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# AN APPROACH TO MATRIX PROBLEMS

In this paper we try to give a general approach to the so-called "matrix problems". The purpose is to construct a transparent theory which would englobe the representation theories of three classes of objects interesting for the author: partially ordered sets, quivers and finite-dimensional algebras. The key point of our approach is the following: we consider the conjugacy classes of elements not in ideals themselves, but in *their cosets.* Due to this the "derivatives" of any matrix problem are again matrix problems of the same type.

I came to the definitions which are exposed below in April–June 2000 during my stay at the University of Bielefeld. My main purpose was to prove the second conjecture of Brauer–Thrall in these terms; I could not do it then, and I cannot do it now. That is why I never made these ideas available to others. But after the XIV International Conference on Representations of Algebras (ICRA XIV, Tokyo, 2010) I have got the impression that this approach could yet have some interest and I decided to make it public.

This paper does not contain any proof or reference. Essentially, it reproduces my e-mail letter to Claus Ringel dated 07.10.2010. In its turn, this letter was based on my ten-years-old text in which I made only minor changes, omitted routine proofs and added some comments.

## 1. Abstract matrix problems

Let $\mathfrak{M}$ be a small additive category, and let $\widetilde{X}(\mathfrak{M}) = \bigcup_{P\in \mathrm{Ob}(\mathfrak{M})} \operatorname{End} P$.

Define on $\widetilde{X}(\mathfrak{M})$ a relation $\sim$ in the following way: if $u \in \operatorname{End} P$, $v \in \operatorname{End} Q$, then we assume that $u \sim v$ if there exists an isomorphism $x : P \to Q$ such that $vx = xu$. It is obvious that $\sim$ is an equivalence on $\widetilde{X}(\mathfrak{M})$; denote by $X(\mathfrak{M})$ the set of equivalence classes. If $P_i \in \mathrm{Ob}(\mathfrak{M})$, $u_i \in \operatorname{End}(P_i)$ $(1 \leqslant i \leqslant 4)$, and $u_1 \sim u_2$, $u_3 \sim u_4$, then $u_1 \oplus u_3 \sim u_2 \oplus u_4$; hence the direct sum of endomorphisms define an operation on $X(\mathfrak{M})$,

*Key words and phrases*: matrix problem, representation of algebra, quiver.

which we call direct sum and denote $\oplus$ as well. The class 0 of the only endomorphism of the zero object of the category $\mathfrak{M}$ is the neutral element for this operation. Thus, $X(\mathfrak{M})$ is a monoid. We say that a class $x \in X(\mathfrak{M})$ is indecomposable if, for any decomposition $x = y \oplus z$, one of the summands $y, z$ is equal to 0.

**Remark.** The class of the zero endomorphism of a non-zero object of $\mathfrak{M}$ is not the zero element 0 of $X(\mathfrak{M})$!

Let now $\mathcal{P} = (\mathfrak{M}, \mathfrak{N}, F, \mathcal{C})$ be a quadruple consisting of additive categories $\mathfrak{M}$, $\mathfrak{N}$, an additive functor $F : \mathfrak{M} \to \mathfrak{N}$ and an endomorphism $\mathcal{C}$ of the identity functor $\mathbf{1} : \mathfrak{N} \to \mathfrak{N}$. The latter means that endomorphisms $\mathcal{C}_X \in \mathrm{End}(X)$ are given for all objects $X$ of the category $\mathfrak{N}$, such that $\mathcal{C}_Y x = x\mathcal{C}_X$ for all morphisms $x : X \to Y$ of the category $\mathfrak{N}$.

**Lemma 1.** *Let $P, Q \in \mathrm{Ob}(\mathfrak{M})$, $u \in \mathrm{End}(P)$, $v \in \mathrm{End}(Q)$.*
(1) *If $u \sim v$ and $F(u) = \mathcal{C}_{F(P)}$, then $F(v) = \mathcal{C}_{F(Q)}$.*
(2) *If $F(u) = \mathcal{C}_{F(P)}$, $F(v) = \mathcal{C}_{F(Q)}$, then $F(u \oplus v) = \mathcal{C}_{F(P \oplus Q)}$.*

It follows from Lemma that the set of endomorphisms $u$ of objects $P$ of the category $\mathfrak{M}$, such that $F(u) = \mathcal{C}_{F(P)}$, is the union of equivalence classes, and that the set $X(\mathcal{P})$ of equivalence classes of all such endomorphisms is a submonoid of the monoid $\widetilde{X}(\mathcal{P})$.

We can now formulate our "matrix problem":

*Describe the monoid $X(\mathcal{P})$ of equivalence classes of endomorphisms $u$ of objects $P$ of the category $\mathfrak{M}$, such that $F(u) = \mathcal{C}_{F(P)}$.*

For shortness we shall denote this problem by the same symbol $\mathcal{P}$ as the data $\mathcal{P} = (\mathfrak{M}, \mathfrak{N}, F, \mathcal{C})$ of the problem. Homomorphisms $\varphi : \mathcal{P} \to \mathcal{P}'$ of matrix problems are defined in a natural way.

## 2. Matrix problem over a field

Let now $k$ be an algebraically closed field. From this moment we consider abstract matrix problems only for such $\mathcal{P} = (\mathfrak{M}, \mathfrak{N}, F, \mathcal{C})$ that $\mathfrak{M}$ and $\mathfrak{N}$ are the categories of right projective finitely generated modules over a finite-dimensional $k$-algebra and over a quotient algebra of this algebra, and $F : \mathfrak{M} \to \mathfrak{N}$ is the natural functor. For any object $P$ of the category $\mathfrak{M}$ denote by $E_P$ the left endomorphism ring $\mathrm{End}_{\mathfrak{M}}(P)$ and by $C_P \subset E_P$ the set of all elements $u \in E_P$ such that $F(u) = \mathcal{C}_{F(P)}$.

**Lemma 2.** *The set $C_P - C_P = \{x - y \mid x, y \in C_P\}$ is a two-sided ideal of $E_P$, and $[E_P, C_P] = \{ux - xu \mid u \in E_P, x \in C_P\} \subseteq C_P - C_P$.*

In other words, for any element $c \in C_P$ the set $C_P$ is the coset $c + I_P$ of the ideal $I_P = C_P - C_P$ in $E_P$, and the class of $c$ modulo $I_P$ is a central element of the ring $E_P/I_P$.

**Proposition 1.** *Let $E$ be a finite-dimensional algebra over $k$ and let $C$ be a subset of $E$ such that $C - C$ is a two-sided ideal of $E$ and $[E, C] \subseteq (C - C)$. Then there exists the data $\mathcal{P} = (\mathfrak{M}, \mathfrak{N}, F, \mathcal{C})$ of a matrix problem and a generator $P$ of the category $\mathfrak{M}$, such that $E = E_P$, $C = C_P$. This problem is unique up to isomorphism.*

**Proof (hint).** $\mathfrak{M}$ and $\mathfrak{N}$ are the categories of right projective finitely generated $E$-modules and $E/I$-modules, where $I = C - C$, and $P = E_E$. □

So, our matrix problem $\mathcal{P}$ is defined not only by the data $(\mathfrak{M}, \mathfrak{N}, F, \mathcal{C})$, but also by the data $(E, C)$.

If $E'$ is an algebra which is Morita-equivalent to the algebra $E$, then for a certain subset $C' \subset E'$ the matrix problem $(E', C')$ is the same as the matrix problem $(E, C)$. But if we require that $E$ was a basic algebra (i.e., if the quotient ring of $E$ over the radical is the direct sum of several copies of $k$) then the pair $(E, C)$ is uniquely determined by the matrix problem.

**Remark.** Representations of quivers, posets and finite-dimensional algebras are matrix problems. For example, any representation of a $k$-algebra $\Lambda$ is the cokernel of an epimorphism $\Lambda^m \to \Lambda^n$, therefore it corresponds to a class of homomorphisms of free $\Lambda$-modules. Thus, the problem of classification of $\Lambda$-modules is the matrix problem $(E, C)$ with

$$E = \begin{pmatrix} \Lambda & \Lambda \\ 0 & \Lambda \end{pmatrix}, \quad C = \begin{pmatrix} 0 & \Lambda \\ 0 & 0 \end{pmatrix}$$

(here the coset $C$ is trivial, i.e. it coincides with the ideal $I = C - C$).

## 3. Epimorphisms for matrix problems

Let $\mathcal{P} = (E, C)$, $\mathcal{P}' = (E', C')$ be matrix problems over a field $k$. We say that an epimorphism $\varphi : E \to E'$ is an epimorphism of the matrix problem $\mathcal{P}$ onto the matrix problem $\mathcal{P}'$ if $\varphi(C) = C'$. Since the corresponding categories $\mathfrak{M}$, $\mathfrak{M}'$ are categories of right projective finitely generated $E$-

and $E'$-modules, the epimorphism $\varphi$ induces functor $\varphi : \mathfrak{M} \to \mathfrak{M}'$. For any object $P$ of the category $\mathfrak{M}$ the epimorphism $\varphi$ uniquely determines the epimorphism $\varphi_P : E_P \to E'_{\varphi(P)}$; denote $C'_{\varphi(P)} = \varphi_P(C_P)$. It is clear that if $P$ is a generator of $\mathfrak{M}$, then the matrix problem $(E'_{\varphi(P)}, C'_{\varphi(P)})$ is the same as the matrix problem $\mathcal{P}' = (E', C')$.

The epimorphism $\varphi$ induces an epimorphism of the monoid $X(\mathcal{P})$ onto the monoid $X(\mathcal{P}')$, and consequently the problem $\mathcal{P}$ is "more difficult" than the problem $\mathcal{P}'$.

## 4. Elementary matrix problems. Differentiation

Let $(E, C)$ be a matrix problem; we can choose the data of this problem so that $E$ was a basic ring. It is the endomorphism ring of $P = P_1 \oplus \ldots \oplus P_n$, where $P_1, \ldots, P_n$ are all pairwise non-isomorphic indecomposable objects of the category $\mathfrak{M}$. Denote by $I$ the ideal $C - C$ of $E$ and by $e_i \in I$ $(1 \leqslant i \leqslant n)$ the primitive idempotent of the ring $E = \mathrm{End}(P)$ which acts as identity on $P_i$ and kills all other $P_j$, $j \neq i$.

We call a matrix problem $(E, C)$ *elementary* if $\dim_k I = 1$. For an epimorphism $\varphi : E \to E'$ of our problem onto an elementary matrix problem $(E', C')$ we shall construct a derivative problem $\partial^{\varphi}\mathcal{P}$ which is "one step easier" than the initial one. Observe first of all that ideal $I' = C' - C' = \varphi(I)$ is a one-dimensional two-sided ideal of $E'$; it is irreducible both as a left and as a right $E'$-module and consequently as a left and a right $E$-module. Therefore $I' = \varphi(e_i)I'\varphi(e_j)$ for some $i, j$ $(1 \leqslant i, j \leqslant n)$.

We distinguish three cases.

(i) $[E', C'] = I'$.

Then $\partial^{\varphi}\mathcal{P} = (E^{\varphi}, C^{\varphi})$, where $E^{\varphi}$ is the subring of $E$ consisting of all elements $x \in E$ such that $\varphi(xc - cx) = 0$ for all $c \in C$, and $C^{\varphi} = C \cap E^{\varphi}$.

(ii) $[E', C'] = 0$, $i \neq j$.

Every object of the category $\mathfrak{M}$ is the direct sum of indecomposable objects each of which is one of the objects $P_1, \ldots, P_n$; let $P \in \mathrm{Ob}(\mathfrak{M})$ be the object in which $P_i$ and $P_j$ enter two times and all other $P_s$ enter exactly once. Denote by $e_{i1}, e_{i2}, e_{j1}, e_{j2}, e_s \in E_P$ the projections onto the corresponding direct summands of $P$ and by $e_i, e_j$ the sums $e_{i1} + e_{i2}$, $e_{j1} + e_{j2}$. The vector space $\varphi_P(e_i I_P e_j)$ is the direct sum of four one-dimensional space $\varphi_P(e_{i1} I_P e_{j1})$, $\varphi_P(e_{i1} I_P e_{j2})$, $\varphi_P(e_{i2} I_P e_{j1})$, $\varphi_P(e_{i2} I_P e_{j2})$. Let $c'_1$ be any non-zero element of $\varphi_P(e_{i1} I_P e_{j1})$; then $\partial^{\varphi}\mathcal{P} = (E^{\varphi}, C^{\varphi})$, where $E^{\varphi}$ is the subring of $E_P$ consisting of all elements $x \in E_P$ such that $\varphi_P(x)c_1' - c_1'\varphi_P(x) = 0$, and $C^\varphi = C_P \cap E^\varphi$.

Describe $C^\varphi$ in slightly different terms. It is easy to prove that there exists a unique element $c' \in C'_{\varphi(P)}$ which is contained in the center of the ring $E'_{\varphi(P)}$; then $C^\varphi = \varphi_P^{-1}(c' + c_1')$.

(iii) $[E', C'] = 0,\ i = j$.

This case is more complicated: to remain in the frame of finite-dimensional algebras we must restrict us to partial differentiations with additional data. Let $h$ be a positive integer and $S$ a finite subset of $k$. We shall define the matrix problem $\partial^\varphi_{h,S}\mathcal{P}$; the complete derivative problem $\partial^\varphi\mathcal{P}$ should be "the union" of all such $\partial^\varphi_{h;S}\mathcal{P}$. For an element $\alpha \in k$ and an integer $r$ denote by $J_r(\alpha)$ the Jordan block of dimension $r$ with the eigenvalue $\alpha$. Let $m$ be the number of elements of $S$, $M = mh(h+1)/2$, and let $P \in \mathrm{Ob}(\mathfrak{M})$ be the object in which $P_i$ enters $M$ times and all other $P_s$ enter exactly once. The dimension of the vector space $\varphi_P(e_i I_P e_i)$ is equal to $M^2$ and this vector space can be naturally identified with the ring $k_M$ of $M \times M$-matrices. Let $c_1$ be the element of $\varphi_P(e_i I_P e_i)$ which corresponds to the Jordan matrix with diagonal boxes $J_r(\alpha)$, where $r$ runs from 1 to $h$ and $\alpha$ runs through the set $S$. Then $\partial^\varphi_{h,S}\mathcal{P} = (E^\varphi_{h,S}, C^\varphi_{h,S})$, where $E^\varphi_{h,S}$ is the subring of $E_P$ consisting of all elements $x \in E_P$ such that $\varphi_P(x)c_1 - c_1\varphi_P(x) = 0$, and $C^\varphi_{h,S} = C_P \cap E^\varphi_{h,S}$.

## 5. Quasi-finite, tame and wild matrix problems

Let $(E, C)$ be a matrix problem, where $E$ is a basic ring. As above, $I = C - C$; denote by $R$ the radical of $E$. We call this problem *etalon-wild*, if $[R, C] = IR = RI = 0$, $\dim_k I \geqslant 2$ and there is a primitive idempotent $e_i$ of the ring $E$ such that $I = e_i I e_i$.

**Lemma 3.** *For any matrix problem $(E, C)$ either $I = C - C = 0$ or there exists an epimorphism of this problem onto an elementary or an etalon-wild matrix problem.*

Thus, any matrix problem either admits a differentiation (and so it can be reduced to "simpler" problem), or there is an epimorphism of this problem onto an etalon-wild problem, and consequently it is "too complicated". We call a matrix problem $\mathcal{P}$ *wild* if there exists a chain of problems $\mathcal{P} = \mathcal{P}_1, \mathcal{P}_2, \dots, \mathcal{P}_N$, each of which is obtained from the precedent by differentiation, such that there is an epimorphism of the problem $\mathcal{P}_N$ onto an etalon-wild problem. Otherwise we call the problem $\mathcal{P}$ tame. A tame problem is called quasi-finite if it and all its consecutive derivatives admit differentiations only of types (i) and (ii).

Thus the alternative tame – wild holds by definition, and it is easy to understand that our notion of tameness coincides with the usual one. Indeed, the objects obtained by differentiation of type (iii) which correspond to almost all different eigenvalues interact with other objects in the same way, and if at least five of them cannot be isolated as direct summands then the problem is wild.

The dimension of an element of $X(\mathcal{P})$ is defined in a natural way: if $\mathfrak{u} \in X(\mathcal{P})$ is the equivalence class of an endomorphism $u$ of the object $P = P_1^{s_1} \oplus \ldots \oplus P_n^{s_n}$, then $\dim \mathfrak{u} = s_1 + \ldots + s_n$. The analogue of the first conjecture of Brauer–Thrall claims that if there are infinitely many indecomposable elements in $X(\mathcal{P})$, then their dimensions are unbounded. But with our definition it is almost trivial: if the problem is not quasi-finite (i.e., it is wild or properly tame), then it is true because the dimensions of Jordan boxes are unbounded, and for the quasi-finite case it follows from the fact that for any $N$ the number of chains of the length $N$ of differentiations of types (i) and (ii) is finite.

Ten years ago my purpose was to find a conceptual proof of the analogue of the second conjecture of Brauer–Thrall, which states that any quasi-finite problem $\mathcal{P}$ is in fact finite, i.e., there are in $X(\mathcal{P})$ only finitely many indecomposable elements. Unfortunately, even now I cannot do that.

## 6. Accessible elements

Let $\mathcal{P} = (E, C)$ be a quasi-finite matrix problem. After differentiation of type (ii) we obtain algebra $E_1$ which instead of $e_i$, $e_j$ has 3 idempotents – what remains of $e_i$ and $e_j$ and the new idempotent which we shall denote $e_i \circ e_j$. Observe that this product does not always exist. After the following differentiation we obtain algebra $E_2$ which instead of $e_i$, $e_j$, $e_i \circ e_j$, $e_p$ has 5 idempotents – what remains of $e_i$, $e_j$, $e_p$, $e_i \circ e_j$ and the new idempotent $(e_i \circ e_j) \circ e_p$. We can do these differentiations in different order. If $e_j \circ e_p$ exists and $E_1'$ is the corresponding algebra, then we can do another differentiation and obtain the algebra $E_2'$ with the idempotent $e_i \circ (e_j \circ e_p)$. It is easy to prove that the rings $E_2$ и $E_2'$ are canonically isomorphic an we can identify them. After this identification the idempotents $(e_i \circ e_j) \circ e_p$ and $e_i \circ (e_j \circ e_p)$ become equal:

$$(e_i \circ e_j) \circ e_p = e_i \circ (e_j \circ e_p).$$

Similarly, if the products $e_i \circ e_j$, $(e_i \circ e_j) \circ e_p$ exist, but $e_j \circ e_p$ does not exist, then

$$(e_i \circ e_j) \circ e_p = (e_i \circ e_p) \circ e_j.$$

Using the above identities which are surrogates of associativity we can prove the following

**Proposition 2.** *Let $\mathcal{P} = (E, C)$ be a quasi-finite matrix problem, and let $e_1, \dots, e_n$ be all pairwise non-isomorphic idempotents of $E$. Any indecomposable element of $X(\mathcal{P})$ corresponds to an idempotent $e$ of an algebra $\bar{E}$ obtained by a sequence of differentiations, and there exist chains of idempotents $d_0, d_1, \dots, d_N$ and indices $1 \leqslant i_0, i_1, \dots i_N \leqslant n$, such that*

$$d_0 = e_{i_0}, \quad d_{s+1} = d_s \circ e_{i_s} \text{ or } d_{s+1} = e_{i_s} \circ d_s \ (0 \leqslant s < N), \quad d_N = e.$$

This Proposition states in fact that all indecomposable elements are "accessible". In the case of representations of algebras of quasi-finite type it means that for any indecomposable module $A$ there exists a chain of indecomposable modules $0 = A_0, A_1, \dots A_N = A$, such that $\dim A_s = s$ and for each $0 \leqslant s < N$ the module $A_s$ is a submodule or a factor module of the module $A_{s+1}$.

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A new general approach to the so-called "matrix problems" is given. With this approach the "derivative" of a matrix problem is again a matrix problem of the same type.

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