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THE YOGA OF COMMUTATORS

Abstract. In the present paper we discuss some recent versions of localization methods for calculations in the groups of points of algebraic-like and classical-like groups. Namely, we describe relative localization, universal localization, and enhanced versions of localization-completion. Apart from the general strategic description of these methods, we state some typical technical results of the conjugation calculus and the commutator calculus. Also, we state several recent results obtained therewith, such as relative standard commutator formulae, bounded width of commutators, with respect to the elementary generators, and nilpotent filtrations of congruence subgroups. Overall, this shows that localization methods can be much more efficient, than expected.

In the present paper we briefly describe three recent versions of localization methods in the study of algebraic-like groups, namely,

- **Relative localization** [24–26]
- **Universal localization** [47],
- Enhanced **localization-completion**, [5, 17, 18, 22, 7],

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and state some recent results obtained therewith.

The term `yoga of commutators` refers to the methods themselves, more precisely, to a large body of calculations and technical facts, conventionally known as the **conjugation calculus** and the **commutator calculus**.

As a matter of fact, the three typical recent applications of these methods, we mention here, also pertain to commutators in algebraic-like groups:

• Standard commutator formulae for congruence subgroups/relative elementary subgroups, [67, 69, 24–26].

• Universal bound for the width of arbitrary commutators in terms of elementary generators [46, 49, 21, 47].

• Nilpotent structure of K_1, for groups over a ring of finite Bass–Serre dimension $d = \delta(R) < \infty$ it can be interpreted as a multiple commutator formula of length $d + 1$, see [5, 17, 18, 22, 7].

This paper is based on our joint talks at the following Conferences:

• Topology, Geometry and Dynamics: Rokhlin Memorial (SPb, January 2010) [20],

• 2nd Group Theory Conference (Mashhad, Iran, March 2010),

• Polynomial Computer Algebra 2010 (SPb, April 2010),

• International Algebra Conference dedicated to the 70th Birthday of A. V. Yakovlev (SPb, June 2010).

There, we described a major project whose goal is to review existing localization methods in the study of groups of points of reductive algebraic groups, classical groups, and related groups. Our main objective is to develop new more powerful and efficient versions of conjugation calculus and commutator calculus, with a view towards new applications.

In this sense, it is a partial update of our survey [65], which was based on our talk at the

• Applications of Computer Algebra 2008 (Linz, July 2008),

• Symbolic and Numeric Scientific Computations 2008 (Linz, July 2008),

• Polynomial Computer Algebra 2009 (SPb, April 2009),

and had more computational flavour. At that time, we were rather skeptical about the use of localization techniques for actual calculations in the groups of points of algebraic groups, with realistic bounds.

However, in the Fall and Winter 2009/2010 we developed new versions of localisation, which allowed us to prove some striking results. For example, it turned out, that for algebraic groups length estimates of various classes of elements, such as commutators, in terms of elementary generators, do not depend on the dimension of the ground ring, but on the type of the group alone.

So far we have not succeeded in getting reasonable polynomial bounds, even less so in converting our methods into working algorithms for calculations in algebraic groups. Still, presently these goals seem slightly less unfeasible, than at the moment we were writing [65].

§1. The groups

Here, we consider algebraic-like or classical-like group functors G. Further, let $G(R)$ be the group of points of G over a ring R. Observe, that groups of types other than A_l only exist over commutative rings. Typically, $G(R)$ is one of the following groups.

A. General linear group $\mathrm{GL}(n, R)$ of degree n over R.

Actually, many results are already new in this context. Moreover, one can consider general linear groups over arbitrary associative rings, and in this case our methods work over quasi-finite rings. Recall, that a ring R is called *almost commutative* (or, sometimes, *module finite*), if it is finitely generated as a module over its centre. *Quasi-finite* rings are direct limits of inductive systems of almost commutative rings.

However, in most of our current papers we work in one of the following more general situations:

B. Bak's unitary groups $\mathrm{GU}(2n, A, \Lambda)$, over a form ring (A, Λ).

The notation we use for these groups, their subgroups and elements are mostly standard. As in [9], in the case of hyperbolic unitary groups we number columns and rows of matrices as follows: $1, \dots, n, -n, \dots, -1$. Recall, that in this setting A is a [not necessarily commutative] ring with involution $\overline{} : A \longrightarrow A$, Λ is the form parameter. To somewhat simplify matters, we usually assume that A is module finite over a commutative ring R. In general, Λ is not an R-module. Thus, R has to be replaced by its subring R_0, generated by $\xi\overline{\xi}$, for $\xi \in R$. In the sequel we usually do not discuss similar technical details, referring to [7, 9, 17, 18, 21, 23, 24, 28] for precise statements and conclusive proofs.

Actually, our favourite setting in this paper is the following one, see [1–3, 58, 64] and references there.

C. Chevalley groups $G(\Phi,R)$ of type Φ over R.

Chevalley groups are indeed *algebraic*, and the ground rings are *commutative* in this case, which usually makes life easier. We illustrate most of our methods in this example.

Together with the algebraic-like group $G(R)$ we consider the following subgroups.

• First of all, the elementary group $E(R)$, generated by elementary unipotents.

In the linear case, the elementary generators are elementary [linear] transvections $t_{ij}(\xi)$, $1 \leqslant i \neq j \leqslant n$, $\xi \in R$. In the unitary case, the elementary generators are elementary unitary transvections $T_{ij}(\xi)$, $1 \leqslant i \neq j \leqslant -1$, $\xi \in A$. In the even hyperbolic case they come in two modifications. They can be short root type, $i \neq \pm j$, when the parameter ξ can be any element of A. On the other hand, for the long root type $i=-j$ and the parameter ξ must belong to [something defined in terms of] the form parameter Λ. Finally, for Chevalley groups, the elementary generators are the elementary root unipotents $x_\alpha(\xi)$ for a root $\alpha \in \Phi$ and a ring element $\xi \in R$.

Further, let $I \trianglelefteq R$ be an ideal of R. We also consider the following relative subgroups.

• The elementary group $E(I)$ of level I, generated by elementary unipotents of level I.

• The relative elementary group $E(R,I)=E(I)^{E(R)}$ of level I.

• The principal congruence subgroups $G(R,I)$ of level I, the kernel of reduction homomorphism $\rho_I : G(R) \longrightarrow G(R/I)$.

• The full congruence subgroups $C(R,I)$ of level I, the inverse image of the centre of $G(R/I)$ with respect to ρ_I.

Recall the usual notation for these groups in the above contexts A–C.

$G(R)$	$\mathrm{GL}(n,R)$	$\mathrm{GU}(n,R,\Lambda)$	$G(\Phi,R)$
$E(R)$	$E(n,R)$	$\mathrm{EU}(n,R,\Lambda)$	$E(\Phi,R)$
$E(I)$	$E(n,I)$	$\mathrm{FU}(n,I,\Gamma)$	$E(\Phi,I)$
$E(R,I)$	$E(n,R,I)$	$\mathrm{EU}(n,I,\Gamma)$	$E(\Phi,R,I)$
$G(R,I)$	$\mathrm{GL}(n,R,I)$	$\mathrm{GU}(n,I,\Gamma)$	$G(\Phi,R,I)$
$C(R,I)$	$C(n,R,I)$	$\mathrm{CU}(n,I,\Gamma)$	$C(\Phi,R,I)$

There are two more general contexts, where Quillen–Suslin localization method has been successfully used by Victor Petrov, Anastasia Stavrova, and Alexander Luzgarev, [42–44, 45, 36].

D. Isotropic reductive groups $G(R)$,

E. Odd unitary groups $U(V,q)$.

We are positive that one could obtain results similar to the ones stated in the present paper also in these contexts, and we are presently working towards it.

§2. Localization

In the present paper we only use commutative localization. First, let us fix some notation. Let R be a commutative ring with 1, S be a multiplicative system in R and $S^{-1}R$ be the corresponding localization. We mostly use localization with respect to the two following types of multiplicative systems.

• *Principal localization*: S coincides with $\langle s\rangle = \{1, s, s^2, \dots\}$, for some nonnilpotent $s \in R$, in this case we usually write $\langle s\rangle^{-1}R = R_s$.

• *Localization at a maximal ideal*: $S = R \setminus M$, for some maximal ideal $M \in \operatorname{Max}(R)$ in R, in this case we usually write $(R \setminus M)^{-1}R = R_M$.

We denote by $F_S : R \longrightarrow S^{-1}R$ the canonical ring homomorphism called the *localization homomorphism*. For the two special cases above, we write $F_s : R \longrightarrow R_s$ and $F_M : R \longrightarrow R_M$, respectively.

When we write an element as a fraction, like a/s or $\dfrac{a}{s}$ we *always* think of it as an element of some localization $S^{-1}R$, where $s \in S$. If s were actually invertible in R, we would have written as^{-1} instead.

Ideologically, all proofs using localizations are based on the interplay of the three following observations:

• Functors of points $R \rightsquigarrow G(R)$ are compatible with localization,

$$g \in G(R) \quad \Longleftrightarrow \quad F_M(g) \in G(R_M) \quad \text{for all} \quad M \in \operatorname{Max}(R).$$

• Elementary subfunctors $R \rightsquigarrow E(R)$ are compatible with factorization, for any $I \trianglelefteq R$ the reduction homomorphism $\rho_I : E(R) \longrightarrow E(R/I)$ is surjective.

• On a [semi-]local ring R the values of semi-simple groups and their elementary subfunctors coincide, $G(R) = E(R)$.

The following property of the functors G and E, will be crucial for what follows: they *commute with direct limits*. In other words, if $R = \varinjlim R_i$, where $\{R_i\}_{i\in I}$ is an inductive system of rings, then

$$X(\Phi, \varinjlim R_i) = \varinjlim X(\Phi, R_i).$$

We use this property in the two following situations.

• *Noetherian reduction*: let R_i be the inductive system of all finitely generated subrings of R with respect to inclusion. Then

$$X = \varinjlim X(\Phi, R_i).$$

This allows to reduce most of the proofs to the case of Noetherian rings.

• *Reduction to principal localizations*: let S be a multiplicative system in R and let R_s, $s \in S$, be the corresponding inductive system with respect to the principal localization homomorphisms: $F_t : R_s \longrightarrow R_{st}$. Then

$$X(\Phi, S^{-1}R) = \varinjlim X(\Phi, R_s).$$

This reduces localization in any multiplicative system to principal localizations.

§3. Injectivity of localization homomorphism

Most localization proofs rely on the injectivity of localization homomorphism F_S. As observed in the previous section, we can only consider *principal* localization homomorphisms F_s. Of course, F_s is injective when s is regular. Thus, localization proofs are particularly easy for integral domains. A large part of what follows are various devices to fight with the presence of zero-divisors.

When s is a zero-divisor, F_s is not injective on the group $G(\Phi, R)$ itself. But its restrictions to appropriate congruence subgroups often are. Here are two important typical cases, Noetherian rings and semi-simple rings.

Lemma 1. *Suppose R is Noetherian and $s \in R$. Then there exists a natural number k such that the homomorphism $F_s : G(\Phi, R, s^kR) \longrightarrow G(\Phi, R_s)$ is injective.*

Proof. The homomorphism $F_s : G(\Phi, R, s^kR) \longrightarrow G(\Phi, R_s)$ is injective whenever $F_s : s^kR \longrightarrow R_s$ is injective. Let $M_i = \mathrm{Ann}_R(s^i)$ be the annihilator of s^i in R. Since R is Noetherian, there exists k such that $M_k = M_{k+1} = \dots$. If s^ka vanishes in R_s, then $s^is^ka = 0$ for some i. But since $M_{k+i} = M_k$, already $s^ka = 0$ and thus s^k_R injects in R_s.

Lemma 2. *If* $\mathrm{Rad}(R) = 0$, *then* $F_s : G(\Phi, R, sR) \longrightarrow G(\Phi, R_s)$ *is injective for all* $s \in R$, $s \neq 0$.

Proof. It suffices to prove that $F_s : sR \longrightarrow R_s$ is injective. Suppose that $s\xi \in sR$ goes to 0 in R_s. Then there exists an $m \in \mathbb{N}$ such that $s^m s\xi = 0$. It follows that $(s\xi)^{m+1} = 0$ and since R is semi-simple, $s\xi = 0$.

In [22] we used reduction to Noetherian rings, whereas in [49] reduction to semi-simple rings was used.

Another important trick to override the presence of zero-divisors consists in throwing in polynomial variables. Namely, instead of the ring R itself we consider the polynomial ring $R[t]$ in the variable t. In that ring t is not a zero-divisor, so that the localization homomorphism F_t is injective. We can use that, and then specialise t to any $s \in R$.

Actually, throwing in polynomial variables has more than one use. The elementary subfunctors $R \rightsquigarrow E(R)$ are not compatible with localization,

$$g \in E(R) \quad \Longrightarrow \quad F_M(g) \in E(R_M) \quad \text{for all } M \in \mathrm{Max}(R),$$

but the converse implication does not hold, for otherwise $E(R)$ would coincide with [the semi-simple part of] $G(R)$ for all commutative rings.

The following remarkable observation was due to Daniel Quillen at the level of K_0, and was first applied by Andrei Suslin at the level of K_1, in the context of solving Serre's conjecture, and its higher analogues [51]. See [30] for a description of Quillen–Suslin's idea in its historical development. We refer to the following result as Quillen–Suslin's lemma.

Theorem 1. *Let* $g \in G(R[t], tR[t])$. *Then,*

$$g \in E(R[t]) \quad \Longleftrightarrow \quad F_M(g) \in E(R_M[t]) \quad \textit{for all} \;\; M \in \mathrm{Max}(R).$$

§4. How localization works

As was already mentioned, **localization and patching** was first used to study the structure of linear groups by Andrei Suslin, back in 1976, see [51]. Among other important early contributors one could mention Suslin's [then] students Vyacheslav Kopeiko [29, 52], and Marat Tulenbaev [55], as well as Leonid Vaserstein [56, 58–60], Eiichi Abe [1–3], Li Fuan [31, 32], Giovanni Taddei [53, 54], You Hong [62].

Let us illustrate how localization works in the classical example, normality of the elementary subfunctor. Namely, we wish to prove that $E(R) \trianglelefteq G(R)$, for any commutative ring R.

To be more specific, below we assume that $G(R) = G(\Phi, R)$ is the simply-connected Chevalley group of type Φ. In this case, $E(R) = E(\Phi, R)$ is generated by the elementary root unipotents $x_\alpha(\xi)$, for $\alpha \in \Phi$ and $\xi \in R$. From here on, we *always* assume that Φ IS REDUCED AND IRREDUCIBLE OF RANK $\geqslant 2$. We do not reproduce this standing assumption in the statements of subsidiary results.

Thus, we wish to prove that for all $g \in G(\Phi, R)$, all $\alpha \in \Phi$ and all $\xi \in R$ one has

$$x = gx_\alpha(\xi)g^{-1} \in E(\Phi, R).$$

All localization proofs are based on *partitions of* 1. In other words, we pick up $\zeta_1, \dots, \zeta_m \in R$ such that $\zeta_1 + \dots + \zeta_m = 1$ and each of $gx_\alpha(\zeta_i\xi)g^{-1}$ already lies in $E(\Phi, R)$. The difference between various localization methods is in how one chooses such a partition.

For our taste, the most elementary way is the following version of **localization and patching** method. Instead of throwing in independent variables, as Quillen and Suslin did originally, and many others after them, we follow Anthony Bak [5], and recourse to Noetherian reduction. As we observed, the functors $G = G(\Phi, \_\ )$ and $E = E(\Phi, \_\ )$ commute with direct limits. Since R is the direct limit of its finitely generated subrings, we can from the very start assume that R is Noetherian. This will allow us to invoke Lemma 1.

Since we work with simply connected groups, for a local ring R the elementary subgroup $E(\Phi, R)$ coincides with the Chevalley group $G(\Phi, R)$. Thus, for any maximal ideal $M \in \operatorname{Max}(R)$ one has $F_M(g) \in E(\Phi, R_M)$. Now, we again invoke the fact that the functors G and E commute with direct limits. Since R_M is the direct limit of R_t, $t \in R \setminus M$, there exists an $s \in R \setminus M$ such that $F_s(g) \in E(\Phi, R_s)$.

We will search for ζ_i's in the desired partition of 1, as multiples of high powers s^l of the above elements s, for various maximal ideals $M \in \operatorname{Max}(R)$ and sufficiently large exponents l. Set $y = gx_\alpha(s^l\eta\xi)g^{-1}$, for some $\eta \in R$. Since the ring R is Noetherian, we can apply Lemma 1, and conclude that for a large power of s, say for s^n, the restriction of F_s to the principal congruence subgroup $G(\Phi, R, s^nR)$ is injective.

First, we argue locally, this part of the proof is called [first] **localization**. Since $F_s(g) \in E(\Phi, R_s)$, it can be written as a product of elementary root unipotents $x_\alpha\big(F_s(\theta/s^k)\big)$, $\theta \in R$, $k \geqslant 0$. From the Chevalley commutator formula it follows that conjugation by such an element is continuous in s-adic topology, this is exactly the **conjugation calculus** we discuss in the next section. In particular, there exists a high power of s, say, s^l,

$l \gg n$, such that

$$F_s(y) = F_s(g)F_t(x_\alpha(s^l\eta\xi))F_s(g)^{-1}$$

can be expressed as a product

$$F_s(y) = \prod_{j=1}^{m} x_{\beta_j}(F_s(s^n\theta_j)) \in E(\Phi, R_s),$$

for some $\theta_j \in R$.

Take the product

$$z = \prod_{i=1}^{m} x_{\beta_i}(s^n c_i) \in E(\Phi, R),$$

By the very definition $F_s(z) = F_s(y)$. On the other hand, since $G(\Phi, R, s^nR)$ is normal in G, one has $y, z \in G(\Phi, R, s^nR)$. Injectivity of F_s implies that $y = z \in E(\Phi, R)$.

The final part of the proof is called **patching**. Since $s^l \notin M$ and the same works for all maximal ideals, we can choose a finite set of such powers $s_i^{l_i}$, $1 \leqslant i \leqslant m$, which generate R as an ideal,

$$\zeta_1 + \ldots + \zeta_m = s_1^{l_1}\eta_1 + \ldots + s_m^{l_m}\eta_m = 1$$

is the desired partition.

Of course, there are some further technical details. For example, when one works with the adjoint group, such as $\mathrm{PGL}(n, R)$ or a diagonal extension of a Chevalley group, such as $\mathrm{GL}(n, R)$, there is an extra toral factor to take care of.

§5. Conjugation calculus

The first main objective of the conjugation calculus is to establish that conjugation by a fixed matrix $g \in G(\Phi, R_s)$ is continuous in s-adic topology. In the proof one uses a base of s-adic neighborhoods of e and establishes that for any such neighborhood V there exists another neighborhood U such that ${}^gU \subseteq V$.

To be more specific, let us state some typical results of conjugation calculus for Chevalley groups. Usually, as the base of neighborhoods of e, one takes

- elementary subgroups $E(\Phi, s^m R)$, or
- relative elementary subgroups $E(\Phi, R, s^m R)$.

For advanced applications, one usually needs more than just continuity of conjugation by g. One has to estimate the *module of continuity*, depending on the size of denominators in expression of g as a product of s-elementary factors. In generation problems, one often has to estimate also the *length* of arising elementary expressions.

To state typical results in this direction, we have to introduce some further notation. Namely, let L be a nonnegative integer and let $E^L(\Phi, I)$ denote the *subset* of $E(\Phi, I)$ consisting of all products of L or fewer elementary root unipotents $x_\alpha(\xi)$, where $\alpha \in \Phi$ and $\xi \in I$. Thus, $E^1(\Phi, I)$ is the set of all $x_\alpha(\xi)$, $\alpha \in \Phi$, $\xi \in I$.

Conjugation calculus and commutator calculus are rare examples of induction results, where the base of induction is *terribly* much harder, than the induction step. Without length estimates the following results have been established by Giovanni Taddei [53, 54], and then, in a stronger and more straightforward form, by the first and the third author [22]. The precise form with explicit length estimates is taken from the paper by the second and the third author [49].

Lemma 3. *If p, q and h are given, there exist o, r such that*

$$ {}^{x}y \in E^{24}(\Phi, s^p t^q R) \quad \textit{for all} \quad x \in E^1\Big(\Phi, \frac{1}{s^h} R\Big), \quad y \in E^1\Big(\Phi, s^o t^r R\Big). $$

For the case, where x and y are not opposite, the claim immediately follows from the Chevalley commutator formula. Indeed, let i_Φ be the largest integer which may appear as i in a root $i\alpha + j\beta \in \Phi$ for all $\alpha, \beta \in \Phi$. Obviously $i_\Phi = 1, 2$ or 3, depending on whether Φ is simply laced, doubly laced or triply laced.

Now, let $\alpha \neq -\beta$ and set $o \geq i_\Phi h + p + 1$, $r \geqslant q$. By the Chevalley commutator formula, one has

$$ \begin{aligned} & x_\alpha\Big(\frac{a}{s^h}\Big) x_\beta\Big(s^o t^r b\Big) x_\alpha\Big(-\frac{a}{s^h}\Big) \\ & \qquad = x_\beta\Big(s^o t^r b\Big) \prod_{i\alpha + j\beta \in \Phi} x_{i\alpha + j\beta}\bigg(N_{\alpha\beta ij}\Big(\frac{a}{s^h}\Big)^i \Big(s^o t^r b\Big)^j\bigg) \end{aligned} $$

and a quick inspection shows that the right-hand side of the above equality is in $E^L(\Phi, s^p t^q R)$, where $L = 2, 3$ or 5, depending on whether Φ is simply laced, doubly laced or triply laced.

For the case of opposite roots, one first has to use the Chevalley commutator formula to express $x_{-\alpha}\Big(s^o t^r b\Big)$ as a product of elementary factors, corresponding to the roots not opposite to α. For example, when $-\alpha = \gamma + \delta$ is the sum of two roots of the same length, one has

$$x_{-\alpha}(s^o t^r b) = \big[x_\gamma(s^{o/2} t^{r/2}), x_\delta(s^{o/2} t^{r/2} b)\big],$$

and we have reduced the problem to the preceding case. For other cases the proof is similar, but slightly fancier, due to the longer products in the Chevalley commutator formula, see [22, 49] for details.

Now, the following general result immediately follows by induction.

Lemma 4. *If* p, q *and* h *are given, there exist* o, r *such that*

$${}^{x}y \in E^{24^L K}(\Phi, s^p t^q R) \quad \textit{for all} \quad x \in E^L\Big(\Phi, \frac{1}{s^h} R\Big), \quad y \in E^K\Big(\Phi, s^o t^r R\Big).$$

Actually, for systems without factors of type G_2 one can even conclude that ${}^{x}y \in E^{13^L K}(\Phi, s^p t^q R)$. For simply laced systems and for F_4, one can conclude that ${}^{x}y \in E^{8^L K}(\Phi, s^p t^q R)$.

§6. Commutator calculus

More sophisticated applications, such as calculation of mutual commutator subgroups, nilpotent filtration, description of various classes of intermediate subgroups, etc., require **second localization**. In other words, we have to be able to simultaneously fight with powers of *two* elements in the denominator.

For $\mathrm{GL}(n, R)$ second localization was used by Anthony Bak in [5], and then generalised to unitary groups in the Thesis of the first named author [17, 18]. For Chevalley groups, it was first implemented by the present authors in [22], and then in a more precise form in [49].

Lemma 5. *Given* $s, t \in R$ *and* $p, q, k, m \in \mathbb{N}$ *there exist* $l, n \in \mathbb{N}$ *and* $L = L(\Phi)$ *such that*

$$[x, y] \in E^L\bigg(\Phi, s^p t^q R\bigg) \quad \textit{for all} \quad x \in E^1\bigg(\Phi, \frac{t^l}{s^k} R\bigg), \quad y \in E^1\bigg(\Phi, \frac{s^n}{t^m} R\bigg).$$

Let $\alpha, \beta \in \Phi$ and $a, b \in R$. We have to prove that the commutator

$$\bigg[x_\alpha\bigg(\frac{t^l}{s^k} a\bigg), x_\beta\bigg(\frac{s^n}{t^m} b\bigg)\bigg] \in E^L\bigg(\Phi, s^p t^q R\bigg),$$

for some specific L. For the case, where $\alpha \neq -\beta$ the proof is easy. Writing the Chevalley commutator formula

$$\left[x_\alpha\left(\frac{t^l}{s^k}a\right), x_\beta\left(\frac{s^n}{t^m}b\right)\right] = \prod_{i,j>0} x_{i\alpha+j\beta}\left(\left(\frac{t^l}{s^k}a\right)^i\left(\frac{s^n}{t^m}b\right)^j\right),$$

we see that one can take l and n large enough to kill the denominators on the right-hand side, and still leave large enough powers of s and t in the numerators. In fact, $l \geqslant i_\Phi m + q + 1$ and $n \geqslant i_\Phi k + p + 1$ would go. Furthermore, the number of factors on the right-hand side of the Chevalley commutator formula is not more than 4. Thus, the product on the right-hand side is in $E^L(\Phi, s^p t^q R)$, where $L = 1, 2$ or 4, depending on whether Φ is simply laced, doubly laced or triply laced.

The proof for opposite roots is *much* fancier, and relies on longer commutator identities.

Again, the general case easily follows by induction, via purely group theoretic arguments, see [49, §9].

Lemma 6. *Let $s, t \in R$ and $p, q, k, m \in \mathbb{N}$ Then there exist $l, n \in \mathbb{N}$ such that*

$$[x, y] \in E^{(L+1)^K-1}(\Phi, s^p t^q R)$$

$$\text{for all } x \in E^1\left(\Phi, \frac{s^l}{t^k}R\right), \quad y \in E^K\left(\Phi, \frac{t^n}{s^m}R\right).$$

By looking inside the proofs in [22] and [49] one gets the following silly length estimates

- $L \leqslant 585$, for simply laced systems,
- $L \leqslant 61882$, for doubly laced systems,
- $L \leqslant 797647204$, for triply laced systems.

In obtaining these stupid bounds we do not look inside the commutators and do not count the actual factors appearing in the Chevalley commutator formula. However, should we do that, the resulting bound for G_2 still would be well in the millions.

Problem 1. *Calculate realistic length bounds for L in these lemmas.*

As far as we can see, calculating in the 7-dimensional or the 8-dimensional representation of the group of type G_2 one gets bounds for L within few dozens, rather than millions.

Let us state a typical target result of the commutator calculus.

Theorem 2. *Fix an element $s \in R$, $s \neq 0$. Then for any p and k there exists an r such that*

$$\Big[E\Big(\frac{1}{s^k}R\Big), F_s(G(\Phi, R, s^r R))\Big] \leqslant E(\Phi, F_s(s^p R)) \leqslant G(\Phi, R_s).$$

Despite its rather technical appearance, it is a very general and powerful result. In fact, in the *trivial* special case, where $s = 1$, this Theorem boils down to the normality of the elementary subgroup!

The general case of the theorem was substantially used in description of overgroups of classical and exceptional groups by the second named author and Victor Petrov [66], and by Alexander Luzgarev [35]. We do not mention any further results in this direction, referring to our surveys [68] and [50].

§7. Relative commutator calculus

For the group $G(R)$ itself, conjugation calculus works marvelously, as one takes $E(s^m R)$ or $E(R, s^m R)$, as the base of s-adic neighbourhoods. But can one *relativise* all occurring calculations? In other words, what happens when we replace the ring R by an ideal $I \trianglelefteq R$? Again, one has to establish that for any neighbourhood V of e in $G(R, I)$ there exists another neighborhood U such that ${}^g U \subseteq V$.

As a first attempt, without much thinking, one tries to replace R by I everywhere in the above calculations. For example, it seems that one should consider the following bases of s-adic neighborhoods of e in $G(R, I)$:

- elementary subgroups $E(s^m I)$,
- relative elementary subgroups $E(R, s^m I)$.

However, both choices are not fully satisfactory in that they lead to extremely onerous calculations. The reason is that the first of these choices is too small as the neighbourhood on the right-hand side, while the second of these choices is too large as the neighbourhood on the left-hand side.

Solving problems posed by two of the present authors in [67], the first and the last authors proposed in [26] a first fully functional version of localization at the relative level. Their idea was to take the following *partially* relativised base of s-adic neighbourhoods.

$$E(s^m R, s^m I) = E(s^m I)^{E(s^m R)}.$$

To convey the flavour of the ensuing results, let us state some typical lemmas from our forthcoming relative Chevalley paper [25]. Similar results for $\mathrm{GL}(n,R)$ and $\mathrm{GU}(2n,R,\Lambda)$ are established in [26] and [24], respectively. Again, the base of induction is the hardest part of the whole argument.

Lemma 7. *If p, q and k are given, there exist h and m such that*

$$E^1\left(\Phi,\tfrac{1}{s^k}R\right)E(\Phi,s^ht^mI)\subseteq E(\Phi,s^pt^qR,s^pt^qI).$$

Such h and m depend on Φ, k, p and q alone, but not on the ideal I.

Observe, that the proofs of this and similar results work in terms of roots alone, and thus one obtains *uniform* estimates for the powers of s and t, which do not depend on the ideal I. In other words, conjugation by $g\in G(\Phi,R_s)$ is *equi-continuous* in all congruence subgroups $G(\Phi,R,I)$, with respect to

$$E(\Phi,s^kR,s^kI)=E(\Phi,s^kI)^{E(\Phi,s^kR)},$$

as the corresponding bases of s-adic neighbourhood.

This is extremely important for applications we have in mind. For example, in the next result we use this to obtain a uniform bound for *two* ideals $A,B\trianglelefteq R$.

Lemma 8. *If p,k are given, then there is an q such that*

$$\begin{aligned}{}^{E^1\left(\Phi,\frac{R}{s^k}\right)}&\big[E(\Phi,s^qR,s^qA),E(\Phi,s^qR,s^qB)\big]\\ &\subseteq\big[E(\Phi,s^pR,s^pA),E(\Phi,s^pR,s^pB)\big].\end{aligned}$$

Similarly, the induction base of the relative commutator calculus looks as follows. Again, in view of applications, we state it for *two* ideals $A,B\trianglelefteq R$.

Lemma 9. *If p,q,k,m are given, then there exist l and n such that*

$$\Big[E^1\Big(\Phi,\frac{t^l}{s^k}A\Big),E^1\Big(\Phi,\frac{s^n}{t^m}B\Big)\Big]\subseteq[E(\Phi,s^pt^qR,s^pt^qA),E(\Phi,s^pt^qR,s^pt^qB)].$$

These l and n depend on Φ,p,q,k,m alone, and do not depend on the choice of ideals A and B.

This is a rather difficult technical result. Also, in the relative case induction step itself is nontrivial. For example, Lemma 9 itself does not suffice even to start the induction. Instead, we have to establish something as follows:

$$\Big[E^1(\Phi, s^q A), {}^{E^1(\Phi,\frac{R}{s^k})} E^1\Big(\Phi, \frac{B}{s^k}\Big)\Big] \subseteq \big[E(\Phi, s^p R, s^p A), E(\Phi, s^p R, s^p B)\big].$$

We do not reproduce the precise target results of the relative commutator calculus, which are far too technical for a casual overview. The interested reader can find such precise statements in our papers [21, 24–26].

§8. Relative commutator formulas

One of the central and most important results in the theory of linear groups over rings are the *absolute* standard commutator formulas

$$[G(R), E(R, I)] = E(R, I) = [E(R), C(R, I)].$$

The first one of them amounts to saying that $E(R, I)$ is normal in $G(R)$, while the second one is a key tool of level reduction. For $\mathrm{GL}(n, R)$ at the stable level these formulae were established by Hyman Bass. Later, Leonid Vaserstein improved the estimate for the stable rank $\mathrm{sr}(R)$ of the ring R by 1.

Soon thereafter, Andrei Suslin [51], Leonid Vaserstein [56], Zenon Borewicz, and the first author, observed that for *almost commutative* rings these formulae hold in $\mathrm{GL}(n, R)$ for any $n \geqslant 3$. Recall, that a ring R is called almost commutative if it is finitely generated as a module over its centre.

In the sequel, Igor Golubchik, Alexander Mikhalev sen., Sergei Khlebutin, and Anthony Bak generalised these formulae to broad classes of noncommutative rings. Vyacheslav Kopeiko and Andrei Suslin [29, 52], Giovanni Taddei, Leonid Vaserstein, You Hong, Anthony Bak, and the present authors generalised these results to other groups. In [6, 8–10, 16, 19, 22, 23, 30, 48, 64, 70] one can find different proofs of these results, *many* further references, and a detailed discussion of their role in the structure theory.

However, much less was known about the *relative* versions of the above formulae. Namely, let $A, B \trianglelefteq R$ be two ideals of the ring R. What can be said about the mutual commutators of congruence subgroups and relative elementary subgroups of levels A and B?

Before our works this problem was only addressed at the stable level, by Alec Mason and Wilson Stothers [37–40].

Let us state some typical results, we can prove by relative versions of localization methods, described in the previous section, see [24–26, 67, 69].

Theorem 3 A. *Let R be a quasi-finite ring, $n \geqslant 3$. Then for any two ideals $A, B \trianglelefteq R$ one has*

$$[E(n,R,A), \mathrm{GL}(n,R,B)] = [E(n,R,A), E(n,R,B)].$$

Theorem 3 B. *Let $n \geqslant 3$, R be a commutative ring, (A,Λ) be a form ring such that A is a quasi-finite R-algebra. Further, let (I,Γ) and (J,Δ) be two form ideals of a form ring (A,Λ). Then*

$$\big[\mathrm{EU}(2n,I,\Gamma), \mathrm{GU}(2n,J,\Delta)\big] = \big[\mathrm{EU}(2n,I,\Gamma), \mathrm{EU}(2n,J,\Delta)\big].$$

Theorem 3 C. *Let Φ be a reduced irreducible root system, $rk(\Phi) \geqslant 2$. Further, let R be a commutative ring, and $A, B \trianglelefteq R$ be two ideals of R. Then*

$$[E(\Phi,R,A), G(\Phi,R,B)] = [E(\Phi,R,A), E(\Phi,R,B)].$$

Observe, that in general one cannot expect the equality

$$[E(R,A), E(R,B)] = E(R,AB).$$

However, the true reason, why this equality holds in the absolute case, is not the fact that one of the ideals A or B coincides with R, but just the fact that A and B are comaximal. Namely, by combining the above relative commutator formulae with commutator identities, like the celebrated Hall –Witt identity, one gets the following results.

Theorem 4 A. *Let R be a quasi-finite ring, $n \geqslant 3$. Then for any two comaximal ideals $A, B \trianglelefteq R$, $A + B = R$, one has*

$$[E(n,R,A), E(n,R,B)] = E(n,R,AB+BA).$$

We do not recall notation pertaining to form ideals [9, 14–18, 23, 24].

Theorem 4 B. *Let $n \geqslant 3$, and (A, Λ) be an arbitrary form ring for which absolute standard commutator formulae are satisfied. Then for any two comaximal form ideals (I, Γ) and (J, Δ) of the form ring (A, Λ), $I + J = A$, one has the following equality*

$$[\mathrm{EU}(2n, I, \Gamma), \mathrm{EU}(2n, J, \Delta)] = \mathrm{EU}(2n, IJ + JI, {}^{J}\Gamma + {}^{I}\Delta + \Gamma_{\min}(IJ + JI)).$$

As for the next result, we also have a more general version, in terms of admissible pairs [2, 3, 16, 64]. We do not reproduce it here, not to overburden the reader with technical details.

Theorem 4 C. *Let Φ be a reduced irreducible root system,* $\mathrm{rk}(\Phi) \geqslant 2$. *Further, let R be a commutative ring, and $A, B \trianglelefteq R$ be two comaximal ideals of R, $A + B = R$, one has the following equality*

$$[E(\Phi, R, A), E(\Phi, R, B)] = E(\Phi, R, AB).$$

§9. Anti-Ore

Over fields, the groups of points of algebraic groups essentially consist of commutators: In fact, the celebrated *Ore conjecture* – now a theorem, [13, 34] – asserts that every element of a [non-Abelian] finite simple group is a single commutator. It is usually only easier to establish similar results for infinite fields.

The results we formulate in this and the next sections go in the opposite direction. Morally, they say that GROUPS OF POINTS OF ALGEBRAIC GROUPS OVER RINGS HAVE VERY FEW COMMUTATORS. In a strict technical sense, they have not more commutators, than elementary generators!

Similar bounded width results have a long history, which we cannot even sketch here. In general, $G(R)$ does not have bounded width with respect to the elementary generators.

• First, it is not even spanned by them! By definition, elementary generators generate the *elementary* subgroup $E(R)$, which is usually strictly smaller, than $G(R)$.

• Even when $G(R) = E(R)$, it does not have to have bounded width with respect to the elementary generators. Wilberd van der Kallen observed that already $\mathrm{SL}(3, \mathbb{C}[x])$ has unbounded width [27].

For some time, it was an open question, whether $E(R)$ has bounded length with respect to commutators. It was settled in the negative by Keith Dennis and Leonid Vaserstein [11, 12].

However, the situation with the commutators turned out to be exactly the opposite to what was expected in the 1980s. The following amazing result is established in the paper by Alexander Sivatsky and the second named author [46].

Theorem 5 A. *Let* $G = \mathrm{GL}(n, R)$, $n \geqslant 3$, *where* R *be a Noetherian ring such that* $\dim \mathrm{Max}(R) = d < \infty$. *Then there exists a natural number* N *depending only on* n *and* d *such that each commutator* $[x, y]$ *of elements* $x \in \mathrm{GL}(n, R)$ *and* $y \in E(n, R)$ *is a product of at most* N *elementary transvections.*

The original proof of that result in [46] depended *both* on localization and a very precise form of decomposition of unipotents, as proven in [48]. It was not at all clear, that it could be generalised to other groups, even the classical ones.

However, the second named author soon came up with an idea to replace the use of decomposition of unipotents by the second localization. Using the machinery developed by the first named and the third named authors in [22] – and, in fact, enhancing it – the second and the third named authors succeeded in generalising the above result to Chevalley groups [49].

Theorem 5 C. *Let* $G = G(\Phi, R)$ *be a Chevalley group of rank* $l \geqslant 2$ *and let* R *be a ring such that* $\dim \mathrm{Max}(R) = d < \infty$. *Then there exists a natural number* N *depending only on* Φ *and* d *such that each commutator* $[x, y]$ *of elements* $x \in G(\Phi, R)$ *and* $y \in E(\Phi, R)$ *is a product of at most* N *elementary root unipotents.*

We do not describe the strategy of the localization proof of this result, since it is expounded in full detail in [49]. Moreover, this result is largely superceded by the truly miraculous result stated in the next section.

Similar result also holds for unitary groups [21], it relies on the full force of localization methods developed in [17, 18, 25].

Theorem 5 B. *Let* $G = \mathrm{GU}(2n, R, \Lambda)$ *be the unitary group of degree* $2n$ *over a commutative form ring* (R, Λ). *Assume that* $n \geqslant 2$ *and* $\dim \mathrm{Max}(R) = d < \infty$. *Then there exists a natural number* N *depending only on* n *and* d *such that each commutator* $[x, y]$ *of elements* $x \in \mathrm{GU}(2n, R, \Lambda)$ *and* $y \in \mathrm{EU}(2n, R, \Lambda)$ *is a product of at most* N *elementary unitary transvections.*

§10. Universal localization

Now, something truly amazing will happen. The results we stated in the previous section relied on the fact that Jacobson dimension $d = \dim(\mathrm{Max}(R))$ of the ground ring R is finite. The length estimates of commutators, in elementary generators, were stated in terms of degree n or type of root system Φ, *and* dimension d.

Recently, the second named author observed, that for *algebraic* groups no finiteness condition on the ground rings is necessary here. In other words, the length estimates do not depend on d. In particular, THERE EXIST UNIVERSAL LENGTH BOUNDS FOR THE LENGTH OF COMMUTATORS, in the group of given type, over an *arbitrary* commutative ring. One does not even have to assume these rings to be Noetherian!

Let us state one of the main results of [47].

Theorem 6 C. *Let Φ be a reduced irreducible roots system of rank $l \geqslant 2$, and let $G = G(\Phi, \_\,)$ be the simply connected Chevalley–Demazure group scheme of type Φ. Then there exists an integer l depending only on Φ, that satisfies the following property. For any commutative ring R, any $x \in G(\Phi, R)$ and any $y \in E(\Phi, R)$ the commutator $[x, y]$ can be written as a product of at most l elementary root unipotents in $G(\Phi, R)$.*

The idea behind this result can be described as follows. There exists a **universal commutator** which is *generic* in the sense that it specialises to any other commutator of this shape. Thus, any elementary expression of this universal commutator provides an upper bound on the length of any such commutator. One expects such a universal commutator to live inside the group of points over the **universal coefficient ring** of the group.

It is easy to guess, what is the universal coefficient ring for the algebraic group itself. Recall, that

$$G(\Phi, R) = \mathrm{Hom}(\mathbb{Z}[G], R),$$

where $\mathbb{Z}[G]$ is the affine ring of $G = G(\Phi, \_\,)$. In other words, a point $h \in G(\Phi, R)$ of the group $G(\Phi, \_\,)$ over the ring R can be identified with a homomorphism $h : \mathbb{Z}[G] \longrightarrow R$.

Clearly, any such homomorphism h can be factored through the identity map $\mathbb{Z}[G] \longrightarrow \mathbb{Z}[G]$. In other words, this means that the point $g \in G(\Phi, \mathbb{Z}[G])$ represented by the identity map $\mathbb{Z}[G] \longrightarrow \mathbb{Z}[G]$, is the

generic element of G in the sense that it specialises to any point $h \in G(\Phi, R)$ over any commutative ring R. Namely, by functoriality,

$$G(h) : G(\Phi, \mathbb{Z}[G]) \longrightarrow G(\Phi, R), \quad g \mapsto h.$$

Similarly, one can define **generic elements** under localization. Namely, for $s \in \mathbb{Z}[G]$ we denote by $g_s \in G(\Phi, \mathbb{Z}[G]_s)$ the element, represented by the localization homomorphism $F_s : \mathbb{Z}[G] \longrightarrow \mathbb{Z}[G]_s$.

As a special case of this construction, we get generic *elementary* root unipotents. Since the affine ring of $\mathbb{G}_a$ is $\mathbb{Z}[t]$, one just has to take an independent variable t as a parameter.

However, it is much harder to figure out, what the generic element of the *elementary* subgroup $E(\Phi, \_\ )$ could be. Also, in the course of localization proof one has to construct generic elements of the *congruence* subgroups $G(\Phi, \_, \_\ )$.

In [47], the second named author finds a way to circumvent these difficulties. Namely, there he manages to produce a universal coefficient ring for *principal* congruence subgroups. In view of the usual direct limit arguments, it suffices to carry through a version of localization method. To be more specific, let us reproduce the precise statement.

Theorem 7. *There exist a commutative ring A, a regular element $s \in A$ and an element $f \in G(\Phi, A, sA)$ with the following property. For any commutative ring R, any regular element $r \in R$ and any $h \in G(\Phi, R, rR)$ there exists a unique ring homomorphism $\phi : A \longrightarrow R$ such that $\phi(s) = r$ and $G(\phi)(f) = h$.*

This is the main new tool, but there are many further details, too technical to be reproduced here. We refer the interested reader to [47].

§11. Completion

Another very important idea, which allows to substantially enhance the scope of localization methods, is to combine it with completion. This idea is due to Anthony Bak [5]. It was applied to unitary groups in the Thesis of the first named author [17, 18], and to Chevalley groups, in the works by the first named and the third named authors [22]. Also, in [17] and [22] we introduced several important simplifications which further enhanced the applicability of this method.

Let $s \in R$. Recall that the s-completion $\widehat{R}_s$ of the ring R is usually defined as the following inverse limit:

$$\widehat{R}_s = \varprojlim R/s^n R, \quad n \in \mathbb{N}.$$

However, this definition is not quite compatible with our purposes. Namely, as always, to control zero divisors, we have to reduce to Noetherian rings first. However, if $R = \varinjlim R_i$ is a direct limit of Noetherian rings, the canonical homomorphism $\varinjlim (\widehat{R}_i)_s \longrightarrow \widehat{R}_s$ is in general neither surjective, nor injective.

This forces us to modify the definition of completion as follows:

$$\widetilde{R}_s = \varinjlim (\widehat{R}_i)_s,$$

where the limit is taken over all finitely generated subrings R_i of R which contain s. Let us denote by $\widetilde{F}_s$ the canonical map $R \longrightarrow \widetilde{R}_s$. For the case, where R is Noetherian $\widetilde{F}_s = \widehat{F}_s$ coincides with the inverse limit of reduction homomorphisms $\pi_{s^n} : R \longrightarrow R/s^n R$

Let R be a commutative ring, Φ be an irreducible root system of rank ≥ 2. Define

$$G(\Phi, R, s^{-1}) = \operatorname{Ker}\big(G(\Phi, R) \longrightarrow G(\Phi, R_s)/E(\Phi, R_s)\big),$$
$$G(\Phi, R, \widehat{s}) = \operatorname{Ker}\big(G(\Phi, R) \longrightarrow G(\Phi, \widetilde{R}_{(s)})/E(\Phi, \widetilde{R}_{(s)})\big).$$

The following theorem embodies the gist of **localization-completion** method. Morally, it tells that the commutator of something, that becomes elementary under localization, with another something, that becomes elementary under completion, is indeed elementary. Clearly, this is something more powerful than just normality of the elementary subgroup. It goes in the direction of proving that the commutator of two *arbitrary* matrices is elementary. Of course, in general this is not the case, but, as we shall see in the next section, for finite dimensional rings it is a very near miss.

Theorem 8 C. *Let R be a commutative ring, Φ be an irreducible root system of rank ≥ 2. Then*

$$[G(\Phi, R, s^{-1}), G(\Phi, R, \widehat{s})] \leqslant E(\Phi, R).$$

Let R_i be the inductive system of all finitely generated subrings of R, containing s. Then

$$G(\Phi, R, s^{-1}) = \varinjlim G(\Phi, R_i, s^{-1}),$$
$$G(\Phi, R, \widehat{s}) = \varinjlim G(\Phi, R_i, \widehat{s}),$$

which again reduces the proof to the case, where R is Noetherian.

Let $x \in G(\Phi, R, s^{-1})$ and $y \in G(\Phi, R, \widehat{s})$. By definition, the condition on x means that $F_s(x) \in E^K(\Phi, a/s^k)$ for some k and K. On the other hand, the condition on y means that $\pi_{s^n}(y) \in E(\Phi, R/s^nR)$ for *all* n, or, what is the same, $y \in E(\Phi, R)G(\Phi, R, s^nR)$.

In other words, for *any* n we can present y as a product $y = uz$, $u \in E(\Phi, R)$ and $z \in G(\Phi, R, s^nR)$. Thus,

$$[x, y] = [x, uz] = [x, u] \cdot {}^u[x, z].$$

The first commutator belongs to $E(\Phi, R)$ together with u since $E(\Phi, R)$ is normal. As for the second commutator, choosing a very large n, and applying Theorem 2, we get $F_s([x, z]) \in E(\Phi, F_s(s^qR))$, for a large q. On the other hand, since $G(\Phi, R, s^qR)$ is normal, $[x, z] \in G(\Phi, R, s^qR)$. Now the usual argument based on the injectivity of the reduction homomorphism convinces us that $[x, z] \in E(\Phi, s^qR)$.

Similar result holds also in the unitary setting. Let (A, Λ) be a form ring, which is module finite over a commutative ring R. Take $s \in R_0$ and define

$$U(2n, A, \Lambda, s^{-1}) = \operatorname{Ker}\Big(U(2n, A, \Lambda) \longrightarrow U(2n, A_s, \Lambda_s)/\operatorname{EU}(2n, A_s, \Lambda_s)\Big),$$

$$U(2n, A, \Lambda, \hat{s}) = \operatorname{Ker}\Big(U(2n, A, \Lambda) \longrightarrow U\big(2n, \widetilde{(A, \Lambda)}_{(s)}\big)/E\big(2n, \widetilde{(A, \Lambda)}_{(s)}\big)\Big).$$

One of the main results of the Thesis by the first named author [17, 18] can be now stated as follows.

Theorem 8 B. *Let (A, Λ) be a module finite form ring over a commutative ring R, and let $s \in R_0$. Then*

$$[U(2n, A, \Lambda, s^{-1}), U(2n, A, \Lambda, \hat{s})] \leqslant \operatorname{EU}(2n, A, \Lambda).$$

§12. Nilpotent filtrations

As another illustration of the power of our methods, let us state some important results obtained by the **localization-completion** method, as developed in [5, 17, 18, 22, 7]. The following theorems imply, in particular, nilpotency of *relative* K_1's.

Theorem 9 B. *Let (A, Λ) be a form ring which is module finite over a commutative ring R of finite Bass–Serre dimension $\delta(R)$, and let (I, Γ) be a form ideal of (A, Λ). Then for any $n \geqslant 3$ the quotient $U(2n, I, \Gamma)/\operatorname{EU}(2n, I, \Gamma)$ is nilpotent by Abelian of nilpotent class at most $\delta(R) + 1$.*

Theorem 9 C. *Let Φ be a reduced irreducible root system of rank $\geqslant 2$, let R be a commutative ring of finite Bass–Serre dimension $\delta(R)$, and let $I \trianglelefteq R$ be its ideal. Then for any Chevalley group $G(\Phi, R)$ of type Φ over R the quotient $G(\Phi, R, I)/E(\Phi, R, I)$ is nilpotent by Abelian of nilpotent class at most $\delta(R) + 1$.*

In fact, in [7] we prove something much more powerful. Namely, without any finiteness assumptions on ground rings, we construct **nilpotent filtrations** of congruence subgroups. For rings of finite Bass–Serre dimension these filtrations are indeed finite.

To state the precise form of these results, we have to recall definitions of certain **higher elementary subgroups**, which play the same role for unitary groups, and for Chevalley groups, as Bak's **very special linear groups** do in the linear case.

Let (A, Λ) be a module finite form ring over a commutative ring R and let (I, Γ) be a form ideal in (A, Λ). Define

$$S^d U(2n, I, \Gamma) = \bigcap_{\phi} \operatorname{Ker}\Big(U(2n, I, \Gamma) \longrightarrow U(2n, I', \Gamma')/\operatorname{EU}(2n, I', \Gamma')\Big),$$

where the intersection is taken over all homomorphisms $A \longrightarrow A'$ of rings with involution, A' is module finite over a commutative ring R' of Bass–Serre dimension $\delta(R') \leqslant d$, Λ' is the form parameter of A' generated by $\phi(\Lambda)$, I' is the involution invariant ideal of A' generated by $\phi(I)$, and, finally, Γ' is the relative form parameter of level I' of the form ring (A', Λ'), generated by $\phi(\Gamma)$.

Theorem 10 B. *Let (A, Λ) be a module finite form ring over a commutative ring R. Further, let (I, Γ) be a form ideal of (A, Λ) and $n \geq 3$. Then*

- *Each $S^d U(2n, I, \Gamma)$ is a normal subgroup of $\operatorname{GU}(2n, R, \Lambda)$.*
- *The sequence*

$$S^0 U(2n, I, \Gamma) \geqslant S^1 U(2n, I, \Gamma) \geqslant S^2 U(2n, I, \Gamma) \geqslant \cdots$$

is a descending $S^0U(2n,R,\Lambda)$*-central series.*

• *The conjugation action of* $\mathrm{GU}(2n,R,\Lambda)$ *on* $U(2n,I,\Gamma)/S^0U(2n,I,\Gamma)$ *is trivial.*

• *If Bass–Serre dimension of* R *is finite,* $\delta(R)<\infty$*, then*

$$S^dU(2n,R,I)=\mathrm{EU}(2n,R,I),$$

whenever $d\geqslant\delta(R)$*.*

Next, we do the same for Chevalley groups. Let R be a commutative ring and let $I\trianglelefteq R$ an ideal of R. Define

$$S^dG(\Phi,R,I)=\bigcap_{\phi}\mathrm{Ker}\big(G(\Phi,R,I)\longrightarrow G(\Phi,A,\phi(I)A)/E(\Phi,A,\phi(I)A)\big),$$

where the intersection is taken over all homomorphisms $\phi:R\longrightarrow A$ to rings of Bass–Serre dimension $\delta(A)\leqslant d$. As usual, we set $S^dG(\Phi,R)=S^dG(\Phi,R,R)$.

Theorem 10 C. *Let* Φ *be an irreducible root system of rank* $\geqslant 2$*, let* R *be a commutative ring, and let* I *be an ideal of* R*. Then*

• *Each* $S^dG(\Phi,R,I)$ *is a normal subgroup of* $G(\Phi,R)$

• *The sequence*

$$S^0G(\Phi,R,I)\geqslant S^1G(\Phi,R,I)\geqslant S^2G(\Phi,R,I)\geqslant\cdots$$

is a descending $S^0G(\Phi,A)$*-central series.*

• *The conjugation action of* $G(\Phi,R)$ *on* $G(\Phi,R,I)/S^0G(\Phi,R,I)$ *is trivial.*

• *If Bass–Serre dimension of* R *is finite,* $\delta(R)<\infty$*, then*

$$S^dG(\Phi,R,I)=E(\Phi,R,I),$$

whenever $d\geqslant\delta(R)$*.*

§13. Where next?

In conclusion, we list some unsolved problems related to the results of the present paper. We have preliminary results in some of these directions, and intend to address them in subsequent publications.

Problem 2. *Obtain explicit length estimates in the relative conjugation calculus and commutator calculus.*

Problem 3. *Obtain explicit length estimates in the universal localization.*

Problem 4. *Develop versions of universal localization in the nonalgebraic setting, in particular, for unitary groups.*

Another important challenge is to improve rank bounds in the commutator calculus for the unitary groups. For the condition below (see [8, 9]).

Problem 5. *Develop conjugation calculus and commutator calculus in the group* $\mathrm{GU}(4, R, \Lambda)$*, provided* $\Lambda R + R\Lambda = \Lambda$.

Problem 6. *Prove relative commutator formulae for the group* $\mathrm{GU}(4, R, \Lambda)$*, provided* $\Lambda R + R\Lambda = \Lambda$.

Another important problem is the description of *subnormal* subgroups of $G(R)$. For the case of $\mathrm{GL}(n, R)$ this problem has a fully satisfactory answer, due to the works by John Wilson, Leonid Vaserstein, and others, see in particular [4, 33, 57, 61, 63].

For unitary groups, there are works by Gerhard Habdank, the fourth author, and You Hong, see, in particular, [14, 15, 71–74]. But there are still a number of loose ends.

Problem 7. *Give localization proofs for the description of subgroups of the unitary group* $\mathrm{GU}(2n, R, \Lambda)$*, normalised by the relative elementary subgroup* $\mathrm{EU}(\Phi, I, \Gamma)$*, for a form ideal* (I, Γ).

Problem 8. *Using relative localization, describe subgroups of a Chevalley group* $G(\Phi, R)$*, normalised by the relative elementary subgroup* $E(\Phi, R, I)$*, for an ideal* $I \trianglelefteq R$.

It would be extremely challenging to fully relativise results concerning nilpotent filtration.

Problem 9. *Let* R *be a ring of finite Bass–Serre dimension*

$$\delta(R) = d < \infty,$$

and let (I_i, Γ_i), $1 \leqslant i \leqslant m$, *be form ideals of* (R, Λ)*. Prove that for any* $m > d$ *one has*

$$[[\ldots[G(\Phi, R, I_1), G(\Phi, R, I_2)], \ldots], G(\Phi, R, I_m)] = [[\ldots[E(\Phi, R, I_1), E(\Phi, R, I_2)], \ldots], E(\Phi, R, I_m)].$$

Problem 10. *Let R be a ring of finite Bass–Serre dimension*

$$\delta(R) = d < \infty,$$

and let (I_i, Γ_i), $1 \leqslant i \leqslant m$, be form ideals of (R, Λ). Prove that for any $m > d$ one has

$$[[\dots[\mathrm{GU}(2n, I_1, \Gamma_1), \mathrm{GU}(2n, I_2, \Gamma_2)], \dots], \mathrm{GU}(2n, I_m, \Gamma_m)]$$
$$= [[\dots[\mathrm{EU}(2n, I_1, \Gamma_1), \mathrm{EU}(2n, I_2, \Gamma_2)], \dots], \mathrm{EU}(2n, I_m, \Gamma_m)].$$

The following two problems are in fact not individual clear cut problems, but rather huge research projects.

Problem 11. *Generalise results of the present paper to odd unitary groups.*

Problem 12. *Obtain results similar to those of the present paper for [groups of points of] isotropic reductive groups.*

In the first one of these settings there are foundational works by Victor Petrov [42–44], while in the second one there are papers by Victor Petrov, Anastasia Stavrova, and Alexander Luzgarev [45, 36], with versions of Quillen–Suslin lemma. But that's about it. Most of the conjugation calculus and the commutator calculus, including relative results, explicit estimates, etc., have to be developed from scratch.

References

1. E. Abe, *Whitehead groups of Chevalley groups over polynomial rings.* — Comm. Algebra **11**, No. 12 (1983), 1271–1308.
2. E. Abe, *Chevalley groups over commutative rings.* In: Proc. Conf. Radical Theory, Sendai (1988), pp. 1–23.
3. E. Abe, *Normal subgroups of Chevalley groups over commutative rings.* — Contemp. Math. **83** (1989), 1–17.
4. A. Bak, *Subgroups of the general linear group normalized by relative elementary groups.* — Lect. Notes Math., Springer, Berlin **967** (1982), 1–22.
5. A. Bak, *Non-Abelian* K*-theory: The nilpotent class of* K_1 *and general stability.* — K–Theory **4** (1991), 363–397.
6. A. Bak, R. Basu, and R. A. Rao, *Local-global principle for transvection groups.* — Proc. Amer. Math. Soc. (2010) (to appear).
7. A. Bak, R. Hazrat, N. Vavilov, *Localization-completion strikes again: relative* K_1 *is nilpotent by Abelian.* — J. Pure Appl. Algebra **213** (2009), 1075–1085.

8. A. Bak, N. Vavilov, *Normality for elementary subgroup functors.* — Math. Proc. Cambridge Phil. Soc. **118**, No. 1 (1995), 35–47.
9. A. Bak, N. Vavilov, *Structure of hyperbolic unitary groups.* I. *Elementary subgroups.* — Algebra Colloq. **7**, No. 2 (2000), 159–196.
10. R. Basu, R. A. Rao, R. Khanna, *On Quillen's local global principle.* — Contemp. Math. **390** (2005), 17–30.
11. R. K. Dennis, L. N. Vaserstein, *On a question of M. Newman on the number of commutators.* — J. Algebra **118** (1988), 150–161.
12. R. K. Dennis, L. N. Vaserstein, *Commutators in linear groups.* — K-theory **2** (1989), 761–767.
13. E. Ellers, N. Gordeev, *On the conjectures of J. Thompson and O. Ore.* — Trans. Amer. Math. Soc. **350** (1998), 3657–3671.
14. G. Habdank, *A classification of subgroups of* Λ*-quadratic groups normalized by relative elementary subgroups.* Dissertation Universität Bielefeld (1987), 1–71.
15. G. Habdank, *A classification of subgroups of* Λ*-quadratic groups normalized by relative elementary subgroups.* — Adv. Math. **110**, No. 2 (1995), 191–233.
16. A. J. Hahn, O. T. O'Meara, *The classical groups and* K*-theory.* Springer, Berlin et al., 1989.
17. R. Hazrat, *Dimension theory and nonstable* K_1 *of quadratic module.* — K-theory **27** (2002), 293–327.
18. R. Hazrat, *On K-theory of classical-like groups.* — Doktorarbeit Univ. Bielefeld (2002), 1–62.
19. R. Hazrat, V. Petrov, N. Vavilov, *Relative subgroups in Chevalley groups.* — J. K-theory **5** (2010), 603–618.
20. R. Hazrat, A. Stepanov, N. Vavilov, Zuhong Zhang, *New versions of localization method.* — Topology, Geometry, and Dynamics: Rokhlin Memorial, St.-Petersburg (2010), 114–116.
21. R. Hazrat, A. Stepanov, N. Vavilov, Zuhong Zhang, *On the length of commutators in unitary groups.* — (2010), 1–24.
22. R. Hazrat, N. Vavilov, K_1 *of Chevalley groups are nilpotent.* — J. Pure Appl. Algebra **179** (2003), 99–116.
23. R. Hazrat, N. Vavilov, *Bak's work on* K*-theory of rings* (*with an appendix by Max Karoubi*). — J. K-Theory **4**, No. 1 (2009), 1–65.
24. R. Hazrat, N. Vavilov, Zuhong Zhang, *Relative commutator calculus in unitary groups, and applications.* — J. Algebra (2010), 1–34.
25. R. Hazrat, N. Vavilov, Zuhong Zhang, *Relative commutator calculus in Chevalley groups, and applications.* — (2010), 1–28.
26. R. Hazrat, Zuhong Zhang, *Generalized commutator formula.* — Comm. Algebra (2010), 1–10.
27. W. van der Kallen, $\mathrm{SL}_3(\mathbb{C}[x])$ *does not have bounded word length.* — Springer Lect. Notes Math. **966** (1982), 357–361.
28. M.-A. Knus, *Quadratic and hermitian forms over rings.* — Springer Verlag, Berlin et al., 1991.
29. V. I. Kopeiko, *The stabilization of symplectic groups over a polynomial ring.* — Math. U.S.S.R. Sbornik **34** (1978), 655–669.

30. Tsit-Yuen Lam, *Serre's problem on projective modules.* Springer Verlag, Berlin et. al., 2006.
31. Li Fuan, *The structure of symplectic group over arbitrary commutative rings.* — Acta Math. Sinica **3**, No. 3 (1987), 247–255.
32. Li Fuan, *The structure of orthogonal groups over arbitrary commutative rings.* — Chinese Ann. Math. Ser. B **10** , No. 3 (1989), 341–350.
33. Li Fuan, Liu Mulan, *Generalized sandwich theorem.* — K-Theory **1** (1987), 171–184.
34. M. Liebeck, E. A. O'Brien, A. Shalev, Pham Huu Tiep, *The Ore conjecture.* — (2009), 1–81; http://www. math.auckland.ac.nz/obrien/research/ore.pdf.
35. A. Yu. Luzgarev, *Overgroups of $E(F_4, R)$ in $G(E_6, R)$.* — St.Petesburg J. Math. **20**, No. 5 (2008), 148–185.
36. A. Yu. Luzgarev, A. K. Stavrova, *Elementary subgroups of isotropic reductive groups are perfect.* — St.Petersburg Math. J. (2010), 1–11.
37. A. W. Mason, *A note on subgroups of* $\mathrm{GL}(n, A)$ *which are generated by commutators.* — J. London Math. Soc. **11** (1974), 509–512.
38. A. W. Mason, *On subgroups of* $\mathrm{GL}(n, A)$ *which are generated by commutators.* II. — J. reine angew. Math. **322** (1981), 118–135.
39. A. W. Mason, *A further note on subgroups of* $\mathrm{GL}(n, A)$ *which are generated by commutators.* — Arch. Math. **37**, No. 5 (1981), 401–405.
40. A. W. Mason, W. W. Stothers, *On subgroups of* $\mathrm{GL}(n, A)$ *which are generated by commutators.* — Invent. Math. **23** (1974), 327–346.
41. D. W. Morris, *Bounded generation of* $\mathrm{SL}(n, A)$. (after D. Carter, G. Keller, and E. Paige). — New York J. Math. **13** (2007), 383–421.
42. V. Petrov, *Overgroups of unitary groups.* — K-Theory **29** (2003), 147–174.
43. V. A. Petrov, *Odd unitary groups.* — J. Math. Sci. **130**, No. 3 (2003), 4752–4766.
44. V. A. Petrov, *Overgroups of classical groups.* — Doktorarbeit Univ. St.Petersburg (2005), 1–129.
45. V. A. Petrov, A. K. Stavrova, *Elementary subgroups of isotropic reductive groups.* — St.Petersburg Math. J. **20**, No. 3 (2008), 160–188.
46. A. Sivatski, A. Stepanov, *On the word length of commutators in* $\mathrm{GL}_n(R)$. — K-theory **17** (1999), 295–302.
47. A. Stepanov, *Universal localization in algebraic groups.* (2010); http://alexei.stepanov.spb.ru/ publicat.html.
48. A. Stepanov, N. Vavilov, *Decomposition of transvections: a theme with variations.* — K-Theory **19** (2000), 109–153.
49. A. Stepanov, N. Vavilov, *On the length of commutators in Chevalley groups.* — Israel J. Math. (2010), 1–20.
50. A. Stepanov, N. Vavilov, You Hong, *Overgroups of semi-simple subgroups via localisation-completion.* — (2010), 1–43.
51. A. A. Suslin, *The structure of the special linear group over polynomial rings.* — Math. USSR Izv. **11**, No. 2 (1977), 235–253.
52. A. A. Suslin, V. I. Kopeiko, *Quadratic modules and orthogonal groups over polynomial rings.* — J. Sov. Math. **20**, No. 6 (1982), 2665–2691.

53. G. Taddei, *Schémas de Chevalley–Demazure, fonctions représentatives et théorème de normalité.* — Thèse, Univ. de Genève (1985).

54. G. Taddei, *Normalité des groupes élémentaire dans les groupes de Chevalley sur un anneau.* — Contemp. Math. **55**, No. 2 (1986), 693–710.

55. M. S. Tulenbaev, *The Steinberg group of a polynomial ring.* — Math. USSR Sb. **45**, No. 1 (1983), 139–154.

56. L. Vaserstein, *On the normal subgroups of the* GL_n *of a ring.* — Algebraic K-Theory, Evanston (1980). Lect. Notes Math. **854**, Springer, Berlin et al. (1981), 454–465.

57. L. Vaserstein, *The subnormal structure of general linear groups.* — Math. Proc. Cambridge Phil. Soc. **99** (1986), 425–431.

58. L. Vaserstein, *On normal subgroups of Chevalley groups over commutative rings.* — Tôhoku Math. J. **36**, No. 5 (1986), 219–230.

59. L. Vaserstein, *Normal subgroups of orthogonal groups over commutative rings.* — Amer. J. Math. **110**, No. 5 (1988), 955–973.

60. L. Vaserstein, *Normal subgroups of symplectic groups over rings.* — K-Theory **2**, No. 5 (1989), 647–673.

61. L. Vaserstein, *The subnormal structure of general linear groups over rings.* — Math. Proc. Cambridge Phil. Soc. **108**, No. 2 (1990), 219–229.

62. L. Vaserstein, You Hong, *Normal subgroups of classical groups over rings.* — J. Pure Appl. Algebra **105**, No. 1 (1995), 93–106.

63. N. Vavilov, *A note on the subnormal structure of general linear groups.* — Math. Proc. Cambridge Phil. Soc. **107**, No. 2 (1990), 193–196.

64. N. Vavilov, *Structure of Chevalley groups over commutative rings.* — In: Proc. Conf. Nonassociative algebras and related topics (Hiroshima – (1990)). World Sci. Publ. London et al. (1991), pp. 219–335.

65. N. Vavilov, A. Luzgarev, A. Stepanov, *Calculations in exceptional groups over rings.* — J. Math. Sci. **373** (2009), 48–72.

66. N. A. Vavilov, V. A. Petrov, *Overgroups of* $\mathrm{Ep}(n, R)$. — St.Petersburg J. Math. **15**, No. 4 (2004), 515–543.

67. N. A. Vavilov, A. V. Stepanov, *Standard commutator formula.* — Vestnik St. Petersburg Univ., ser.1 **41**, No. 1 (2008), 5–8.

68. N. A. Vavilov, A. V. Stepanov, *Overgroups of semi-simple groups.* — Vestnik Samara State Univ., Ser. Nat. Sci. No. 3 (2008), 51–95.

69. N. A. Vavilov, A. V. Stepanov, *Standard commutator formulae, revisited.* — Vestnik St.Petersburg State Univ., ser.1 **43**, No. 1 (2010), 12–17.

70. N. A. Vavilov, A. V. Stepanov, *Linear groups over general rings* I. *Main Structure Theorems.* — Vestnik Samara State Univ., Ser. Nat. Sci. (2010), 1–87.

71. Hong You, *Subgroups of classical groups normalised by relative elementary groups.* — J. Pure Appl. Algebra (2010), 1–16.

72. Zuhong Zhang, *Lower K-theory of unitary groups.* — Doktorarbeit Univ. Belfast (2007), 1–67.

73. Zuhong Zhang, *Stable sandwich classification theorem for classical-like groups.* — Math. Proc. Cambridge Phil. Soc. **143**, No. 3 (2007), 607–619.

74. Zuhong Zhang, *Subnormal structure of nonstable unitary groups over rings.* — J. Pure Appl. Algebra **214** (2010), 622–628.

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