and third authors: The results of this paper have been proved together with our friend P. C. Tsekrekos which passed away from a heart attack in Fall of 2009, at the age of 64. We would like to express our deepest sorrow for his sudden death.

where $\widehat{A}$ is one well known correct self-adjoint operator, C is a $m \times m$ matrix, we investigate the operator B_4 corresponding to the boundary problem:

$$\begin{aligned} B_4 x = \widehat{A}^4 x - V\langle \widehat{A}x, F^t\rangle_{H^m} - Y\langle \widehat{A}^2 x, F^t\rangle_{H^m} - S\langle \widehat{A}^3 x, F^t\rangle_{H^m} \\ - G\langle \widehat{A}^4 x, F^t\rangle_{H^m} = f, \quad D(B_4) = D(\widehat{A}^4), \end{aligned} \tag{1.1}$$

where the vectors $V \in \mathbb{H}^m$, $Y \in D(\widehat{A})^m$, $S \in D(\widehat{A}^2)^m$, $G \in D(\widehat{A}^3)^m$, $F \in D(\widehat{A}^4)^m$ and V, S, Y satisfy (3.12).

We show that the operator B_4 is biquadratic, i.e. $B_4 = B^4$ and prove a criterion of correctness and selfadjointness of the problem (1.1) in terms of the matrices C. We give also representations for the unique solution of this problem which is essentially simpler than in the general case of non-biquadratic operators. Note that the self-adjointness of B_4 can be proved by more general method developed in [2] or [3]. But here we don't need the full strength of this method and prove it in a simpler and straightforward way.

The paper is organized as follows. In Section 2 we recall some basic terminology and notation about operators. In Section 3 we prove the main result and give one example of integro-differential equations which shows the usefulness of our results.

2. Terminology and notation

By $\langle x, f\rangle_H$ we denote the inner product of elements x, f of a complex Hilbert space $\mathbb{H}$. For operators $A : \mathbb{H} \to \mathbb{H}$ we write $D(A)$ and $R(A)$ for the domain and the range of A respectively. An operator $\widehat{A}$ is called *correct* if $R(\widehat{A}) = \mathbb{H}$ and the inverse $\widehat{A}^{-1}$ exists and is continuous. An operator B_1 is called *biquadratic* if there exists an operator B such that $B_1 = B^4$. Let A be an operator with domain $D(A)$ dense in $\mathbb{H}$. The *adjoint* operator $A^* : \mathbb{H} \longrightarrow \mathbb{H}$ of A with domain $D(A^*)$ is defined by the equation $\langle Ax, y\rangle_H = \langle x, A^*y\rangle_H$ for every $x \in D(A)$ and every $y \in D(A^*)$. The domain $D(A^*)$ of A^* consists of all $y \in \mathbb{H}$ for which the functional $x \longmapsto \langle Ax, y\rangle_H$ is continuous on $D(A)$. An operator A is called *selfadjoint* if $A = A^*$. If an operator $B : H \to H$ is correct (resp. selfadjoint), then we say that *the problem $Bx = f$ is correct (resp. selfadjoint)*. Let $F_i, g_i \in \mathbb{H}, i = 1, \ldots, m$. Then $F = (F_1, \ldots, F_m)$, $G = (g_1, \ldots, g_m)$ and $AF = (AF_1, \ldots, AF_m)$ are vectors of $\mathbb{H}^m$. Let $\mathcal{F} = (\widehat{A}^{-3}F, \widehat{A}^{-2}F, \widehat{A}^{-1}F, F) = (\widehat{A}^{-3}F_1, \ldots, \widehat{A}^{-3}F_m, \widehat{A}^{-2}F_1, \ldots, \widehat{A}^{-2}F_m, \widehat{A}^{-1}F_1, \ldots, \widehat{A}^{-1}F_m, F_1, \ldots, F_m)$

is a vector of $\mathbb{H}^{4m}$ and $\widehat{A}^{-4} = (\widehat{A}^{-1})^4$. We also write F^t and $\langle Ax, F^t\rangle_{H^m}$ for the column vectors $col(F_1, \ldots, F_m)$ and $col(\langle Ax, F_1\rangle_H, \ldots, \langle Ax, F_m\rangle_H)$ respectively. We denote by $\overline{M}$ (resp. M^t) the complex conjugate (resp. transpose) matrix of M and by $\langle G^t, F\rangle_{H^m}$ the $m \times m$ matrix whose i, j-th entry is the inner product $\langle g_i, F_j\rangle_H$. Note that $\langle G^t, F\rangle_{H^m}$ define the matrix inner product and has the properties: $\langle CG^t, F\rangle_H = C\langle G^t, F\rangle_H$, $\langle G^t, FC\rangle_H = \langle G^t, F\rangle_H\overline{C}$, $\langle G^t, F\rangle_H = \overline{\langle F^t, G\rangle}_H^t$, where C-is a $m \times m$ constant matrix. It is obvious that $\langle f, F^t\rangle_H = \overline{\langle F^t, f\rangle}_H$. We also denote by I_m and $[0]_m$ the identity $m \times m$ and the zero $m \times m$ matrix respectively.

3. Correct and self-adjoint problems for biquadratic operators

Next theorem is Theorem 3.1 of [13].

Theorem 3.1. *Let $B : \mathbb{H} \to \mathbb{H}$ and*

$$Bx = \widehat{A}x - (\widehat{A}F)C\langle \widehat{A}x, F^t\rangle_{H^m} = f, \quad D(B) = D(\widehat{A}), \tag{3.1}$$

where $\widehat{A}$ is correct and self-adjoint on $\mathbb{H}$, C is a $m \times m$ matrix with rank $C = n \le m$ and $F_1, \ldots, F_m$ linearly independent elements of $D(\widehat{A})$. Then:

(i) *B is self-adjoint operator if and only if C is a Hermitian operator,*

(ii) *B is a correct operator if and only if*

$$\det\left[I_m - \overline{\langle \widehat{A}F^t, F\rangle}_{H^m} C\right] \ne 0. \tag{3.2}$$

(iii) *If B is a correct operator, then $\dim R(B - \widehat{A}) = n$,*

(iv) *The unique solution of (3.1), where B is a correct operator, for every $f \in \mathbb{H}$ is given by the formula*

$$x = B^{-1}f = \widehat{A}^{-1}f + FC\left[I_m - \overline{\langle \widehat{A}F^t, F\rangle}_{H^m} C\right]^{-1}\langle f, F^t\rangle_{H^m}. \tag{3.3}$$

Remark 3.2. The correctness of B and the solution (3.3) of (3.1) in Theorem 3.1 do not depend on the linear independence of the components of $F_1, \ldots, F_m$.

This statement follows immediately from Remark 3.1 of [11], if we suppose that any functional $\Phi_0 \in \mathbb{H}^*$ can be identified in Hilbert space $\mathbb{H}$ by theorem Riesz with unique element $F_0 \in \mathbb{H}$ such that

$$\Phi_0(x) = (\Phi_0, x)_H = \langle x, F_0\rangle_H \quad \text{for all} \quad x \in \mathbb{H}, \tag{3.4}$$

so from (3.4) easy follows that any vector $\Phi = (\Phi_1, \dots, \Phi_m) \in \mathbb{H}^{*m}$, can be identified with unique vector $F = (F_1, \dots, F_m) \in \mathbb{H}^m$ such that for all $x \in \mathbb{H}$

$$\begin{aligned} &\Phi_i(x) = (\Phi_i, x)_H = \langle x, F_i \rangle_H, \; i = 1, \dots, m, \quad \text{or} \\ &\Phi^t(x) = (\Phi^t, x)_H = \langle x, F^t \rangle_H \end{aligned} \tag{3.5}$$

and for all vectors $G = (g_1, \dots, g_m) \in \mathbb{H}^m$

$$\Phi^t(G) = (\Phi^t, G)_{H^m} = \langle G^t, F \rangle^t_{H^m} = \overline{\langle F^t, G \rangle}_{H^m}. \tag{3.6}$$

Now it is evident that $\Phi_1, \dots, \Phi_m$ is a set of linearly independent elements of $\mathbb{H}^{*m}$ if and only if $F_1, \dots, F_m$ linearly independent on $\mathbb{H}^m$.

By Theorem 3.1, since $\widehat{A}^4$ is a correct self-adjoint operator and the components of $\mathcal{F}$ linearly independent, it follows easily the next theorem

Theorem 3.3. *Let $B_1 : \mathbb{H} \to \mathbb{H}$ and*

$$B_1 x = \widehat{A}^4 x - (\widehat{A}^4 \mathcal{F}) \mathbb{C}_{4m} \langle \widehat{A}^4 x, \mathcal{F}^t \rangle_{H^{4m}} = f, \quad D(B_1) = D(\widehat{A}^4),$$

where $\widehat{A}$ as in Theorem 3.1, $\mathbb{C}_{4m}$ is a $(4m) \times (4m)$ matrix with rank $\mathbb{C}_{4m} = n \le 4m$ and the components of the vector $\mathcal{F} = (\widehat{A}^{-3} F, \widehat{A}^{-2} F, \widehat{A}^{-1} F, F)$ are linearly independent elements of $D(\widehat{A}^4)$. Then:

(i) *B_1 is a self-adjoint operator if and only if $\mathbb{C}_{4m}$ is Hermitian,*

(ii) *B_1 is a correct operator if and only if*

$$\det L_1 = \det \left[I_{4m} - \overline{\langle \widehat{A}^4 \mathcal{F}^t, \mathcal{F} \rangle}_{H^{4m}} \mathbb{C}_{4m} \right] \neq 0. \tag{3.8}$$

(iii) *If B_1 is a correct operator, then $\dim R(B_1 - \widehat{A}^4) = n$,*

(iv) *The unique solution of (3.7), where B_1 is a correct operator, for every $f \in \mathbb{H}$ is given by the formula*

$$x = B_1^{-1} f = \widehat{A}^{-4} f + \mathcal{F} \mathbb{C}_{4m} \left[I_{4m} - \overline{\langle \widehat{A}^4 \mathcal{F}^t, \mathcal{F} \rangle}_{H^{4m}} \mathbb{C}_{4m} \right]^{-1} \langle f, \mathcal{F}^t \rangle_{H^{4m}}. \tag{3.9}$$

Lemma 3.4. *Let the operators $B, B_4 : \mathbb{H} \to \mathbb{H}$ be defined by*

$$Bx = \widehat{A} x - G \langle \widehat{A} x, F^t \rangle_{H^m} = f, \quad D(B) = D(\widehat{A}), \tag{3.10}$$

$$\begin{aligned} B_4 x = \widehat{A}^4 x - V \langle \widehat{A} x, F^t \rangle_{H^m} - Y \langle \widehat{A}^2 x, F^t \rangle_{H^m} - S \langle \widehat{A}^3 x, F^t \rangle_{H^m} - \\ - G \langle \widehat{A}^4 x, F^t \rangle_{H^m} = f, \quad D(B_4) = D(\widehat{A}^4), \end{aligned} \tag{3.11}$$

where $\widehat{A}$ is a linear operator on $\mathbb{H}$, G is a vector of $D(\widehat{A}^3)^m$, the vectors V, Y, S, G satisfy the equations

$$V = \widehat{A}Y - G\overline{\langle F^t, \widehat{A}Y\rangle}_{H^m}, \quad Y = \widehat{A}S - G\overline{\langle F^t, \widehat{A}S\rangle}_{H^m},$$
$$S = \widehat{A}G - G\overline{\langle F^t, \widehat{A}G\rangle}_{H^m} \tag{3.12}$$

and the components of the vector $F = (F_1, \ldots, F_m)$ belong to $D(\widehat{A}^4)$. Then $B_4 = B^4$, i.e. B_4 is a biquadratic operator.

Proof. From (3.10) and equations (3.12), we get

$$BG = \widehat{A}G - G\overline{\langle F^t, \widehat{A}G\rangle}_{H^m} = S,$$
$$BS = \widehat{A}S - G\overline{\langle F^t, \widehat{A}S\rangle}_{H^m} = Y,$$
$$BY = \widehat{A}Y - G\overline{\langle F^t, \widehat{A}Y\rangle}_{H^m} = V.$$

Taking this into account and the relation (3.10) for every $x \in D(\widehat{A}^4) \bigcap D(B^4)$ from (3.11) we have:

$$\begin{aligned} B_4 x &= \widehat{A}^4 x - BY\langle \widehat{A}x, F^t\rangle_{H^m} - BS\langle \widehat{A}^2 x, F^t\rangle_{H^m} \\ &\quad - BG\langle \widehat{A}^3 x, F^t\rangle_{H^m} - G\langle \widehat{A}^4 x, F^t\rangle_{H^m} \\ &= B(\widehat{A}^3 x) - BY\langle \widehat{A}x, F^t\rangle_{H^m} - BS\langle \widehat{A}^2 x, F^t\rangle_{H^m} - BG\langle \widehat{A}^3 x, F^t\rangle_{H^m} \\ &= B\big(\widehat{A}^3 x - Y\langle \widehat{A}x, F^t\rangle_{H^m} - S\langle \widehat{A}^2 x, F^t\rangle_{H^m} - G\langle \widehat{A}^3 x, F^t\rangle_{H^m}\big). \end{aligned}$$

In [13, Lemma 3.3] we have showed that for the operator B_3 defined by

$$B_3 x = \widehat{A}^3 x - Y\langle \widehat{A}x, F^t\rangle_{H^m} - S\langle \widehat{A}^2 x, F^t\rangle_{H^m} - G\langle \widehat{A}^3 x, F^t\rangle_{H^m} = f,$$
$$D(B_3) = D(\widehat{A}^3), \tag{3.13}$$

hold $B_3 = B^3$ and $D(B^3) = D(\widehat{A}^3)$. So $B_4 x = B^4 x$ for every $x \in D(\widehat{A}^4) \bigcap D(B^4)$. Now we show that $D(B^4) = D(\widehat{A}^4)$. From $D(B^3) = D(\widehat{A}^3)$ we have $D(B^4) = \{x \in D(\widehat{A}^3) : B^3 x \in D(\widehat{A})\}$. Let $x \in D(\widehat{A}^4)$. Then from (3.13) since $Y, S, G \in D(\widehat{A})$ we get $x \in D(B^4)$. Let now $x \in D(B^4)$. Again from (3.13) since $Y, S, G \in D(\widehat{A})$ we conclude that $x \in D(\widehat{A}^4)$. So, $D(B^4) = D(\widehat{A}^4)$ and $B_4 = B^4$.

We now present the main result of this paper. For biquadratic operator B_4 we prove a criterion of correctness and selfadjointness in terms of the matrices C and give explicit representations for the unique solution of the equation $B_4 x = f$ which is essentially simpler than in the general case of non-biquadratic operators.

Theorem 3.5. *Let the operators $\widehat{A}, B_4 : \mathbb{H} \to \mathbb{H}$ and vectors G, S, Y, V be defined as in Lemma 3.4. We also assume that $\widehat{A}$ is a correct operator, $G = (\widehat{A}F)C$, where C is a $m \times m$ matrix with rank $C = n \leq m$ and the components of vector $\mathcal{F} = (\widehat{A}^{-3}F, \widehat{A}^{-2}F, \widehat{A}^{-1}F, F)$ $\big($resp. $\widehat{A}^3\mathcal{F} = (F, \widehat{A}F, \widehat{A}^2F, \widehat{A}^3F)\big)$ are linearly independent elements of $D(\widehat{A}^4)$ $\big($resp. $D(\widehat{A})\big)$. Then:*

(i) *B_4 is a self-adjoint operator if and only if C is Hermitian,*

(ii) *B_4 is a correct operator if and only if holds*

$$\det L = \det \left[I_m - \overline{\langle \widehat{A}F^t, F\rangle}_{H^m} C\right] \neq 0. \tag{3.14}$$

(iii) *If B_4 is a correct operator, then $\dim R(B_4 - \widehat{A}^4) = 4n$ $(n \leq m)$,*

(iv) *The unique solution of the problem (3.11) , where B_4 is correct, for every $f \in \mathbb{H}$ is given by*

$$\begin{aligned} x = B_4^{-1}f = \widehat{A}^{-4}f \\ + \Big[\widehat{A}^{-3}F + (\widehat{A}^{-2}F)CL^{-1}\overline{\langle F^t, F\rangle}_{H^m} + (\widehat{A}^{-1}F)W + FCL^{-1}\big(\overline{\langle \widehat{A}^{-2}F^t, F\rangle}_{H^m} \\ + \overline{\langle \widehat{A}^{-1}F^t, F\rangle}_{H^m} CL^{-1}\overline{\langle F^t, F\rangle}_{H^m} + \overline{\langle F^t, F\rangle}_{H^m} W\big)\Big] CL^{-1} \times \langle f, F^t\rangle_{H^m} \\ + \Big[\widehat{A}^{-2}F + (\widehat{A}^{-1}F)CL^{-1}\overline{\langle F^t, F\rangle}_{H^m} + FW\Big] CL^{-1}\langle f, \widehat{A}^{-1}F^t\rangle_{H^m} \\ + \Big[\widehat{A}^{-1}F + FCL^{-1}\overline{\langle F^t, F\rangle}_{H^m}\Big] CL^{-1}\langle f, \widehat{A}^{-2}F^t\rangle_{H^m} \\ + FCL^{-1}\langle f, \widehat{A}^{-3}F^t\rangle_{H^m}, \end{aligned} \tag{3.15}$$

where $W = CL^{-1}\Big[\overline{\langle \widehat{A}^{-1}F^t, F\rangle}_{H^m} + \overline{\langle F^t, F\rangle}_{H^m} CL^{-1}\overline{\langle F^t, F\rangle}_{H^m}\Big]$.

Proof. (i), (iii) Let

$$\begin{aligned} T &= \overline{\langle F^t, F\rangle}_{H^m}, \quad D = \overline{\langle \widehat{A}F^t, F\rangle}_{H^m}, \\ K &= \overline{\langle \widehat{A}^2F^t, F\rangle}_{H^m}, \quad P = \overline{\langle \widehat{A}^3F^t, F\rangle}_{H^m}, \\ M &= \overline{\langle \widehat{A}^4F^t, F\rangle}_{H^m}, \quad H = \overline{\langle \widehat{A}^{-1}F^t, F\rangle}_{H^m} \\ N &= \overline{\langle \widehat{A}^{-2}F^t, F\rangle}_{H^m}, \quad Z = C(P - KCK)C. \end{aligned}$$

Then the matrix L in (3.14) is written as $L = I_m - DC$, the vectors S, Y, V in (3.11) as

$$\begin{aligned}
S &= (\widehat{A}^2F)C - (\widehat{A}F)CKC,\\
Y &= (\widehat{A}^3F)C - (\widehat{A}^2F)CKC - (\widehat{A}F)Z,\\
V &= (\widehat{A}^4F)C - (\widehat{A}^3F)CKC - (\widehat{A}^2F)Z - (\widehat{A}F)C(MC - PCKC - KZ).
\end{aligned}$$

The equation (3.11) can also be written in matrix notation as

$$\begin{gathered}
B_4x = \widehat{A}^4x - (\widehat{A}F, \widehat{A}^2F, \widehat{A}^3F, \widehat{A}^4F)\\
\times \begin{pmatrix} -C(MC-PCKC-KZ) & -Z & -CKC & C \\ -Z & -CKC & C & [0]_m \\ -CKC & C & [0]_m & [0]_m \\ C & [0]_m & [0]_m & [0]_m \end{pmatrix} \begin{pmatrix} \langle \widehat{A}x, F^t\rangle_{H^m} \\ \langle \widehat{A}^2x, F^t\rangle_{H^m} \\ \langle \widehat{A}^3x, F^t\rangle_{H^m} \\ \langle \widehat{A}^4x, F^t\rangle_{H^m} \end{pmatrix} = f \quad (3.16)\\
\text{or} \quad B_4x = \widehat{A}^4x - (\widehat{A}^4\mathcal{F})\mathbb{C}_{4m}\langle \widehat{A}^4x, \mathcal{F}^t\rangle_{H^{4m}} = f,
\end{gathered}$$

where $\mathcal{F} = (\widehat{A}^{-3}F, \widehat{A}^{-2}F, \widehat{A}^{-1}F, F)$,

$$\mathbb{C}_{4m} = \begin{pmatrix} -C(MC - PCKC - KZ) & -Z & -CKC & C \\ -Z & -CKC & C & [0]_m \\ -CKC & C & [0]_m & [0]_m \\ C & [0]_m & [0]_m & [0]_m \end{pmatrix}.$$

It is easy to verify that $\mathbb{C}_{4m}$ is a Hermitian matrix with rank $\mathbb{C}_{4m} = 4n$ if and only if C is Hermitian with rank $C = n$. Then, by Theorem 3.3, $\dim R(B_4 - \widehat{A}^4) = 4n\,(n \le m)$ and the operator B_4 is self-adjoint if and only if C is Hermitian.

(ii) Let $Q = C(MC - PCKC - KZ)$. By Theorem 3.3, the operator B_4 is correct if and only if (3.8) holds true with B_1 replaced by B_4 and L_1 by L_4. We find

$$L_4 = I_{4m} - \overline{\langle \widehat{A}^4\mathcal{F}^t, \mathcal{F}\rangle}_{H^{4m}}\mathbb{C}_{4m} = \qquad (3.17)$$

$$= I_{4m} - \overline{\begin{pmatrix} \langle F^t, \widehat{A}^{-2}F\rangle_{H^m} & \langle F^t, \widehat{A}^{-1}F\rangle_{H^m} & \langle F^t, F\rangle_{H^m} & \langle \widehat{A}F^t, F\rangle_{H^m} \\ \langle F^t, \widehat{A}^{-1}F\rangle_{H^m} & \langle F^t, F\rangle_{H^m} & \langle \widehat{A}F^t, F\rangle_{H^m} & \langle \widehat{A}^2F^t, F\rangle_{H^m} \\ \langle F^t, F\rangle_{H^m} & \langle \widehat{A}F^t, F\rangle_{H^m} & \langle \widehat{A}^2F^t, F\rangle_{H^m} & \langle \widehat{A}^3F^t, F\rangle_{H^m} \\ \langle \widehat{A}F^t, F\rangle_{H^m} & \langle \widehat{A}^2F^t, F\rangle_{H^m} & \langle \widehat{A}^3F^t, F\rangle_{H^m} & \langle \widehat{A}^4F^t, F\rangle_{H^m} \end{pmatrix}} \mathbb{C}_{4m}$$

$$= I_{4m} + \begin{pmatrix} N & H & T & D \\ H & T & D & K \\ T & D & K & P \\ D & K & P & M \end{pmatrix} \begin{pmatrix} Q & Z & CKC & -C \\ Z & CKC & -C & [0]_m \\ CKC & -C & [0]_m & [0]_m \\ -C & [0]_m & [0]_m & [0]_m \end{pmatrix}$$

$$= \begin{pmatrix} J_1 & NZ + HCKC - TC & NCKC - HC & -NC \\ J_2 & I_m + HZ + TCKC - DC & HCKC - TC & -HC \\ J_3 & TZ + DCKC - KC & I_m + TCKC - DC & -TC \\ J_4 & DZ + KCKC - PC & DCKC - KC & I_m - DC \end{pmatrix},$$

where

$$\begin{aligned} J_1 &= I_m + NQ + HZ + TCKC - DC, \\ J_2 &= HQ + TZ + DCKC - KC, \\ J_3 &= TQ + DZ + KCKC - PC, \\ J_4 &= DQ + KZ + PCKC - MC. \end{aligned}$$

Multiplying the elements of the second column by KC and adding to the corresponding elements of the first column we get

$$\det L_4 = \det \begin{pmatrix} X_1 & NZ{+}HCKC{-}TC & NCKC{-}HC & -NC \\ X_2 & I_m{+}HZ{+}TCKC{-}DC & HCKC{-}TC & -HC \\ X_3 & TZ{+}DCKC{-}KC & I_m{+}TCKC{-}DC & -TC \\ X_4 & DZ{+}KCKC{-}PC & DCKC{-}KC & I_m{-}DC \end{pmatrix},$$

where

$$\begin{aligned} X_1 &= L + NCMC - NCKCPC + HCPC, \\ X_2 &= HCMC - HCKCPC + TCPC, \\ X_3 &= TCMC - TCKCPC - LPC, \\ X_4 &= DCMC - DCKCPC + KCPC - MC. \end{aligned}$$

Multiplying the elements of the third column by PC and adding to the corresponding elements of the first column we get

$$\det L_4 = \det \begin{pmatrix} L{+}NCMC & NZ{+}HCKC{-}TC & NCKC{-}HC & -NC \\ HCMC & L{+}HZ{+}TCKC & HCKC{-}TC & -HC \\ TCMC & TZ{+}DCKC{-}KC & L{+}TCKC & -TC \\ -LMC & DZ{+}KCKC{-}PC & DCKC{-}KC & L \end{pmatrix}.$$

Multiplying the elements of the fourth column by MC and adding to the corresponding elements of the first column we get

$$\det L_4 = \det \begin{pmatrix} L & NC(P-KCK)C+HCKC-TC & NCKC-HC & -NC \\ [0]_m & L+HC(P-KCK)C+TCKC & HCKC-TC & -HC \\ [0]_m & TC(P-KCK)C-LKC & L+TCKC & -TC \\ [0]_m & DC(P-KCK)C+KCKC-PC & DCKC-KC & L \end{pmatrix}.$$

Multiplying the elements of the third column by KC and adding to the corresponding elements of the second column we get

$$\det L_4 = \det \begin{pmatrix} L & NCPC-TC & NCKC-HC & -NC \\ [0]_m & L+HCPC & HCKC-TC & -HC \\ [0]_m & TCPC & L+TCKC & -TC \\ [0]_m & -LPC & -LKC & L \end{pmatrix}.$$

Multiplying the elements of the fourth column by PC and adding to the corresponding elements of the second column we get

$$\det L_4 = \det \begin{pmatrix} L & -TC & NCKC-HC & -NC \\ [0]_m & L & HCKC-TC & -HC \\ [0]_m & [0]_m & L+TCKC & -TC \\ [0]_m & [0]_m & -LKC & L \end{pmatrix}.$$

Multiplying the elements of the fourth column by KC and adding to the corresponding elements of the third column we get

$$\begin{aligned} \det L_4 = \det \begin{pmatrix} L & -TC & -HC & -NC \\ [0]_m & L & -TC & -HC \\ [0]_m & [0]_m & L & -TC \\ [0]_m & [0]_m & [0]_m & L \end{pmatrix} \\ = (\det L)^4 \neq 0 \Leftrightarrow \det L \neq 0. \end{aligned} \tag{3.18}$$

So, by Theorem 3.3, because of (3.17) and (3.18), the operator B_4 is correct if and only if (3.14) holds true.

(iv) In [13, Theorem 3.4] we have showed that

$$\begin{aligned} B_3^{-1} f = \widehat{A}^{-3} f + \big[\widehat{A}^{-2} F + (\widehat{A}^{-1} F) C L^{-1} \overline{\langle F^t, F\rangle}_{H^m} \\ + FCL^{-1}\big(\overline{\langle \widehat{A}^{-1} F^t, F\rangle}_{H^m} + \overline{\langle F^t, F\rangle}_{H^m} C L^{-1} \overline{\langle F^t, F\rangle}_{H^m}\big)\big] C L^{-1} \langle f, F^t\rangle_{H^m} \\ + \big[\widehat{A}^{-1} F + FCL^{-1}\overline{\langle F^t, F\rangle}_{H^m}\big] C L^{-1} \langle f, \widehat{A}^{-1} F^t\rangle_{H^m} \\ + FCL^{-1}\langle f, \widehat{A}^{-2} F^t\rangle_{H^m}. \end{aligned} \tag{3.19}$$

Let $g = B^{-3}f$. Then, since (3.3) and (3.19), we find

$$\begin{aligned}
B^{-4}f = B^{-1}g &= \widehat{A}^{-1}g + FCL^{-1}\overline{\langle F^t, g\rangle}_{H^m} \\
&= \widehat{A}^{-1}\big\{\widehat{A}^{-3}f + \big[\widehat{A}^{-2}F + (\widehat{A}^{-1}F)CL^{-1}T \\
&\quad + FCL^{-1}\big(H + TCL^{-1}T\big)\big]CL^{-1}\langle f, F^t\rangle_{H^m} \\
&\quad + \big(\widehat{A}^{-1}F + FCL^{-1}T\big)CL^{-1}\langle f, \widehat{A}^{-1}F^t\rangle_{H^m} + FCL^{-1}\langle f, \widehat{A}^{-2}F^t\rangle_{H^m}\big\} \\
&\quad + FCL^{-1}\big\{\overline{\langle F^t, \widehat{A}^{-3}f\rangle}_{H^m} + \big(N + HCL^{-1}T + TW\big)CL^{-1}\langle f, F^t\rangle_{H^m} \\
&\quad + (H + TCL^{-1}T)CL^{-1}\langle f, \widehat{A}^{-1}F^t\rangle_{H^m} + TCL^{-1}\langle f, \widehat{A}^{-2}F^t\rangle_{H^m}\big\} \\
&= \widehat{A}^{-4}f + \big[\widehat{A}^{-3}F + (\widehat{A}^{-2}F)CL^{-1}T + (\widehat{A}^{-1}F)W\big]CL^{-1}\langle f, F^t\rangle_{H^m} \\
&\quad + \big[\widehat{A}^{-2}F + (\widehat{A}^{-1}F)CL^{-1}T\big]CL^{-1}\langle f, \widehat{A}^{-1}F^t\rangle_{H^m} \\
&\quad + (\widehat{A}^{-1}F)CL^{-1}\langle f, \widehat{A}^{-2}F^t\rangle_{H^m} + FCL^{-1}\big[\langle f, \widehat{A}^{-3}F^t\rangle_{H^m} \\
&\quad + (N + HCL^{-1}T + TW)CL^{-1}\langle f, F^t\rangle_{H^m} \\
&\quad + (H + TCL^{-1}T)CL^{-1}\langle f, \widehat{A}^{-1}F^t\rangle_{H^m} + TCL^{-1}\langle f, \widehat{A}^{-2}F^t\rangle_{H^m}\big] \\
&= \widehat{A}^{-4}f + \big[\widehat{A}^{-3}F + (\widehat{A}^{-2}F)CL^{-1}T + (\widehat{A}^{-1}F)W \\
&\quad + FCL^{-1}(N + HCL^{-1}T + TW)\big]CL^{-1}\langle f, F^t\rangle_{H^m} \\
&\quad + \big[\widehat{A}^{-2}F + (\widehat{A}^{-1}F)CL^{-1}T + FW\big]CL^{-1}\langle f, \widehat{A}^{-1}F^t\rangle_{H^m} \\
&\quad + (\widehat{A}^{-1}F + FCL^{-1}T)CL^{-1}\langle f, \widehat{A}^{-2}F^t\rangle_{H^m} + FCL^{-1}\langle f, \widehat{A}^{-3}F^t\rangle_{H^m}
\end{aligned}$$

which gives (3.15). The theorem thus has been proved. □

From the proof of the previous theorem and remark 3.2 immediately follows the next remark

Remark 3.6. The correctness of B_4 and the solution of $B_4x = f$ in the Theorem 3.5 do not depend on the linear independence of the components of the vector $\mathcal{F}$,

Remark 3.7. In applications we encounter operators B_1 of the form

$$\begin{aligned}
B_1u = \widehat{A}^4u - W_{1m}\langle u, J_1^t\rangle_{H^m} - W_{2m}\langle u, J_2^t\rangle_{H^m} - W_{3m}\langle u, J_3^t\rangle_{H^m} \\
- W_{4m}\langle u, J_4^t\rangle_{H^m} = f, \quad D(B_1) = D(\widehat{A}^4),
\end{aligned} \tag{3.20}$$

where the vectors $J_i, W_{im} \in \mathbb{H}^m$, $i = 1,2,3,4$. Then we are interested in knowing whether the operator B_1 is a B_4-type operator defined by

(3.11) and, therefore, Theorem 3.5 applies. For this purpose, we proceed as follows:

1. We show that the operator $\widehat{A}$ in (3.20) is correct and self-adjoint.

2. We find a vector $F \in D(\widehat{A}^4)^m$ and $m \times m$ matrices M_i, $i = 1, 2, 3, 4$ with constant elements such that:

$$\langle u, J_1^t \rangle_{H^m} = M_1 \langle \widehat{A}u, F^t \rangle_{H^m},$$
$$\langle u, J_2^t \rangle_{H^m} = M_2 \langle \widehat{A}^2 u, F^t \rangle_{H^m},$$
$$\langle u, J_3^t \rangle_{H^m} = M_3 \langle \widehat{A}^3 u, F^t \rangle_{H^m},$$

and

$$\langle u, J_4^t \rangle_{H^m} = M_4 \langle \widehat{A}^4 u, F^t \rangle_{H^m}.$$

3. We find vectors $V = W_{1m} M_1 \in \mathbb{H}^m$, $Y = W_{2m} M_2 \in D(\widehat{A})^m$, $S = W_{3m} M_3 \in D(\widehat{A}^2)^m$ and $G = W_{4m} M_4 \in D(\widehat{A}^3)^m$ to satisfy the equations $V = \widehat{A}Y - G\overline{\langle F^t, \widehat{A}Y \rangle}_{H^m}$, $Y = \widehat{A}S - G\overline{\langle F^t, \widehat{A}S \rangle}_{H^m}$ and $S = \widehat{A}G - G\overline{\langle F^t, \widehat{A}G \rangle}_{H^m}$. If one of these steps fails, then B_1 is not identified as a B_4-type operator and, therefore, the theory can not be applied.

Below $H^i(0,1)$ denote the Sobolev spaces of all complex functions from $L_2(0,1)$ that have generalized derivatives up to i -th order that are Lebesque integrable, $i = 1, 2, 3, 4$. In the example presented bellow, we have used the programs Derive and Mathematica 6 for computing integrals and some complex expressions. We recall [10, p. 780] that the operator $\widehat{A} : L_2(0,1) \to L_2(0,1)$ defined by

$$\widehat{A}u = iu' = f, \quad D(\widehat{A}) = \{u(t) \in H^1(0,1) : u(0) + u(1) = 0\} \qquad (3.21)$$

is correct and self-adjoint and the unique solution u of the problem (3.21) is given by the formula

$$u(t) = \widehat{A}^{-1} f(t) = \frac{i}{2} \int_0^1 f(x)\,dx - i \int_0^t f(x)\,dx \quad \text{for all } f \in H. \qquad (3.22)$$

Then [13, p. 424] the operator $\widehat{A}^2$ defined by

$$\begin{aligned} &\widehat{A}^2 u = -u'' = f, \\ &D(\widehat{A}^2) = \{u \in H^2(0,1) : u(0) + u(1) = 0,\ u'(0) + u'(1) = 0\}, \end{aligned} \qquad (3.23)$$

is correct and self-adjoint, and for every $f \in L_2(0,1)$ the unique solution u of the problem (3.23) is given by the formula

$$u(t) = \widehat{A}^{-2} f(t) = -\int_0^t (t-x) f(x)\,dx + \frac{1}{4}\int_0^1 (2t-2x+1) f(x)\,dx. \quad (3.24)$$

Also [13, Proposition 3.6] the operator $\widehat{A}^3$ defined by

$$\widehat{A}^3 u = -iu''' = f, \quad (3.25)$$
$$D(\widehat{A}^3)=\{u \in H^3(0,1) \,:\, u(0)+u(1)=0,\, u'(0)+u'(1)=0,\, u''(0)+u''(1)=0\},$$

is correct and self-adjoint, and for every $f \in L_2(0,1)$ the unique solution u of the problem (3.25) is given by the formula

$$u(t) = \widehat{A}^{-3} f(t) = \frac{i}{2}\int_0^t (t-x)^2 f(x)\,dx - \frac{i}{4}\int_0^1 (t-x)(t-x+1) f(x)\,dx. \quad (3.26)$$

Proposition 3.8. *Let the operator $\widehat{A}$ defined by* (3.21). *Then the operator $\widehat{A}^4 : L_2(0,1) \to L_2(0,1)$ defined by*

$$\widehat{A}^4 u = u^{(4)} = f, \quad (3.27)$$
$$D(\widehat{A}^4) = \{u \in H^4(0,1) \,:\, u^{(k)}(0) + u^{(k)}(1) = 0,\, k = 0,1,2,3\},$$

$\widehat{A}^4$ is correct and self-adjoint and for every $f \in L_2(0,1)$ the unique solution u of the problem (3.27) *is given by the formula*

$$u(t) = \widehat{A}^{-4} f(t) = \frac{1}{6}\int_0^t (t-x)^3 f(x)\,dx \quad (3.28)$$
$$+ \frac{1}{48}\int_0^1 \big[4x^3 - 6x^2(2t+1) + 12tx(t+1) - 4t^3 - 6t^2 + 1\big] f(x)\,dx$$

Proof. Correctness and self-adjointness of $\widehat{A}$ imply correctness and self-adjointness of $\widehat{A}^3$. Now we will prove the formula (3.28). Let $y(x) =$

$\widehat{A}^{-3}f(x)$. Then by (3.22), (3.26) and Fubini's theorem we have

$$\widehat{A}^{-4}f(t)=\widehat{A}^{-1}\big(\widehat{A}^{-3}f(t)\big)=\widehat{A}^{-1}y(t)=\frac{i}{2}\int_0^1 y(z)\,dz-i\int_0^t y(z)\,dz$$

$$=\frac{i}{2}\int_0^1\Big[\frac{i}{2}\int_0^z(z-x)^2f(x)\,dx-\frac{i}{4}\int_0^1(z-x)(z-x+1)f(x)\,dx\Big]\,dz$$

$$-i\int_0^t\Big[\frac{i}{2}\int_0^z(z-x)^2f(x)\,dx-\frac{i}{4}\int_0^1(z-x)(z-x+1)f(x)\,dx\Big]\,dz$$

$$=\frac{-1}{4}\int_0^1dz\int_0^z(z-x)^2f(x)\,dx+\frac{1}{8}\int_0^1dz\int_0^1(z-x)(z-x+1)f(x)\,dx$$

$$+\frac{1}{2}\int_0^t dz\int_0^z(z-x)^2f(x)\,dx-\frac{1}{4}\int_0^t dz\int_0^1(z-x)(z-x+1)f(x)\,dx$$

$$=-\frac{1}{4}\int_0^1 f(x)\,dx\int_x^1(z-x)^2\,dz+\frac{1}{8}\int_0^1 f(x)\,dx\int_0^1(z-x)(z-x+1)\,dz$$

$$+\frac{1}{2}\int_0^t f(x)\,dx\int_x^t(z-x)^2\,dz-\frac{1}{4}\int_0^1 f(x)\,dx\int_0^t(z-x)(z-x+1)\,dz$$

$$=\frac{1}{12}\int_0^1(x-1)^3f(x)\,dx+\frac{1}{48}\int_0^1(6x^2-12x+5)f(x)\,dx$$

$$+\frac{1}{6}\int_0^t(t-x)^3f(x)\,dx$$

$$-\frac{t}{24}\int_0^1\big[6x^2-6x(t+1)+2t^2+3t\big]f(x)\,dx=\frac{1}{6}\int_0^t(t-x)^3f(x)\,dx$$

$$+\frac{1}{48}\int_0^1\big[4x^3-6x^2(2t+1)+12tx(t+1)-4t^3-6t^2+1\big]f(x)\,dx \tag{3.29}$$

which gives (3.28). □

Example 3.9 The operator $B_1 : L_2(0,1) \to L_2(0,1)$ which corresponds to the problem

$$B_1 u = u^{(4)} + 80\Big\{\frac{620}{21}(t^2-t) - \frac{65654570}{250047}(t^4-2t^3+t) - i\Big[6t-3-\frac{310^2}{63^2}(4t^3-6t^2+1)\Big]\Big\}\int_0^1 (2x^5-5x^4+5x^2-1)u'(x)\,dx + 80\Big\{\frac{155}{63}(4t^3-6t^2+1) - i\Big[3t^2-3t-\frac{310^2}{63^2}(t^4-2t^3+t)\Big]\Big\} \times \int_0^1 u''(x)(2x^5-5x^4+5x^2-1)\,dx - 20i\Big[\frac{620i}{63}(t^4-2t^3+t)+4t^3-6t^2+1\Big] \times \int_0^1 u'''(x)(2x^5-5x^4+5x^2-1)\,dx + 200i(t^4-2t^3+t)\int_0^1 u'''(x)(x^4-2x^3+x)\,dx = f(t), \quad D(B_1) = D(\widehat{A}^4) \tag{3.30}$$

is correct and self-adjoint and the unique solution of (3.30), for every $f \in L_2(0,1)$, is given by the formula

$$u(t) = \widehat{A}^{-4}f(t) + \frac{2}{3}\Big[\frac{i}{56}(t^8-4t^7+14t^5-28t^3+17t) - \frac{691}{38808}(8t^7-28t^6 +70t^4-84t^2+17) - \frac{691^2 i}{693^2}(t^6-3t^5+5t^3-3t) + \frac{9452636909}{2884375494}(2t^5-5t^4 +5t^2-1)\Big]\int_0^1 (2x^5-5x^4+5x^2-1)f(x)\,dx + \frac{2i}{9}\Big[\frac{1}{56}(8t^7-28t^6+70t^4 -84t^2+17) + \frac{691i}{693}(t^6-3t^5+5t^3-3t) - \frac{691^2}{160083}(2t^5-5t^4+5t^2-1)\Big] \times \int_0^1 (x^6-3x^5+5x^3-3x)f(x)\,dx - \frac{1}{252}\Big[-i(t^6-3t^5+5t^3-3t)$$

$$+\frac{691}{231}(2t^5-5t^4+5t^2-1)\Big]\int_0^1(8x^7-28x^6+70x^4-84x^2+17)f(x)\,dx$$

$$+\frac{i}{84}(2t^5-5t^4+5t^2-1)\int_0^1(x^8-4x^7+14x^5-28x^3+17x)f(x)\,dx, \tag{3.31}$$

where $\widehat{A}^{-4}f(t)$ is defined by (3.28).

Proof. We refer to Theorem 3.5. If we compare Eq. (3.30) with Eq. (3.11), it is natural to take $\widehat{A}^4u = u^{(4)}$ with $D(\widehat{A}^4) = D(B_1)$, $m = 1$, $F = 2t^5-5t^4+5t^2-1$. Then we can get $\widehat{A}$ to be defined by (3.21), $\widehat{A}^2$ defined by (3.23), and $\widehat{A}^3$ by (3.25). It is obvious that $F \in D(\widehat{A}^4)$, $\widehat{A}F = 10i(t^4-2t^3+t)$, $\widehat{A}^2F = -10(4t^3-6t^2+1)$, $\widehat{A}^3F = -120i(t^2-t)$, $\widehat{A}^4F = 120(2t-1)$, and that

$$\langle \widehat{A}u, F\rangle_H = \int_0^1 iu'(x)(2t^5-5t^4+5t^2-1)\,dx,$$

$$\langle \widehat{A}^2u, F\rangle_H = -\int_0^1 u''(x)(2t^5-5t^4+5t^2-1)\,dx,$$

$$\langle \widehat{A}^3u, F\rangle_H = -i\int_0^1 u'''(x)(2t^5-5t^4+5t^2-1)\,dx,$$

$$\langle \widehat{A}^4u, F\rangle_H = \int_0^1 u^{(4)}(x)(2x^5-5x^4+5x^2-1)\,dx.$$

Then

$$\int_0^1 u'(x)(2x^5-5x^4+5x^2-1)\,dx = -i\langle \widehat{A}u, F\rangle_H,$$

$$\int_0^1 u''(x)(2x^5-5x^4+5x^2-1)\,dx = -\langle \widehat{A}^2u, F\rangle_H,$$

$$\int_0^1 u'''(x)(2x^5-5x^4+5x^2-1)\,dx = i\langle \widehat{A}^3u, F\rangle_H.$$

Integrating by parts, we have

$$\langle \widehat{A}^4 u, F\rangle_H = -10\int_0^1 u'''(x)(x^4-2x^3+x)\,dx,$$

Then

$$\int_0^1 u'''(x)(x^4-2x^3+x)\,dx = -\frac{1}{10}\langle \widehat{A}^4 u, F\rangle_H.$$

Substituting these formulas into (3.30), we obtain:

$$\begin{aligned} B_1 u = \widehat{A}^4 u &- 80i\Big\{\frac{620}{21}(t^2-t) - \frac{65654570}{250047}(t^4-2t^3+t) \\ &- i\Big[6t-3-\frac{310^2}{63^2}(4t^3-6t^2+1)\Big]\Big\}\langle \widehat{A}u, F\rangle_H \\ &-80\Big\{\frac{155}{63}(4t^3-6t^2+1)-i\Big[3t^2-3t-\frac{310^2}{63^2}(t^4-2t^3+t)\Big]\Big\}\langle \widehat{A}^2 u, F\rangle_H \\ &+20\Big[\frac{620i}{63}(t^4-2t^3+t)+4t^3-6t^2+1\Big]\langle \widehat{A}^3 u, F\rangle_H \\ &-20i(t^4-2t^3+t)\langle \widehat{A}^4 u, F\rangle_H = f(t). \end{aligned} \tag{3.32}$$

Again, comparing (3.32) with (3.11), we get

$$\begin{aligned} V =&80i\Big\{\frac{620}{21}(t^2-t) - \frac{65654570}{250047}(t^4-2t^3+t) \\ &- i\Big[6t-3-\frac{310^2}{63^2}(4t^3-6t^2+1)\Big]\Big\}, \\ Y =&80\Big\{\frac{155}{63}(4t^3-6t^2+1)-i\Big[3t^2-3t-\frac{310^2}{63^2}(t^4-2t^3+t)\Big]\Big\}, \\ S =&-20\Big[\frac{620i}{63}(t^4-2t^3+t)+4t^3-6t^2+1\Big] \quad\text{and}\quad G = 20i(t^4-2t^3+t). \end{aligned}$$

It is obvious that $G \in D(\widehat{A}^3)$. The vectors $F, \widehat{A}F, \widehat{A}^2F, \widehat{A}^3F$ are linearly independent elements of $D(\widehat{A})$, since the corresponding determinant of the Gram matrix is nonzero. Using Derive, we obtain

$$\begin{aligned} \widehat{A}G - G\overline{\langle F^t, \widehat{A}G\rangle}_{H^m} &= -20(4t^3-6t^2+1) - 20i(t^4-2t^3+t)\frac{620}{63} = S, \\ \widehat{A}S - G\overline{\langle F^t, \widehat{A}S\rangle}_{H^m} &= -20i\Big[\frac{620i}{63}(4t^3-6t^2+1)+12(t^2-t)\Big] \\ &\quad -20i(t^4-2t^3+t)\Big(-\frac{620^2}{63^2}\Big) = Y \end{aligned}$$

and

$$\widehat{A}Y - G\overline{\langle F^t, \widehat{A}Y\rangle}_{H^m} = 80i\Big\{\frac{155}{63}(12t^2 - 12t) - i\Big[6t - 3 - \frac{310^2}{63^2}(4t^3 - 6t^2 + 1)\Big]\Big\} - 20i(t^4 - 2t^3 + t)\frac{262618280}{250047} = V.$$

The last three equalities, by Lemma 3.4, show that the operator B_1 is biquadratic, i.e. $B_1 = B_4$. From $G = (\widehat{A}F)C$ it follows that $20i(t^4 - 2t^3 + t) = 10i(t^4 - 2t^3 + t)C$. This equation implies that $C = 2$. Using the program Derive, we find $\langle F^t, F\rangle_H = \frac{691}{1386}$ $\langle \widehat{A}F^t, F\rangle_H = 0$. By Theorem 3.5, the operator B_1 is correct and self-adjoint, since $C = 2$ is a real number and

$$\det L = \det[I_m - \overline{\langle \widehat{A}F^t, F\rangle}_{H^m} C] = 1 - 0 = 1 \neq 0.$$

Then $L^{-1} = 1$. If we substitute in (3.22), (3.24) and (3.26) $f = F = 2t^5 - 5t^4 + 5t^2 - 1$, we receive

$$\widehat{A}^{-1}F = -\frac{i}{3}(t^6 - 3t^5 + 5t^3 - 3t),$$

$$\widehat{A}^{-2}F = -\frac{1}{168}(8t^7 - 28t^6 + 70t^4 - 84t^2 + 17)$$

and

$$\widehat{A}^{-3}F = \frac{i}{168}(t^8 - 4t^7 + 14t^5 - 28t^3 + 17t).$$

Then

$$\langle f, \widehat{A}^{-1}F\rangle_H = -\frac{i}{3}\int_0^1 (x^6 - 3x^5 + 5x^3 - 3x)f(x)\,dx,$$

$$\langle f, \widehat{A}^{-2}F\rangle_H = -\frac{1}{168}\int_0^1 (8x^7 - 28x^6 + 70x^4 - 84x^2 + 17)f(x)\,dx$$

and

$$\langle f, \widehat{A}^{-3}F\rangle_H = \frac{i}{168}\int_0^1 (x^8 - 4x^7 + 14x^5 - 28x^3 + 17x)f(x)\,dx.$$

Using the program Derive, we have $\langle \widehat{A}^{-1}F, F\rangle_H = 0$, $\langle \widehat{A}^{-2}F, F\rangle_H = \frac{5461}{108108}$ and from (3.15) $W = \frac{4\cdot 691^2}{1386^2}$. As a result of this and (3.15) we get the solution (3.31) of the problem (3.30). □

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