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BIG AND SMALL ELEMENTS IN CHEVALLEY GROUPS

Abstract. Let $\widetilde{G}$ be a reductive algebraic group which is defined and split over a field K. Here we consider the Zariski open subset $\mathfrak{B}$ of the group $\widetilde{G}$ which consists of elements such that their conjugacy classes intersect the Big Bruhat Cell. In particular, we give a description of the set $\mathfrak{B}(K)$ in the case $\widetilde{G} = \mathrm{GL}_n, \mathrm{SL}_n$.

1. Introduction

Let $\widetilde{G}$ be a reductive algebraic group that is defined and split over a field K and let $\widetilde{B}$ be a fixed Borel subgroup of $\widetilde{G}$ that is defined over K. Further, let $G = \widetilde{G}(K)$ and $B = \widetilde{B}(K)$. The groups $\widetilde{G}$ and G have Bruhat decompositions

$$\widetilde{G} = \bigcup_{w\in W} \widetilde{B}\dot{w}\widetilde{B}, \quad G = \bigcup_{w\in W} B\dot{w}B,$$

where W is the Weyl group corresponding to $\widetilde{G}$ and $\dot{w}$ is a preimage of $w \in W$ in the normalizer of a fixed maximal torus of $\widetilde{B}$ (we assume $\dot{w} \in G$). The question "when does a given conjugacy class of $\widetilde{G}$ (respectively, G) intersects a given Bruhat cell $\widetilde{B}\dot{w}\widetilde{B}$ (respectively, $B\dot{w}B$)?" is investigated, in particular, in [4-6, 8–10, 14–16]. The complete solution of this problem seems to be very complicated. Here we are interested in the following part of the question "when is $\widetilde{C} \cap \widetilde{B}\dot{w}_0\widetilde{B} \neq \varnothing$ (respectively, $C \cap B\dot{w}_0B \neq \varnothing$), where $\widetilde{C}$ (respectively, C) is a conjugacy class of $\widetilde{G}$ (respectively, G) and w_0 is the longest element of the Weyl group?" that is, "when does a conjugacy class of a Chevalley group intersect the big Bruhat cell?".

Key words and phrases: reductive algebraic group, Chevalley group, conjugacy class, Big Bruhat Cell.

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However, even this particular question seems to be difficult to answer. Here we give an answer only for the cases $G = \mathrm{GL}_n(K), \mathrm{SL}_n(K)$. Namely, the conjugacy class C_g of an element $g \in \mathrm{GL}_n(K)$ (respectively, $g \in \mathrm{SL}_n(K)$) intersects the big Bruhat cell of $\mathrm{GL}_n(K)$ (respectively, $\mathrm{SL}_n(K)$) if and only if

$$\mathrm{rank}\,(g - \alpha E_n) \geq [\frac{n}{2}] \quad \text{for every} \quad \alpha \in K^*; \tag{$*$}$$

(here E_n is the identity matrix of $GL_n(K)$ and $[x] = \max\{m \in \mathbb{N} \mid m \leq x\}$). For an algebraically closed field K this result was obtained in [4]. It is easy to extend this result to the case where K is an infinite field (see, Theorem 2.3, below). However for finite fields such extension cannot be obtained by the same arguments.

Here we give a proof of $(*)$ which holds for all fields.

The proof is based on the following construction. Let Φ be a simple root system corresponding to $\widetilde{G}$ and let $w_\alpha, w \in W$, where w_α is the reflection that corresponds to the root $\alpha \in \Phi$. Further, let $w' = w_\alpha w w_\alpha$. We say that there is *a short descent* $w \to w'$ if $l(w') \leq l(w)$; (here $l(w)$ is the length of w with respect to the set of basic reflections $\{w_\alpha \mid \alpha \in \Phi\}$). A *descent* $w \to w'$ is a sequence of short descents $w \to w_1 \to \cdots \to w_n = w'$. We say that a short descent $w \to w'$ is *strict* if $l(w') < l(w)$. In the latter case, we have two *jumps* $w \rightsquigarrow w_\alpha w,\ w \rightsquigarrow w w_\alpha$. We say that there is *a way* $w \mapsto w'$, where $w' \in W$, if there is a sequence $w_1, \dots, w_m \in W_n$ such that $w_1 = w$, $w_m = w'$, and for every pair w_i, w_{i+1} there is a descent $w_i \to w_{i+1}$ or a jump $w_i \rightsquigarrow w_{i+1}$. If $w \mapsto w'$ is a way, then for a conjugacy class C of G

$$C \cap B\dot{w}'B \neq \varnothing \Rightarrow C \cap B\dot{w}B \neq \varnothing$$

(see [6, Propositions 2.2 and 2.10]; note, that in [6] we considered only jumps of the form $w \rightsquigarrow w w_\alpha$, but Proposition 2.2 in [6] shows that we also may consider the jumps $w \rightsquigarrow w_\alpha w$). Thus, to show for a conjugacy class C of G (with condition $(*)$) that $C \cap B\dot{w}_0 B \neq \varnothing$, we construct a way $w_0 \mapsto w$ to an appropriate element of W such that $C \cap B\dot{w}B \neq \varnothing$. This gives us the sufficiency of $(*)$. The necessity of $(*)$ follows from a simple observation on matrices belonging to $\dot{w}_0 B$.

The problem of describing the elements whose conjugacy classes intersect the big Bruhat cell can be reformulated as follows. For an element $g \in \widetilde{G}$ put

$$\mathfrak{B}_g = g(\widetilde{B}\dot{w}_0\widetilde{B})g^{-1} \quad \text{and} \quad \widehat{\mathfrak{B}}_g = \widetilde{G} \setminus \mathfrak{B}_g.$$

We define the sets

$$\mathfrak{B} = \bigcup_{g\in\widetilde{G}} g(\widetilde{B}\dot{w}_0\widetilde{B})g^{-1} \quad \text{and} \quad \widehat{\mathfrak{B}} = \widetilde{G}\setminus\mathfrak{B} = \bigcap_{g\in\widetilde{G}} \widehat{\mathfrak{B}}_g,$$

which we call the *set of big elements* and the *set of small elements* of $\widetilde{G}$, respectively. The set $\mathfrak{B}$ is an open subset of $\widetilde{G}$; it consists of the elements $g\in\widetilde{G}$ such that the conjugacy class C_g of g has a nonempty intersection with the big Bruhat cell $\widetilde{B}\dot{w}_0\widetilde{B}$, and the set $\widehat{\mathfrak{B}}$ is the closed subset of $\widetilde{G}$ that consists of the elements whose conjugacy classes have no intersection with the big cell. We also define an open and a closed subset of $\widetilde{G}$

$$\mathfrak{B}_K = \bigcup_{g\in G} g(\widetilde{B}\dot{w}_0\widetilde{B})g^{-1} \text{ and } \widehat{\mathfrak{B}}_K = \widetilde{G}\setminus\mathfrak{B}_K = \bigcap_{g\in G} \widehat{\mathfrak{B}}_g,$$

which we call the *set of K-big elements* and the *set of K-small elements* of $\widetilde{G}$, respectively. We show that the closed subsets $\widehat{\mathfrak{B}}$ and $\widehat{\mathfrak{B}}_K$ are defined over K, and if K is an infinite field, $\widehat{\mathfrak{B}} = \widehat{\mathfrak{B}}_K$. This implies, in particular, if K is an infinite field and $x\in G$, then

$$gxg^{-1}\in\widetilde{B}\dot{w}_0\widetilde{B} \quad \text{for some} \quad g\in\widetilde{G} \Leftrightarrow gxg^{-1}\in B\dot{w}_0B \quad \text{for some} \quad g\in G.$$

We also describe the closed set $\widehat{\mathfrak{B}}_K$ for $\widetilde{G} = \mathrm{GL}_n, \mathrm{SL}_n, \mathrm{Sp}_4$.

Throughout the paper we use the notation that we established in the Introduction.

We identify the group $\widetilde{G}$ with the group of points $\widetilde{G}(\mathfrak{K})$ for some algebraically closed field $\mathfrak{K}\supset K$; all fields considered below are assumed to be subfields of $\mathfrak{K}$.

Further, $\overline{F}$ is the algebraic closure of a field F;

$\overline{Y}$ is the Zariski closure of a subset $Y\subset X$ of an algebraic variety X;

e is the identity of G;

E_n is the identity matrix in GL_n;

$\mathbf{0}_{k\times m}$ is the zero $k\times m$-matrix;

$C_\Gamma(x)$ is the centralizer of an element x in the group Γ;

F_p is the field consisting of p elements, where p is a prime.

2. The sets $\mathfrak{B}$, $\mathfrak{B}_K$, $\widehat{\mathfrak{B}}$, $\widehat{\mathfrak{B}}_K$

Proposition 2.1. *For $g \in G$ the closed subset $\widehat{\mathfrak{B}}_g$ of $\widetilde{G}$ is defined over K. Moreover,*

$$\widehat{\mathfrak{B}}_g(K) = \bigcup_{w \neq w_0} g(B\dot{w}B)g^{-1}.$$

Proof. Since the map $x \to gxg^{-1}$ is an isomorphism of the affine variety $\widetilde{G}$ onto itself that is defined over K, it suffices to deal with the case $g = e$. Consider the closed subset

$$\widehat{\mathfrak{B}}_e = \widetilde{G} \setminus \mathfrak{B}_e = \bigcup_{w \neq w_0} \widetilde{B}\dot{w}\widetilde{B}$$

of $\widetilde{G}$ (we assume $\dot{w} \in G$). For every extension F/K we have

$$\bigcup_{w \neq w_0} \widetilde{B}(F)\dot{w}\widetilde{B}(F) \subset \widehat{\mathfrak{B}}_e \cap \widetilde{G}(F), \quad \widetilde{B}(F)\dot{w}_0\widetilde{B}(F) \subset \mathfrak{B}_e \cap \widetilde{G}(F), \tag{2.1}$$

$$\widetilde{G}(F) = \left(\bigcup_{w \neq w_0} \widetilde{B}(F)\dot{w}\widetilde{B}(F) \right) \cup (\widetilde{B}(F)\dot{w}_0\widetilde{B}(F)). \tag{2.2}$$

From (2.1) and (2.2),

$$\widehat{\mathfrak{B}}_e \cap \widetilde{G}(F) = \bigcup_{w \neq w_0} \widetilde{B}(F)\dot{w}\widetilde{B}(F), \quad \mathfrak{B}_e \cap \widetilde{G}(F) = \widetilde{B}(F)\dot{w}_0\widetilde{B}(F). \tag{2.3}$$

Let F be an infinite field. Since $\widetilde{G}$ is a split group, the group $\widetilde{B}$ is a connected, split, solvable group, thus the group $\widetilde{B}$ is a unirational variety (see [12, Theorem 14.3.8]) and therefore the set $\widetilde{B}(F)$ is dense in $\widetilde{B}$ ([12, 13.2.6]). Thus, $\overline{\widetilde{B}(F)} = \widetilde{B}$ and, by (2.3),

$$\overline{\widehat{\mathfrak{B}}_e \cap \widetilde{G}(F)} = (\overline{\bigcup_{w \neq w_0} \widetilde{B}(F)\dot{w}\widetilde{B}(F)}) \supset (\bigcup_{w \neq w_0} \overline{\widetilde{B}(F)}\dot{w}\overline{\widetilde{B}(F)}) = \widehat{\mathfrak{B}}_e. \tag{2.4}$$

Thus, if K is an infinite field we may put $F = K$ and get a dense subset $\widehat{\mathfrak{B}}_e \cap \widetilde{G}(K)$ in $\widehat{\mathfrak{B}}_e$ (this follows from (2.4)) and therefore the closed set $\widehat{\mathfrak{B}}_e$ is defined over K ([12, 11.2.4, ii]). Now let K be a finite field and put $F = \overline{K}$. Again (2.4) implies that $\widehat{\mathfrak{B}}_e$ is defined over $\overline{K}$ and $\mathfrak{B}_e(\overline{K})$ is a dense subset of $\widehat{\mathfrak{B}}_e$. Also, the set $\widehat{\mathfrak{B}}_e(\overline{K})$ is $\mathrm{Gal}(\overline{K}/K)$-stable. Hence, $\widehat{\mathfrak{B}}_e$ is K-defined (see [12, 11.2.8]).

The second assertion of the proposition follows from (2.3). □

Proposition 2.2. *The closed subsets $\widehat{\mathfrak{B}}$, $\widehat{\mathfrak{B}}_K$ of $\widetilde{G}$ are defined over K.*

Proof. Let char $K = p \neq 0$. Since $\widetilde{G}$ is split over K we may assume that $\widetilde{G}$ is defined and split over the prime field F_p. For the algebraically closed field $\mathfrak{K}$ the map $\gamma : \mathfrak{K} \to \mathfrak{K}$ given by the formula $\gamma(a) = a^p$ is an automorphism of $\mathfrak{K}$. If $\Gamma = \langle\gamma\rangle$, then $\mathfrak{K}^\Gamma = F_p$.

Now we assume that $\widetilde{G}$ is a closed subset of $\mathrm{GL}_n(\mathfrak{K})$ and the corresponding embedding $i : \widetilde{G} \hookrightarrow \mathrm{GL}_n(\mathfrak{K})$ is an F_p-defined morphism.

Let $F_p[\mathrm{GL}_n]$ be the coordinate ring of the F_p-group GL_n. The automorphism

$$l \otimes f \to \gamma(l) \otimes f$$

of $\mathfrak{K}[\mathrm{GL}_n] = \mathfrak{K} \otimes_{F_p} F_p[\mathrm{GL}_n]$, where $l \in \mathfrak{K}$ and $f \in F_p[\mathrm{GL}_n]$, will also be denoted by γ. Thus, the group $\Gamma = \langle\gamma\rangle$ acts on $\mathfrak{K}[\mathrm{GL}_n]$. Consider the map

$$\widetilde{\gamma} : \mathrm{GL}_n(\mathfrak{K}) \to \mathrm{GL}_n(\mathfrak{K})$$

such that $\widetilde{\gamma}(\{a_{ij}\}) = \{a_{ij}^p\}$. Since the group $\widetilde{G}$ is F_p-defined,

$$\widetilde{\gamma}(\widetilde{G}) = \widetilde{G}, \quad \widetilde{\gamma}(G) \subset G.$$

Let

$$I_g = \{f \in \mathfrak{K}[\mathrm{GL}_n] \quad | \quad f_{|\widehat{\mathfrak{B}}_g} \equiv 0\}$$

be the ideal of functions vanishing on $\widehat{\mathfrak{B}}_g$. If $g = e$ this ideal is generated by polynomials with coefficients in F_p. Hence the ideal I_g of functions vanishing on $\widehat{\mathfrak{B}}_g = g\widehat{\mathfrak{B}}_e g^{-1}$ is generated by polynomials whose coefficients are rational functions of entries in the matrix $i(g) \in \mathrm{GL}_n(\mathfrak{K})$. Now $\gamma(I_g) = I_{\widetilde{\gamma}(g)}$ and $\widetilde{\gamma}(g) \in \widetilde{G}$ (respectively, $\widetilde{\gamma}(g) \in G$ if $g \in G$). Hence, the ideal $I = \sum\limits_{g\in\widetilde{G}} I_g$ (respectively, $I_K = \sum\limits_{g\in G} I_g$) is Γ-invariant, and, therefore, the ideal I (respectively, I_K) is generated as a vector subspace of $\mathfrak{K}[\mathrm{GL}_n]$ by elements from $F_p[\mathrm{GL}_n]$, because $\mathfrak{K}^\Gamma = F_p$ (see [12, 11.1.4]). Since $\widehat{\mathfrak{B}} = V(I)$ (respectively, $\widehat{\mathfrak{B}}_K = V(I_K)$) and since F_p is a perfect field, the set $\widehat{\mathfrak{B}}$ (respectively, $\widehat{\mathfrak{B}}_K$) is defined over F_p (see [7, 34.1]) and therefore it is defined over K.

Let char $K = 0$. Then K is a perfect field and therefore the intersection of K-defined closed sets $\widehat{\mathfrak{B}}_g = \bigcup_{w\neq w_0} g(B\dot{w}B)g^{-1}$, $g \in G$, (Proposition 2.1) is also K-defined (see [12, 11.2.13]). Thus, the closed set $\widehat{\mathfrak{B}}_K$ is K-defined.

Further, since $\widehat{\mathfrak{B}}_K$ is K-defined, the set $\widehat{\mathfrak{B}}_K(\overline{K})$ is dense in $\widehat{\mathfrak{B}}_K$.

Now we show the implication

$$x \in \widehat{\mathfrak{B}}_K(\overline{K}) \Rightarrow x \in \widehat{\mathfrak{B}} \cap \widetilde{G}(\overline{K}). \tag{2.5}$$

Suppose $x \notin \widehat{\mathfrak{B}} \cap \widetilde{G}(\overline{K})$. Then $x \in \mathfrak{B} \cap \widetilde{G}(\overline{K})$ and therefore $gxg^{-1} \in \widetilde{B}\dot{w}_0\widetilde{B}$ for some $g \in \widetilde{G}$ (by the definition of $\mathfrak{B}$). Hence the conjugacy class $\widetilde{C}_x$ of the element x in $\widetilde{G}$ has a nontrivial intersection U_x with the open subset $\widetilde{B}\dot{w}_0\widetilde{B}$, and therefore, the set U_x contains an open subset of the closure of $\widetilde{C}_x$. Hence the subset U_x of the conjugacy class $\widetilde{C}_x$ has a nontrivial intersection with any dense subset of $\widetilde{C}_x$. But the set $V_x = \{g^{-1}xg \mid g \in G\}$ is dense in $\widetilde{C}_x = \{g^{-1}xg \mid g \in \widetilde{G}\}$, because K is an infinite field and, therefore, G is dense in $\widetilde{G}$ (see [1, 18.3]). Thus $U_x \cap V_x \neq \varnothing$. If $g^{-1}xg \in U_x \cap V_x$, then $x \in g\widetilde{B}\dot{w}_0\widetilde{B}g^{-1}$, where $g \in G$. Hence $x \in \mathfrak{B}_K$ and therefore $x \notin \widehat{\mathfrak{B}}_K$, which contradicts our assumption. This confirms (2.5).

Since $\widehat{\mathfrak{B}} \subset \widehat{\mathfrak{B}}_K$, the implication (2.5) yields

$$\widehat{\mathfrak{B}}_K(\overline{K}) = \widehat{\mathfrak{B}} \cap \widetilde{G}(\overline{K}). \tag{2.6}$$

The set $\widehat{\mathfrak{B}} \cap \widetilde{G}(\overline{K})$ is dense in $\widehat{\mathfrak{B}}$ (this follows from (2.6) and the density of $\widehat{\mathfrak{B}}_K(\overline{K})$ in $\widehat{\mathfrak{B}}_K \supset \widehat{\mathfrak{B}}$). Thus $\widehat{\mathfrak{B}}$ is $\overline{K}$-defined (see [1, AG, 14.4]). Now let $\Gamma = \mathrm{Gal}\,(\overline{K}/K)$ be the Galois group of the extension $\overline{K}/K$. The set $\widehat{\mathfrak{B}}_K(\overline{K}) = \widehat{\mathfrak{B}} \cap \widetilde{G}(\overline{K})$ is Γ-stable. Hence $\widehat{\mathfrak{B}}$ is K-defined (see [12, 11.2.8, i]). □

Theorem 2.3. *If K is an infinite field, then*

(i) $\widehat{\mathfrak{B}}_K = \widehat{\mathfrak{B}}$;

(ii) *for $\sigma \in G$ the following statements are equivalent:*

(a) $g\sigma g^{-1} \in \widetilde{B}\dot{w}_0\widetilde{B}$ *for some* $g \in \widetilde{G}$;

(b) $g\sigma g^{-1} \in B\dot{w}_0B$ *for some* $g \in G$.

Proof. (i) We may apply here the same arguments as in the proof of (2.5). Namely, if $x \in \widehat{\mathfrak{B}}_K$, then the conjugacy class C_x of x in $\widetilde{G}$ intersects $\widetilde{B}\dot{w}_0\widetilde{B}$ trivially (otherwise, we get a contradiction to the assumption $x \in \widehat{\mathfrak{B}}_K$ as we did in the proof of (2.5)), and therefore we get $x \in \widehat{\mathfrak{B}}$. Since $\widehat{\mathfrak{B}} \subset \widehat{\mathfrak{B}}_K$ we get (i).

(ii) The implication (b) $\Rightarrow$ (a) is obvious. Now we assume (a). Then $\sigma \in \mathfrak{B}$. Hence $\sigma \in \mathfrak{B}_K$ and therefore $g\sigma g^{-1} \in \widetilde{B}\dot{w}_0\widetilde{B}$ for some $g \in G$. Since $g\sigma g^{-1} \in G = \cup_{w \in W} B\dot{w}B$ and $B = \widetilde{B}(K) \subset \widetilde{B}$, the element $g\sigma g^{-1}$ can belong only to the Bruhat cell $B\dot{w}_0B$. This establishes (b). □

3. Example I: $\widetilde{G} = \mathrm{GL}_n, \mathrm{SL}_n$

Let G be $\mathrm{GL}_n(K)$ or $\mathrm{SL}_n(K)$ and let $w_0 \in W \approx S_n$ be the element of maximal length. Consider the big Bruhat cell $B\dot{w}_0B$ of G. Note that a conjugacy class C_g of $g \in G$ intersects $B\dot{w}_0B$ if and only if it intersects the set $\dot{w}_0B$, which is the set of matrices of the form:

$$\begin{pmatrix} 0 & 0 & \cdots & 0 & a_{1n} \\ 0 & 0 & \cdots & a_{2\,n-1} & a_{2n} \\ \cdots & \cdots & \cdots & \cdots & \\ 0 & a_{n-1\,2} & \cdots & a_{n-1\,n-1} & a_{n-1\,n} \\ a_{n\,1} & a_{n\,2} & \cdots & a_{n\,n-1} & a_{nn} \end{pmatrix}. \tag{3.1}$$

Now, if a matrix $g \in G$ has the form (3.1), then

$$\mathrm{rank}\,(g - \alpha E_n) \geq \left[\frac{n}{2}\right] \tag{3.2}$$

for every $\alpha \in K^*$.

In particular, if g is a split semisimple element, the condition (3.2) means that the multiplicity of eigenvalues of g is less than or equal to $[\frac{n+1}{2}]$.

Theorem 3.1. *For $g \in G$,*

$$C_g \cap B\dot{w}_0B \neq \varnothing \quad \Leftrightarrow \ \mathrm{rank}\,(g - \alpha E_n) \geq \left[\frac{n}{2}\right] \quad \textit{for every} \quad \alpha \in K^*.$$

Proof. We use the following notation:

We denote the symmetric group corresponding to the interval $[1, n]$ by S_n and also by $S[1, n]$ to identify imbeddings of symmetric subgroups of smaller degree. For instance, the symmetric subgroup S_k of degree $k < n$ can be identified with any subgroup of all permutations of the subinterval $[i, j] \subset [1, n]$, where $j - i = k - 1$. In this case, we denote such subgroup by $S[i, j]$. Thus, if $1 \leq i \leq j \leq n$, $j - i = k - 1$ we have the imbedding

$$S_k \hookrightarrow S[i, j] \leq S[1, n] = S_n.$$

We also identify the symmetric group S_n with the Weyl group $W_n = W(A_{n-1})$ with the standard set of simple reflections $w_{\alpha_1}, w_{\alpha_2}, \ldots, w_{\alpha_{n-1}}$, where $\Phi = \{\alpha_1 = \epsilon_1 - \epsilon_2, \ldots, \alpha_{n-1} = \epsilon_{n-1} - \epsilon_n\}$ is the standard simple root system (see [2, Table I]). We also identify w_{α_i} with the transposition $(i\, i+1)$ and for every root $\alpha = \epsilon_i - \epsilon_j$, we write $w_\alpha = (ij)$.

We denote the length of $w \in W_n = S_n$ with respect to the generating set $\{w_{\alpha_1}, \dots , w_{\alpha_{n-1}}\}$ by $l(w)$. The number of nonunit eigenvalues of the element $w \in W_n$, which is considered as a linear operator in the standard linear representation of $W_n = S_n$ induced by permutations of a basis of the n-dimensional linear space, will be denoted by $i(g)$. Let

$$w_0 = \begin{cases} (1\ n)(2\ n-1)\cdots(l\quad l+1) & \text{if } \ n = 2l, \\ (1\ n)(2\ n-1)\cdots(l\quad l+2) & \text{if } \ n = 2l+1. \end{cases}$$

Then w_0 is the element of maximal length $\frac{n(n-1)}{2}$ and $i(w_0) = [\frac{n}{2}]$.

Proposition 3.2. *Let $w \in W_n$. If $i(w) \geq [\frac{n}{2}]$, then there is a way $w_0 \mapsto w'$, where $w' \in W_n$ is an element that is in the conjugacy class C_w of w in W, and $l(w') = \min\{l(w'') \mid w'' \in C_w\}$.*

Proof. Now we state the assumption of the induction:

$\flat$: Let $\omega' \in W_m = S[1, m] = \langle w_{\alpha_1}, \dots , w_{\alpha_{m-1}}\rangle$, $1 < m < n$, be an element satisfying the following conditions:

(a) $i(\omega') \geq [\frac{m}{2}]$.

(b) Let e be the number of stable points of the permutation ω'. There exists an element $\omega \in S[e + 1, m]$, which is conjugate to ω' in W_m and which satisfies the following conditions:

1. there is a way $w_0' \mapsto \omega$ where w_0' is the element of maximal length in W_m with respect to the generating set $\{w_{\alpha_1}, \dots , w_{\alpha_{m-1}}\}$;

2. $\omega = \prod_{\alpha \in X} w_\alpha$, where $X \subset \{\alpha_{e+1}, \dots , \alpha_{m-1}\}$ and each $w_\alpha, \alpha \in X$, occurs only once;

3. if $\omega = \omega_1\omega_2 \cdots \omega_d$ is the decomposition of ω into a product of disjoint cycles of lengths $r_1, \dots , r_d$, respectively, then $r_1 = \min\{r_i\}$ and $\omega_1 \in S[e + 1, e + r_1]$. □

For $n = 2, 3$ and 4 the assumption $\flat$ can be checked by simple calculation.

We need the following lemmas.

Lemma 3.3. *Let $1 \leq i < j \leq m$. Further, let $\omega = \mu\nu \in W_m$, where $\mu \in S[i, j]$ and where $\nu \in W_m$ is an element that stabilizes every element* *in $[i,j]$. If there is a way $\mu \mapsto \mu' \in S[i,j]$ in the group $S[i,j]$, then there is a way $\omega \mapsto \mu'\nu$ in the group W_m.*

Proof. Let $\zeta \to \zeta' = w_{\alpha_l}\zeta w_{\alpha_l}$ be a descent in $S[i,j]$. We may assume $\zeta(\alpha_l) \neq \alpha_l$ (otherwise, $\zeta\nu \to w_{\alpha_l}\zeta\nu w_{\alpha_l} = \zeta\nu$ is a nonstrict descent). Then either $\zeta(\alpha_l) < 0$ or $\zeta^{-1}(\alpha_l) < 0$ (see [3, Prop. 2.2.8]). Since ν stabilizes every element in $[i,j]$ and $\nu(\alpha_l) = \alpha_l$, either $\zeta\nu(\alpha_l) = \zeta(\alpha_l) < 0$ or $(\zeta\nu)^{-1}(\alpha_l) = \zeta^{-1}(\alpha_l) < 0$ and therefore $\omega \to \mu'\nu$ is a descent.

Now suppose $\zeta \to \zeta' = w_{\alpha_l}\zeta w_{\alpha_l}$ is a strict descent. Then $\zeta = w_{\alpha_l}\zeta_1 w_{\alpha_l}$, where $0 < \zeta_1(\alpha_l) \neq \alpha_l$, $0 < \zeta_1^{-1}(\alpha_l) \neq \alpha_l$ (see [3, Prop. 2.2.8]). Furthermore $0 < \zeta_1\nu(\alpha_l) \neq \alpha_l$, $0 < \zeta_1^{-1}\nu^{-1}(\alpha_l) \neq \alpha_l$ and therefore $\zeta\nu \to w_{\alpha_l}\zeta\nu w_{\alpha_l} = \zeta_1\nu$ is a strict descent and $\zeta\nu \rightsquigarrow w_{\alpha_1}\zeta\nu$ and $\zeta\nu \rightsquigarrow \zeta\nu w_{\alpha_1}$ are jumps. □

Lemma 3.4. *Let $\omega \in W_m = S[1,m]$ be an element satisfying the conditions $\flat$: (b) $2,3$; (here e is the number of stable points of ω). If $e \geq 1$, then in the group $W_{m+1} = S[1,m+1]$ there is a descent*

$$(1 \quad m+1)\omega \to (e \quad e+r_1+1)\omega_1\widetilde{\omega},$$

where $\widetilde{\omega} \in S[e+r_1+2, m+1]$ is the product of disjoint cycles of lengths $r_2, \dots, r_d$. Moreover,

$$\widetilde{\omega} = \prod_{\alpha \in X'} w_\alpha,$$

where $X' \subset \{\alpha_{e+r_1+2}, \dots, \alpha_m\}$ and each w_α, $\alpha \in X'$, occurs only once.

Proof. Let $i < e$, $\alpha_i = \epsilon_i - \epsilon_{i+1}$. Clearly

$$[(i\ m+1)\omega](\alpha_i) = \epsilon_{m+1} - \epsilon_{i+1} < 0 \Rightarrow$$

$$\Rightarrow l(w_{\alpha_i}[(i\ m+1)\omega]w_{\alpha_i}) \leq l([(i \quad m+1)\omega]) \Rightarrow$$

$$\Rightarrow [(i\ m+1)\omega] \to w_{\alpha_i}[(i\ m+1)\omega]w_{\alpha_i} = (i+1\ m+1)\omega.$$

Thus

$$(1 \quad m+1)\omega \to (e \quad m+1)\omega$$

is a descent.

Now let $e + r_1 + 2 \leq j \leq m+1$. Put

$$D_j = \{e+r_1+1, \dots, j-1, j+1, \dots, m+1\}$$

(if $j = m+1$, then $D_j = \{e + r_1 + 1, \dots, m\}$).

Suppose there is a descent

$$(e\ m+1)\omega \to (ej)\omega_1\widetilde{\omega}',$$

where $\widetilde{\omega}'$ is a permutation of the set D_j that is conjugate to $\omega_2\omega_3\cdots\omega_d$ in W_m. Moreover, we suppose that $\widetilde{\omega}'$ is a product of transpositions of type w_{α_k}, where $k \neq j-1$, j and, possibly, the transposition $(j-1\ j+1)$ and each such transposition can occur not more than once. Therefore

$$\begin{aligned}
&[(ej)\omega_1\widetilde{\omega}']^{-1}(\epsilon_{j-1} - \epsilon_j) = \epsilon_l - \epsilon_e, \quad l > e \Rightarrow \\
&\Rightarrow l(w_{\alpha_{j-1}}[(ej)\omega_1\widetilde{\omega}']w_{\alpha_{j-1}}) \leq l([(ej)\omega_1\widetilde{\omega}']) \Rightarrow \\
&\Rightarrow [(ej)\omega_1\widetilde{\omega}'] \to w_{\alpha_{j-1}}[(ej)\omega_1\widetilde{\omega}']w_{\alpha_{j-1}} = (e\ j-1)\omega_1\widetilde{\omega}''.
\end{aligned}$$

Here $\widetilde{\omega}'' = w_{\alpha_{j-1}}\widetilde{\omega}'w_{\alpha_{j-1}}$. Note that among the factors of $\widetilde{\omega}'$ only $w_{\alpha_{j-2}}$ and $(j-1\quad j+1)$ do not commute with $w_{\alpha_{j-1}}$. But

$$\begin{aligned}
w_{\alpha_{j-1}}w_{\alpha_{j-2}}w_{\alpha_{j-1}} &= (j-2\quad j), \\
w_{\alpha_{j-1}}(j-1\quad j+1)w_{\alpha_{j-1}} &= (j\quad j+1) = w_{\alpha_j}.
\end{aligned}$$

Hence the element $\widetilde{\omega}''$ is a product of transpositions of type w_{α_k}, where $k \neq j-2$, $j-1$ and, possibly, the transposition $(j-2\ j)$, and each such transposition can occur only once.

Thus we get a descent

$$(e\ m+1)\omega \to (e\quad e+r_1+1)\omega_1\widetilde{\omega},$$

where $\widetilde{\omega}$ satisfies the condition of the lemma. □

Lemma 3.5. *If $\nu = (1m)\nu' \in W_m$, where $\nu' \in S[2, m-1]$ is an $(m-2)$-cycle with $m-2 \geq 2$, then there is a way $\nu \mapsto \mu$, where μ is an m-cycle and $l(\mu) > m-1$.*

Proof. Clearly $\nu(\epsilon_1 - \epsilon_2) = \epsilon_m - \epsilon_k < 0$ and therefore $l(\nu w_{\alpha_1}) < l(\nu)$. Further, $(\nu w_{\alpha_1})^{-1}(\epsilon_1 - \epsilon_2) = \epsilon_m - \epsilon_{k'} < 0$. Hence $l(w_{\alpha_1}\nu w_{\alpha_1}) < l(\nu w_{\alpha_1})$ and $\nu \rightsquigarrow \mu = w_{\alpha_1}\nu$ is a jump.

Further, the transposition $(1m)$ is the representative of the minimal length of the coset $(1m)S[2, m-1]$, because $(1m)(\alpha_k) = \alpha_k$ for every $k = 2, \dots, m-2$ and therefore $l(\nu) = l((1m)) + l(\nu')$ (see [3, Prop. 2.3.3]). Clearly

$$l(\nu) = l((1m)) + l(\nu') \geq 2m - 3 + m - 3 = 3m - 6 \geq m + 2.$$

Hence $l(\mu) \geq m+1$. □

Lemma 3.6. *If $\mu \in W_m$ is an m-cycle with $l(\mu) > m-1$, then there is a way $\mu \mapsto \widetilde{\mu} \in S[2,m]$, where $\widetilde{\mu}$ is an $(m-1)$-cycle and $l(\widetilde{\mu}) = m-2$.*

Proof. Since $\mu \mapsto \mu'$ where μ' is an m-cycle with $l(\mu') = m-1$ (see [5, Proposition 3.3]), we may assume $l(\mu) = m+1$. Hence

$$\mu = w_\alpha \mu' w_\alpha$$

for some $\alpha \in \Phi$, where $\Phi = \{\alpha_1, \dots, \alpha_{m-1}\}$ is the standard simple root system and for some m-cycle μ' with $l(\mu') = m-1$. Further, there exists a partition $\Phi = \Phi_1 \cup \Phi_2 \cup \{\alpha\}$, where Φ_1, $\Phi_2 \neq \varnothing$, $\Phi_1 \cap \Phi_2 = \varnothing$, $\alpha \notin \Phi_1$, Φ_2, such that

$$\mu = w_\alpha \Big(\prod_{\beta \in \Phi_1} w_\beta \Big) w_\alpha \Big(\prod_{\gamma \in \Phi_2} w_\gamma \Big) w_\alpha .$$

Note,

$$w_\alpha \Big(\prod_{\beta \in \Phi_1} w_\beta \Big) \neq \Big(\prod_{\beta \in \Phi_1} w_\beta \Big) w_\alpha, \quad w_\alpha \Big(\prod_{\gamma \in \Phi_2} \gamma \Big) \neq \Big(\prod_{\gamma \in \Phi_2} w_\gamma \Big) w_\alpha,$$

because otherwise $l(\mu) = m-1$. Hence $w_\alpha \neq (12)$, $(m\ m-1)$, and if $w_\alpha = (i\ i+1)$, $i \neq 1, m-1$, then each set

$$\{w_\beta\}_{\beta \in \Phi_1}, \quad \{w_\gamma\}_{\gamma \in \Phi_2}$$

contains only one transposition in the set $\{(i-1\ i), (i+1\ \ i+2)\}$ (because only those simple transpositions do not commute with $(i\ i+1)$).

Put

$$\mu_1 = \begin{cases} w_\alpha \Big(\prod\limits_{\beta \in \Phi_1} w_\beta \Big) w_\alpha \Big(\prod\limits_{\gamma \in \Phi_2} w_\gamma \Big) & \text{if} \quad (i-1\ i) \in \{w_\beta\}_{\beta \in \Phi_1}, \\ \Big(\prod\limits_{\beta \in \Phi_1} w_\beta \Big) w_\alpha \Big(\prod\limits_{\gamma \in \Phi_2} w_\gamma \Big) w_\alpha & \text{if} \quad (i-1\ i) \in \{w_\gamma\}_{\gamma \in \Phi_2}. \end{cases}$$

Then there is a jump $\mu \rightsquigarrow \mu_1$, where μ_1 is an $(m-1)$-cycle in the set $\{1, 2, \dots, i-1, i+1, \dots, m\}$. Moreover, $l(\mu_1) = m$ and

$$\mu_1 = \Big(\prod_{\beta \in \Psi_1} w_\beta \Big) (i-1 \quad i+1) \Big(\prod_{\gamma \in \Psi_2} w_\gamma \Big),$$

where $\Psi_1 \cup \Psi_2 = \Psi = \Phi \setminus \{\epsilon_{i-1} - \epsilon_i, \quad \epsilon_i - \epsilon_{i+1})\}$, $\Psi_1 \cap \Psi_2 = \varnothing$. By commuting with w_β, where $\beta \in \Psi_1$, we may have a non-strict descent $\mu_1 \to \mu_2$, where

$$\mu_2 = (i-1 \quad i+1)\Big(\prod_{\zeta \in \Psi} w_\zeta\Big), \quad l(\mu_2) = m.$$

Put $\delta = \epsilon_{i-1} - \epsilon_i$. Suppose $i - 1 \neq 1$. Then $\mu_2(\delta) = \epsilon_k - \epsilon_i > 0$, $k \leq i-2$ (because among the roots in Ψ there is the root $\epsilon_{i-2} - \epsilon_{i-1}$), and $\mu_2^{-1}(\delta) = \epsilon_l - \epsilon_i < 0$, $l \geq i+1$. Hence $l(w_\delta \mu_2 w_\delta) = l(\mu_2) = m$. Put $\mu_3 = w_\delta \mu_2 w_\delta$. We have $\mu_2 \to \mu_3$, where

$$\mu_3 = (i \quad i+1)\Big(\prod_{\zeta' \in \Psi'} w_{\zeta'}\Big)(i-2 \quad i)\Big(\prod_{\zeta'' \in \Psi''} w_{\zeta''}\Big),$$

where $\Psi' \cup \Psi'' = \Psi \setminus \{\epsilon_{i-2} - \epsilon_{i-1}\}$, $\Psi' \cap \Psi'' = \varnothing$. Similar as in the case of the descent $\mu_1 \to \mu_2$ we can get a descent $\mu_3 \to \mu_4$, where

$$\mu_4 = (i-2 \quad i)\Big(\prod_{\chi \in \Delta} w_\chi\Big), \quad l(\mu_4) = m,$$

where $\Delta = \Phi \setminus \{\epsilon_{i-2} - \epsilon_{i-1}, \epsilon_{i-1} - \epsilon_i\}$. Thus, acting similarly, we can get a descent $\mu_4 \to \mu'$, where μ' is an $(m-1)$-cycle of the form

$$\mu' = (13) \prod_{\psi \in \Sigma} w_\psi,$$

where $\Sigma = \Phi \setminus \{\epsilon_1 - \epsilon_2, \epsilon_2 - \epsilon_3\}$. Let $w_{\alpha_1} = w_{\epsilon_1 - \epsilon_2}$. Then

$$\widetilde{\mu} \stackrel{\text{def}}{=} w_{\alpha_1} \mu' w_{\alpha_1} = (23) \prod_{\psi \in \Sigma} w_\psi = \prod_{\phi \in \Phi \setminus \{(12)\}} w_\phi.$$

Obviously, $\widetilde{\mu}$ is an $(m-1)$-cycle in $S[2, m]$ and $l(\widetilde{\mu}) = m - 2$. □

Lemma 3.7. *If $\omega = (1m) \in W_m$, then for every k with $1 \leq k \leq m-1$ there is a way $\omega \mapsto \mu$, where $\mu = (k\ k+1 \ldots m)$.*

Proof. Conjugating ω successively by $(12), (23), \ldots, (k-1 \quad k)$ we get a descent $\omega \to (km)$. Now our statement follows from ([6, Proposition 4.1]). □

Let $w \in W_n$ with $i(w) \geq [\frac{n}{2}]$, and let k be the number of stable points of w. Further, assume

$$w = u_1 \cdots u_s$$

is the decomposition of w into a product of disjoint cycles. Also, let $l_1, \ldots, l_s$ be the degrees of the cycles $u_1, \ldots, u_s$, respectively. We assume $l_1 = \min\{l_i\}_{i=1}^s$.

Case 1. $k \geq 1$, $l_1 > 2$.

Let u_1' be a cycle of length $l_1 - 1$. Put $w_1 = u_1' u_2 \cdots u_s$. Then the number of stable points of w_1 is equal to $k+1$ and $i(w_1) = i(w) - 1 \geq [\frac{n-2}{2}]$. Since the condition of the proposition for the element w and the statement concern all elements of the conjugacy class of w in W_n, we may assume $w_1 \in S[2, n-1]$ (because $k > 1$).

By assumption $\flat$, there is a way $w_0' \mapsto w_2$, where w_0' is the element of maximal length in the group $S[2, n-1]$ and w_2 is an element in the group $S[2, n-1]$ that is conjugate to w_1 in W_n and that satisfies conditions (2) and (3) of $\flat$. By Lemma 3.3, there is a way

$$w_0 = (1n) w_0' \mapsto w_3 = (1n) w_2,$$

where $w_2 = \omega_1 \omega_2 \cdots \omega_s \in S[k+1, n-1]$ is a product of disjoint cycles of degree $l_1 - 1, l_2, \ldots, l_s$. Moreover, $\omega_1, \ldots, \omega_s$ are products of simple reflections w_{α_i}, where each such reflection can occur not more than once. Also, ω_1 is an $(l_1 - 1)$-cycle in the set $[k+1, k+l_1-1]$. The element w_2 satisfies the conditions of Lemma 3.4 (with $w_2 = \omega$, $l_1 - 1 = r_1$, $l_i = r_i$, $i > 2$. $s = d$, $n = m+1$, $e = k$). Hence there is a descent

$$w_3 = (1n) w_2 \to w_4 = (k \quad k + l_1) \omega_1 \widetilde{\omega},$$

where $\widetilde{\omega} \in S[k+l_1+1, n]$ is conjugate to $\omega_2 \omega_3 \cdots \omega_s$ and $\widetilde{\omega} \in S[k+l_1+1, n]$ is a product of basic reflections, where each such reflection can occur not more than once. By Lemmas 3.5 and 3.6, there is a way

$$(k \quad k + l_1) \omega_1 \mapsto \omega_1' \in S[k+1, k+l_1],$$

where ω_1' is an l_1-cycle and where $l(\omega_1') = l_1 - 1$. By Lemma 3.3, there is a way

$$w_4 = (k \quad k + l_1) \omega_1 \widetilde{\omega} \mapsto w_5 = \omega_1' \widetilde{\omega} \in S[k+1, n].$$

The process of the construction shows that the element w_5 satisfies the conditions for w' of the proposition.

Case 2. $k = 0$, $s > 1$.

Claim. $i(u_2 \cdots u_s) \geq [\frac{n-2}{2}]$.

Proof. We have

$$i(u_2 \cdots u_s) = (l_2 - 1) + \cdots + (l_s - 1)$$

$$= n - l_1 - s + 1 \geq \frac{n-2}{2} \Leftrightarrow n \geq 2(l_1 + s - 2).$$

Since $l_1 \geq 2$, $s \geq 2$, and $l_1 = \min\{l_i\}$, we obtain

$$n \geq l_1 s = l_1[(s-2)+2] = l_1(s-2)+2l_1 \geq 2(s-2)+2l_1 = 2(l_1+s-2). \quad \square$$

The same arguments as above yield the way

$$w_0 \mapsto (1l_1)\widetilde{\omega} \in S[1, n],$$

where $\widetilde{\omega} \in S[l_1 + 1, n]$ is an element that is conjugate to $u_2 \cdots u_s$ and, using Lemma 3.7, we get the way

$$(1l_1)\widetilde{\omega} \in S[k+1, n] \mapsto w',$$

where w' satisfies the conditions of the proposition.

Case 3. $k = 0$, $l_1 > 2$, $s = 1$.

Again the same arguments as above yield the way

$$w_0 \mapsto (1n)\zeta',$$

where $\zeta' \in S[2, n-1]$ is an $(n-2)$-cycle of length $n-3$. Thus there is a jump

$$(1n)\zeta' \rightsquigarrow \zeta = (12)(1n)\zeta',$$

where ζ is an n-cycle. Therefore there is a descent

$$\zeta \to w',$$

where w' is an n-cycle of length $n-1$. $\square$

Proposition 3.8. *Let $G = \mathrm{GL}_n(K)$ or $G = \mathrm{SL}_n(K)$. If $g \in G$ and*

$$\operatorname{rank}(g - \alpha E_n) \geq \left[\frac{n}{2}\right] \quad \text{for every} \quad \alpha \in K^*,$$

then g is conjugate in G to a block-diagonal matrix

$$R = \mathrm{di}(R_1, R_2, \dots, R_s), \tag{3.3}$$

where each R_i is a cyclic matrix of size n_i (possibly, $n_i = 1$)

$$\begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0\cdots & 0 \\ \cdots & & & & \\ 0 & 0 & 0 & 0\cdots & 1 \\ a_1 & a_2 & a_3 & \cdots & a_{n_i} \end{pmatrix} \tag{3.4}$$

and

$$\sum_{i=1}^{s}(n_i - 1) \geq \left[\frac{n}{2}\right]. \tag{3.5}$$

Proof. Note, that every matrix in G is conjugate to a matrix of the form (3.3) (we may take the rational form). We may assume that we cannot join any two blocks R_i, R_j into one block of the form (3.4). Suppose

$$\sum_{i=1}^{s}(n_i - 1) < \left[\frac{n}{2}\right]. \tag{3.6}$$

Inequality (3.6) implies that there is a block R_i of size one, $n_i = 1$, i.e., $R_i = \alpha \in K^*$. Consider any block R_j, $j \neq i$. If α is not an eigenvalue of R_j, then we can join the blocks R_i, R_j into one block of the form (3.4), which is a contradiction to our assumption. Thus α is an eigenvalue of every block R_j, and therefore

$$\operatorname{rank}(R - \alpha E_n) \leq \sum_{i=1}^{s}(n_i - 1). \tag{3.7}$$

Now we have a contradiction of (3.6), (3.7) with the assumption of the proposition. □

Now we can finish the proof of the theorem.

Let $g \in G$ be an element satisfying condition (3.2), and let C_g be its conjugacy class. By Proposition 3.8, $C_g \cap B\dot{w}'B \neq \varnothing$ for some $w' \in W_n$ with $i(w') \geq \left[\frac{n}{2}\right]$. By Proposition 3.7, there is a way $w_0 \mapsto w$, where $w \in W_n$ is an element in the same conjugacy class as w'. Note, that in Proposition 3.8 we can take the block diagonal matrix R corresponding to w. Thus, we may assume $C_g \cap B\dot{w}B \neq \varnothing$. This implies $C_g \cap B\dot{w}_0B \neq \varnothing$ (see [6] and the Introduction). □

Remark. If $g \in \mathrm{GL}_n(K) \leq \mathrm{GL}_n(\overline{K})$, then

$$\begin{aligned} \operatorname{rank}(g - \alpha E_n) \geq \left[\frac{n}{2}\right] \quad \text{for every} \quad \alpha \in K^* \Leftrightarrow \\ \Leftrightarrow \operatorname{rank}(g - \alpha E_n) \geq \left[\frac{n}{2}\right] \quad \text{for every} \quad \alpha \in \overline{K}^*. \end{aligned}$$

Indeed, if $\alpha \in \overline{K} \setminus K$, then α can be an eigenvalue only for blocks of the form (3.4) of size ≥ 2. Hence the inequality $\operatorname{rank}(g - \alpha E_n) \geq \left[\frac{n}{2}\right]$ holds for every such α.

In the following proposition we describe the structure of the affine variety $\widehat{\mathfrak{B}}$. We assume here $K = \mathfrak{K}$ is an algebraically closed field. Let $g \in \widehat{\mathfrak{B}}$ be a semisimple element. Then Theorem 3.1 implies $\operatorname{rank}(g - \alpha_0 E_n) < [\frac{n}{2}]$ for some $\alpha_0 \in K^*$. This means that g has an eigenvalue α_0 with multiplicity $m = \left[\frac{n+3}{2}\right]$. Let T be the group of diagonal matrices in G and let

$$T_m = \{\mathrm{di}(\underbrace{\alpha, \alpha, \dots, \alpha}_{m-\text{times}}, \beta_1, \beta_2, \dots, \beta_{n-m}) \mid \alpha, \beta_i \in K\}.$$

Since the semisimple element g has eigenvalue α_0 with multiplicity m, it is conjugate to an element in T_m.

Note, that T_m is a subtorus of T if $m < n$ or $G = \mathrm{GL}_n(K)$, that is T_m is a connected algebraic group. The cases $m = n$ are possible only for $n = 2, 3$, and in these cases the set $\widehat{\mathfrak{B}}$ coincides with the center of the group G.

Proposition 3.9. *Let K be an algebraically closed field and let $G = \mathrm{GL}_n(K)$, $\mathrm{SL}_n(K)$. If T is the group of diagonal matrices in G, then*

$$\widehat{\mathfrak{B}} = \overline{\bigcup_{g \in G} gT_m g^{-1}}. \tag{3.8}$$

In particular, if $n > 3$ or $G = \mathrm{GL}_n(K)$, the set $\widehat{\mathfrak{B}}$ is irreducible and

$$\dim \widehat{\mathfrak{B}} = \begin{cases} n^2 - m^2 + 1 & \text{if} \quad G = \mathrm{GL}_n(K), \\ n^2 - m^2 & \text{if} \quad G = \mathrm{SL}_n(K). \end{cases} \tag{3.9}$$

Proof. Let H_1, $H_2 \leq G$ be the subgroups consisting of matrices of the form

$$H_1 = \left\{ \begin{pmatrix} X & | & \mathbf{0}_{(n-m)\times m} \\ -- & | & -- \\ \mathbf{0}_{m\times(n-m)} & | & \alpha E_m \end{pmatrix} \ \middle| \ X \in \mathrm{GL}_{n-m}(K), \alpha \in K^* \right\};$$

(note that $\alpha^m \det X = 1$, $m < n$ if $G = \mathrm{SL}_n(K)$),

$$H_2 = \left\{ \begin{pmatrix} E_{n-m} & | & Y \\ -- & | & -- \\ \mathbf{0}_{m\times(n-m)} & | & E_m \end{pmatrix} \ \middle| \ Y \in M_{(n-m)\times m}(K) \right\}.$$

Obviously, H_1 and H_2 are connected groups and the set $H = H_1 H_2 = \{h_1 h_2 \mid h_1 \in H_1,\ h_2 \in H_2\}$ is also a group. Thus H is a connected subgroup of G. Further, if S is a maximal torus of H_1, then S is also a maximal torus of H, and the centralizer of S in H coincides with S. Hence the elements that are conjugate to S are dense in H (see [7, 2.2]). On the other hand, the torus S is conjugate to T_m in G (this follows from the definitions of T_m and H). Thus

$$H \subset \overline{\bigcup_{g \in G} gT_m g^{-1}}. \tag{3.10}$$

Further, if $x \in \widehat{\mathfrak{B}}$, then the linear operator x satisfies the inequality $\operatorname{rank}(x - \alpha E_n) < [\frac{n}{2}]$ for some $\alpha \in K^*$, and therefore x has at least $m = [\frac{n+3}{2}]$ eigenvectors corresponding to the eigenvalue α. Hence the operator x is conjugate to an element in H. Thus

$$\widehat{\mathfrak{B}} = \bigcup_{g \in G} gHg^{-1}. \tag{3.11}$$

Now (3.10) and (3.11) imply

$$\widehat{\mathfrak{B}} = \overline{\bigcup_{g\in G} gT_m g^{-1}}.$$

The variety $G \times T_m$ is irreducible. Hence the closure of the image of the morphism $\phi : G \times T_m \to G$, given by the formula $\phi(g,t) = gtg^{-1}$, is irreducible. Thus $\widehat{\mathfrak{B}}$ is an irreducible affine variety.

Let $\phi(g_1 \times t_1) = \phi(g_2 \times t_2)$. Then $g_2^{-1} g_1(t_1) g_1^{-1} g_2 = t_2$. Further, since t_1, $t_2 \in T$ are conjugate, $\dot{w} t_2 \dot{w}^{-1} = t_1$ for some $w \in W$. Hence $g_2 = g_1 c \dot{w}^{-1}$ for some $c \in C_G(t_1)$, and therefore

$$\dim \phi^{-1}(\phi(g_1 \times t_1)) = \dim C_G(t_1). \tag{3.12}$$

Let $t = \mathrm{di}(\underbrace{\alpha, \alpha, \dots, \alpha}_{m-\text{times}}, \beta_1, \beta_2, \dots, \beta_{n-m})$, where $\alpha \neq \beta_i$ for every i and $\beta_i \neq \beta_j$. The set of such elements t is dense in T_m. Therefore (3.12) implies

$$\dim \widehat{\mathfrak{B}} = \dim T_m + \dim G - \dim C_G(t). \tag{3.13}$$

Further,

$$\dim T_m = \begin{cases} n-m+1 & \text{if } G = \mathrm{GL}_n, \\ n-m & \text{if } G = \mathrm{SL}_n, \end{cases} \tag{3.14}$$

$$\dim C_G(y) = \begin{cases} n-m+m^2 & \text{if } G = \mathrm{GL}_n, \\ n-m-1+m^2 & \text{if } G = \mathrm{SL}_n. \end{cases} \tag{3.15}$$

The formula for $\dim \widehat{\mathfrak{B}}$ follows from (2.13)–(2.15). □

4. Example II: $Sp_4(K)$

In this section, we consider the case $\mathfrak{K} = K$ and $\widetilde{G} = G = Sp_4(K) \leq GL(V)$, $\dim V = 4$.

Here $\Phi = \Phi(C_2) = \{\epsilon_1 - \epsilon_2, 2\epsilon_2\}$ is the standard simple root system of the root system C_2 (see [2, Table III]). The weights of the representation $G \hookrightarrow GL(V)$ are $\pm\epsilon_1$, $\pm\epsilon_2$. The highest weight is ϵ_1. We fix the basis e_1, e_2, e_{-2}, e_{-1} for V, where $e_{\pm i}$ is the weight vector of the weight $\pm\epsilon_i$. Here the corresponding bilinear form is given by

$$\langle e_i, e_{-i} \rangle = 1, \quad i = 1, 2, \quad \langle e_i, e_j \rangle = 0, \quad j \neq -i.$$

The following is a presentation of root subgroups of G:

$$x_{\epsilon_1-\epsilon_2}(x) = \begin{pmatrix} 1 & x & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -x \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad x_{\epsilon_1+\epsilon_2}(y) = \begin{pmatrix} 1 & 0 & y & 0 \\ 0 & 1 & 0 & y \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$x_{2\epsilon_1}(s) = \begin{pmatrix} 1 & 0 & 0 & s \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad x_{2\epsilon_2}(t) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & t & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Further, if char $K \neq 2$, there exist the following nontrivial unipotent conjugacy classes in G:

$C_{\text{reg}} = \{$the conjugacy class of regular unipotent elements$\}$
$= $ the conjugacy class of $x_{\epsilon_1-\epsilon_2}(1)x_{2\epsilon_2}(1)$;

$C_{\epsilon_1-\epsilon_2} = \{$the conjugacy class of short root element $x_{\epsilon_1-\epsilon_2}(1)\}$
$= \{$conjugacy class of $\mathfrak{u} = x_{2\epsilon_1}(1)x_{2\epsilon_2}(1)\}$;

$C_{2\epsilon_2} = \{$the conjugacy class of the long root element $x_{2\epsilon_2}(1)\}$;

$C_1 = \{$the conjugacy class of E_4 $\}$.

Moreover we have the following inclusion:

$$C_1 \subset \overline{C}_{2\epsilon_2} \subset \overline{C}_{\epsilon_1-\epsilon_2} \subset \overline{C}_{\text{reg}}.$$

(see [3, p. 435] and [11, Tables]). Let C be a unipotent class and $u \in C$; the class of $-u$ will be denoted by $-C$.

The group $H = \langle h_{2\epsilon_1}(\alpha), h_{2\epsilon_2}(\beta)\rangle$ is a maximal torus of G and the presentation of elements of H by matrices is the following:

$$h_{2\epsilon_1}(\alpha)h_{2\epsilon_2}(\beta) = \begin{pmatrix} \alpha & 0 & 0 & 0 \\ 0 & \beta & 0 & 0 \\ 0 & 0 & \beta^{-1} & 0 \\ 0 & 0 & 0 & \alpha^{-1} \end{pmatrix}.$$

We emphasize the element

$$h_0 = h_{2\epsilon_1}(-1)h_{2\epsilon_2}(1),$$

which is conjugate to $-h_0 = h_{2\epsilon_1}(1)h_{2\epsilon_2}(-1)$. We denote the conjugacy class of h_0 by C_{h_0}.

The general matrix corresponding to the Borel subgroup has the form

$$B(\alpha,\beta,x,y,t,s) = h_{2\epsilon_1}(\alpha)h_{2\epsilon_2}(\beta)x_{2\epsilon_1}(s)x_{2\epsilon_2}(t)x_{\epsilon_1-\epsilon_2}(x)x_{\epsilon_1+\epsilon_2}(y)$$
$$= \begin{pmatrix} \alpha & \alpha x & \alpha y & \alpha c \\ 0 & \beta & \beta t & \beta y - \beta t x \\ 0 & 0 & \beta^{-1} & -\beta^{-1}x \\ 0 & 0 & 0 & \alpha^{-1} \end{pmatrix},$$

where c is a polynomial in α,β,x,y,t,s such that for every fixed α,β,x,y,t we can get every value of c in K changing the parameter s. Further, we choose

$$\dot{w}_0 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}.$$

Hence

$$\dot{w}_0 B(\alpha,\beta,x,y,t,s) = \begin{pmatrix} 0 & 0 & 0 & \alpha^{-1} \\ 0 & 0 & \beta^{-1} & -\beta^{-1}x \\ 0 & -\beta & -\beta t & -\beta y + \beta t x \\ -\alpha & -\alpha x & -\alpha y & -\alpha c \end{pmatrix}. \tag{4.1}$$

Note,

$$g \in \mathfrak{B} \Leftrightarrow g \ \text{ is conjugate to a matrix of the form (4.1).}$$

Thus

$$g \in \mathfrak{B} \Rightarrow \operatorname{rank}(g - \alpha E_4) \geq 2 \quad \text{for every} \quad \alpha \in K^*. \tag{4.2}$$

Proposition 4.1. *Let $G = Sp_4(K)$. If* char $K \neq 2$, *then*

$$\widehat{\mathfrak{B}} = \pm C_1 \cup C_{h_0} \cup \pm C_{2\epsilon_2}.$$

Proof.

Lemma 4.2. *If $g \in G$ is an element that has no eigenvalues ± 1, then $g \in \mathfrak{B}$.*

Proof. Let $g = g_s g_u$ be the Jordan decomposition. We may assume $g_s = h_{2\epsilon_1}(\alpha)h_{2\epsilon_2}(\beta)$. Then $\alpha,\beta \neq \pm 1$. Also $x g_s x^{-1} = \dot{w}_{2\epsilon_1}\dot{w}_{2\epsilon_2}u$ for some $x \in \langle X_{\pm 2\epsilon_1}\rangle \times \langle X_{\pm 2\epsilon_2}\rangle$, $u \in X_{2\epsilon_1} \times X_{2\epsilon_2}$. Thus $g_s \in \mathfrak{B}$. If $g \notin \mathfrak{B}$, then $g \in \widehat{\mathfrak{B}}$. Since $\widehat{\mathfrak{B}}$ is closed and G-invariant, the closure of the conjugacy class of g is also in $\widehat{\mathfrak{B}}$. But g_s is in this closure (see[13, II]). This is a contradiction. Hence $g \in \mathfrak{B}$. □

Lemma 4.3. *If* $\mathfrak{u} = x_{2\epsilon_1}(1)x_{2\epsilon_2}(1)$ *and if* $g \in G$ *is an element that is conjugate to* $\pm\mathfrak{u}$, $\pm h_0\mathfrak{u}$, *then* $g \in \mathfrak{B}$.

Proof. The same arguments as in the proof of Lemma 4.2. □

Lemma 4.4. *If* $\alpha \neq \pm 1$, *then* $h_{2\epsilon_1}(\alpha)h_{2\epsilon_2}(\pm 1)x_{2\epsilon_2}(1) \in \mathfrak{B}$.

Proof. The same arguments as in the proof of Lemma 4.2. □

Lemma 4.5. *If* u *is a regular unipotent element, then* $u \in \mathfrak{B}$.

Proof. This follows from Lemma 4.3 and the inclusion $C_{\mathfrak{u}} \subset C_{\text{reg}}$ (see also [8]). □

Lemma 4.6. $h_0 \in \widehat{\mathfrak{B}}$.

Proof. Consider the natural surjection $\phi : \mathrm{Sp}_4(K) \to \mathrm{SO}_5(K)$. Consider the natural representation of $\mathrm{SO}_5(K)$. One can easily check that $\phi(h_0) = \mathrm{di}(-1,-1,-1,-1,1)$. Also, $\phi(\mathfrak{B}_{\mathrm{Sp}_4}) = \mathfrak{B}_{\mathrm{SO}_5}$ (here $\mathfrak{B}_{\mathrm{Sp}_4}$ and $\mathfrak{B}_{\mathrm{SO}_5}$ are the variety $\mathfrak{B}$ for $\mathrm{Sp}_4(K)$ and $\mathrm{SO}_5(K)$, respectively) and if $g \in \mathfrak{B}_{\mathrm{SO}_5}$, then $\mathrm{rank}\,(g + E_5) \geq 2$. □

Lemma 4.7. *If* δ, $t \in K$, $\delta \neq \pm 1$, $t \neq 0$, *then*

$$h_{2\epsilon_1}(\delta)h_{2\epsilon_2}(\pm 1), \quad \pm h_0 x_{2\epsilon_2}(t) \in \mathfrak{B}.$$

Proof. Let g_x and g_{-x} be two matrices of the form (4.1) (i.e., $g_{\pm x} \in \dot{w}_0 B$) with the following values of parameters $\alpha = \beta = 1, t = 2, y = x, c = 2 - x^2$:

$$g_x = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & -x \\ 0 & -1 & -2 & x \\ -1 & -x & -x & x^2 - 2 \end{pmatrix},$$

and $\alpha = \beta = 1,\ t = -2,\ y = -x,\ c = x^2 - 2$:

$$g_{-x} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & -x \\ 0 & -1 & 2 & -x \\ -1 & -x & x & 2 - x^2 \end{pmatrix}.$$

Consider the matrices

$$g_x + E_4 = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & -x \\ 0 & -1 & -1 & x \\ -1 & -x & -x & x^2-1 \end{pmatrix},$$

$$g_x - E_4 = \begin{pmatrix} -1 & 0 & 0 & 1 \\ 0 & -1 & 1 & -x \\ 0 & -1 & 1 & -x \\ -1 & -x & x & 1-x^2 \end{pmatrix}.$$

It is easy to see that rank $(g_x + E_4) = 2$ and rank $(g_{-x} - E_4) = 2$. Hence the set of eigenvalues of g_x is $\{-1, -1, \delta, \delta^{-1}\}$ and the set of eigenvalues of g_{-x} is $\{1, 1, \delta, \delta^{-1}\}$. Varying the parameter x we can get any value for $\operatorname{tr} g_{\pm x}$ and, therefore, we can get any value for δ.

If $\delta \neq \pm 1$, then $g_{\pm x}$ are semisimple elements (otherwise the elements $g_{\pm x}$ are conjugate to $h_{2\epsilon_1}(\delta)h_{2\epsilon_2}(\pm 1)x_{2\epsilon_2}(d)$ for some $d \neq 0$, and then rank $(g_{\pm x} \pm 1) > 2$). Thus, if $\delta \neq \pm 1$, there are semisimple elements $g_{\pm x}$ of the form (4.1) (i.e., $g_{\pm x} \in \mathfrak{B}$) that are conjugate to $h_{2\epsilon_1}(\delta)h_{2\epsilon}(\pm 1)$.

Now we put $x = 2$ and get $\operatorname{tr} g_x = 0$. Then the element g_2 has eigenvalues $\{-1, -1, 1, 1\}$ and therefore the semisimple part of the Jordan decomposition of g_x is conjugate to h_0. Since $h_0 \notin \mathfrak{B}$ (Lemma 4.6) the unipotent part of g_x is not trivial. There are two possibilities: g_x is conjugate to $\pm h_0 x_{2\epsilon_2}(t)$ or to $\pm h_0 \mathfrak{u}$. But in the latter case rank $(g_x + E_4) = 3$. Hence there is only the possibility that g_x is conjugate to $\pm h_0 x_{2\epsilon_2}(t)$. □

Now we can prove our statement. Obviously, $\pm C_1 = \{\pm E_4\} \subset \widehat{\mathfrak{B}}$. Further, if $g \in \pm C_{2\epsilon_2}$, then rank $(g \pm E_4) = 1$. Hence $\pm C_{2\epsilon_2} \subset \widehat{\mathfrak{B}}$, and, by Lemma 4.6, $C_{h_0} \subset \mathfrak{B}$.

Now let $g \in \widehat{\mathfrak{B}}$ and let $g = g_s g_u$ be its Jordan decomposition. By Lemmas 4.2 and 4.7, the eigenvalues of the element g_s can only be 1 or -1. Thus, $g_s = \pm E_4$ or g_s is conjugate to h_0. In the latter case, $g_u = 1$, by Lemma 4.7. If $g_s = \pm E_4$ then Lemmas 4.3 and 4.5 imply that the unipotent part g_u is either trivial or it is conjugate to $x_{2\epsilon_2}(1)$.

Now the proposition has been proved. □

Now we consider the case char $K = 2$. Here we have the following diagram of unipotent conjugacy classes

$$C_{\mathrm{reg}}$$

$$\downarrow$$

$$C_{2\epsilon_1 2\epsilon_2}$$

$$\swarrow \quad \searrow$$

$$C_{\epsilon_1-\epsilon_2} \qquad\qquad C_{2\epsilon_2}$$

$$\searrow \quad \swarrow$$

$$C_1$$

where $C_{2\epsilon_1 2\epsilon_2}$ is the conjugacy class of $x_{2\epsilon_1}(1)x_{2\epsilon_2}(1)$ and where $C_a \to C_b$ means $C_b \subset \overline{C}_a$ (see [11, Tables]).

Proposition 4.8. *Suppose* char $K = 2$. *If* $G = \mathrm{Sp}_4(K)$, *then*

$$\widehat{\mathfrak{B}} = C_1 \cup C_{2\epsilon_2} \cup C_{\epsilon_1-\epsilon_2}.$$

Proof. Let $g \in G$ and let $g = g_s g_u$ be the Jordan decomposition. If $g_s \neq 1$, then $g_s \notin \widehat{\mathfrak{B}}$ (the proof is the same as in the case char $K \neq 2$). Thus we need to check only the unipotent classes. The same arguments as in the case char $K \neq 2$ show that $C_{2\epsilon_1 2\epsilon_2} \subset \mathfrak{B}$, $C_{reg} \subset \mathfrak{B}$, $C_{2\epsilon_2} \subset \widehat{\mathfrak{B}}$. If $C_{\epsilon_1-\epsilon_2} \subset \mathfrak{B}$, then $c = \dot{w}_0 u$ for some $c \in C_{\epsilon_1-\epsilon_2}$, $\quad u \in U$. Since $c^2 = 1$ we have

$$1 = \underbrace{(\dot{w}_0 u \dot{w}_0)}_{\in U^-} u \Rightarrow u = 1 \Rightarrow c = \dot{w}_0 = \dot{w}_{2\epsilon_1}\dot{w}_{2\epsilon_2}, \quad \dot{w}_{2\epsilon_1}^2 = \dot{w}_{2\epsilon_2}^2 = 1.$$

The involution $x_{2\epsilon_1}(1)$ is conjugate in $\langle X_{\pm 2\epsilon_1}\rangle$ to $\dot{w}_{\epsilon_1}$ and the involution $x_{2\epsilon_2}(1)$ is conjugate in $\langle X_{\pm 2\epsilon_2}\rangle$ to $\dot{w}_{\epsilon_2}$. Hence the involution $x_{2\epsilon_1}(1)x_{2\epsilon_2}(1)$ is conjugate to c. Therefore, $c \in C_{2\epsilon_1 2\epsilon_2}$ and $c \in C_{\epsilon_1-\epsilon_2}$. This is a contradiction and therefore $C_{\epsilon_1-\epsilon_2} \not\subseteq \mathfrak{B}$. □

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