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SOME POINCARÉ-TYPE INEQUALITIES FOR FUNCTIONS OF BOUNDED DEFORMATION INVOLVING THE DEVIATORIC PART OF THE SYMMETRIC GRADIENT

Abstract. If $\Omega \subset \mathbb{R}^n$ is a bounded Lipschitz domain, we prove the inequality $\|u\|_1 \leqslant c(n)\,\mathrm{diam}(\Omega)\int\limits_\Omega |\varepsilon^D(u)|$ being valid for functions of bounded deformation vanishing on $\partial\Omega$. Here $\varepsilon^D(u)$ denotes the deviatoric part of the symmetric gradient and $\int\limits_\Omega |\varepsilon^D(u)|$ stands for the total variation of the tensor-valued measure $\varepsilon^D(u)$. Further results concern possible extensions of this Poincaré-type inequality.

**Dedicated to the jubilee of
G. A. Seregin**

Poincaré-type inequalities established by Strauss [25], Temam and Strang [27] and by Anzellotti and Giaquinta [2], in which certain integral norms of the deformation u are estimated in terms of the total variation of the strain tensor $\varepsilon(u)$, play an important role in the mathematical analysis of continuum media models. In particular, they create a technical ingredient of the fundamental research of Seregin on the regularity theory for problems from plasticity theory (see, e.g., [12–23] and [6]). The purpose of our note is to show that it is sometimes possible to replace $\varepsilon(u)$ in these inequalities through its deviatoric part. To be precise, suppose that we are given a bounded Lipschitz domain $\Omega \subset \mathbb{R}^n$, $n \geqslant 2$, and a field $u : \Omega \to \mathbb{R}^n$. We introduce the symmetric gradient of u

$$\varepsilon(u) := (\varepsilon_{ij}(u))_{1\leqslant i,j\leqslant n}\,, \quad \varepsilon_{ij}(u) := \frac{1}{2}(\partial_i u^j + \partial_j u^i),$$

and its deviatoric part

$$\varepsilon^D(u) := \varepsilon(u) - \frac{1}{n}(\mathrm{div}\, u)\mathbf{1}, \quad \mathbf{1} = (\delta_{ij})_{1\leqslant i,j\leqslant n},$$

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whenever these expressions make sense. We further denote by $W^1_s(\Omega;\mathbb{R}^n)$ the Sobolev space of all fields $u:\Omega\to\mathbb{R}^n$, which together with their first weak partial derivatives are in the Lebesgue class $L^s(\Omega;\mathbb{R}^n)$ for some exponent $s\in[1,\infty)$. Moreover, we consider the subspace $\overset{\circ}{W}{}^1_s(\Omega;\mathbb{R}^n)$ of $W^1_s(\Omega;\mathbb{R}^n)$ consisting of all functions vanishing on the boundary of Ω. For a more detailed definition and further properties of these spaces the reader is referred to the monograph of Adams [1]. Our first result is in some sense an extension of a Sobolev–Poincaré inequality obtained by Strauss (see [25, Theorem 1]):

Theorem 1. *There is a constant $c(n)$ such that*

$$\|u\|_{L^1(\Omega)} \leqslant c(n)\,\mathrm{diam}(\Omega)\|\varepsilon^D(u)\|_{L^1(\Omega)} \tag{1}$$

holds for any function $u\in\overset{\circ}{W}{}^1_1(\Omega;\mathbb{R}^n)$. If p is some number in $[1,\frac{n}{n-1})$, then for a suitable constant $c(n,p)$ we have

$$\|u\|_{L^p(\Omega)} \leqslant c(n,p)\,\mathrm{diam}(\Omega)^{1-n+\frac{n}{p}}\|\varepsilon^D(u)\|_{L^1(\Omega)} \tag{2}$$

for all fields $u\in\overset{\circ}{W}{}^1_1(\Omega;\mathbb{R}^n)$.

Remark 1. In his work Strauss discusses fields from the space $\overset{\circ}{W}{}^1_{\frac{n}{n-1}}(\Omega;\mathbb{R}^n)$ and proves ([25, Theorem 1])

$$\|u\|_{L^{\frac{n}{n-1}}(\Omega)} \leqslant C\|\varepsilon(u)\|_{L^1(\Omega)},$$

whereas in our case only the deviatoric part of the symmetric gradient occurs on the right-hand side. However, on the left-hand side of inequality (2) our techniques do not allow us to include the limit exponent $p=\frac{n}{n-1}$, and so it remains an interesting open question, if (2) is true for this choice of p.

Remark 2. Assume that $n\geqslant 3$ and fix an exponent $s\in(1,\infty)$. Then, according to Theorem 2 of Reshetnyak's deep paper [11], we have the Korn–type inequality

$$\|v-P(v)\|_{W^1_s(\Omega)} \leqslant C\|\varepsilon^D(v)\|_{L^s(\Omega)} \tag{3}$$

valid for all $v\in W^1_s(\Omega;\mathbb{R}^n)$ with a finite constant C depending on n, s and Ω. Here $P(v)$ denotes the projection of v on the kernel of ε^D (space of Killing vectors), which is of finite dimension. If v is smooth having, in addition, compact support in Ω, then it can be deduced from the representation formula (2.20) in [11] that $P(v)$ is constant (cf. proof of Theorem 1 for details), and we infer from (3)

$$\|\nabla v\|_{L^s(\Omega)} \leqslant C\|\varepsilon^D(v)\|_{L^s(\Omega)} \tag{4}$$

for all $v \in \overset{\circ}{W}{}^1_s(\Omega;\mathbb{R}^n)$. Combining (4) with Poincaré's inequality we obtain

$$\|v\|_{L^s(\Omega)} \leqslant C\|\varepsilon^D(v)\|_{L^s(\Omega)}$$

again for $v \in \overset{\circ}{W}{}^1_s(\Omega;\mathbb{R}^n)$, which is the "$L^s$-variant" of (1) for exponents $s > 1$. We emphasize that it is not possible to derive inequality (1) along these lines, since even $\int_\Omega |\varepsilon(u)|\,dx$ does not dominate each quantity $\int_\Omega |\partial u^i/\partial x_j|\,dx$ for arbitrary fields from $\overset{\circ}{W}{}^1_1(\Omega;\mathbb{R}^n)$. Counterexamples can be traced in the works of Mitjagin [9], de Leeuw and Mirkil [4] and of Ornstein [10].

Next we pass to the space $\mathrm{BD}(\Omega)$ consisting of all fields $u \in L^1(\Omega;\mathbb{R}^n)$ having bounded deformation introduced by Suquet [26] and by Matthies, Strang, and Christiansen [8] and further investigated by, e.g., Temam and Strang [27], and Anzellotti and Giaquinta [2] in the context of plasticity theory. According to Proposition 1.2 of [2] it holds

$$\int_{\mathbb{R}^n} |u|\,dx \leqslant c(n)\,\mathrm{diam}(\mathrm{spt}\,u)\int_{\mathbb{R}^n} |\varepsilon(u)|$$

for $u \in \mathrm{BD}(\mathbb{R}^n)$ having compact support, and we can state:

Theorem 2. *There is a constant $c(n)$ such that*

$$\|u\|_{L^1(\Omega)} \leqslant c(n)\,\mathrm{diam}(\Omega)\int_\Omega |\varepsilon^D(u)| \tag{5}$$

is satisfied for all fields $u \in \mathrm{BD}(\Omega)$ with $u|_{\partial\Omega} = 0$.

Here $u|_{\partial\Omega}$ denotes the trace of the function u in the sense of [27, Theorem 1.1]. The proof of Theorem 2 is easily obtained, if we accept Theorem 1 for the moment and follow the remarks stated in [2] after the proof of their Theorem 1.3: given u as above, there exists a sequence $u_m \in C^\infty(\Omega;\mathbb{R}^n) \cap \mathrm{BD}(\Omega)$ such that

(i) $u_m \to u$ in $L^1(\Omega;\mathbb{R}^n)$,

(ii) $u_m\big|_{\partial\Omega} = u\big|_{\partial\Omega} = 0$,

(iii) $\int_\Omega |\varepsilon^D(u_m)|\,dx \to \int_\Omega |\varepsilon^D(u)|$ as $m \to \infty$.

On account of (ii), we have inequality (1) for the sequence u_m, and by (i), (iii) we may pass to the limit $m \to \infty$ in order to obtain our claim (5). Before we present the proof of Theorem 1, we want to mention an additional related result being valid provided Ω is a finite union of domains Ω_i each of them being a star region with respect to a ball $B_i \subset \Omega_i$ (see [11]):

Theorem 3. *For a finite constant $c(n)$ we have the inequality ($\kappa = \kappa(u)$)*

$$\|u-\kappa\|_{L^1(\Omega)} \leqslant c(n)\ \mathrm{diam}(\Omega)\int_\Omega |\varepsilon^D(u)|$$

valid for all $u \in \mathrm{BD}(\Omega)$. In case $n = 2$, κ denotes a suitable holomorphic function, whereas for $n \geqslant 3$ κ is a Killing vector as explained, for example in [3, p. 537]. *For $p \in [1, \frac{n}{n-1})$, we also have*

$$\|u-\kappa\|_{L^p(\Omega)} \leqslant c(n,p)\,\mathrm{diam}(\Omega)^{1-n+\frac{n}{p}}\int_\Omega |\varepsilon^D(u)|\,.$$

We wish to note that the estimates from Theorem 3 correspond to the inequalities obtained in [27] and [2], in which the BD-distance of fields u from $\mathrm{BD}(\Omega)$ to the space of rigid motions is controlled through the total variation of the tensor-valued measure $\varepsilon(u)$. A proof of Theorem 3 for domains $\Omega \subset \mathbb{R}^2$ and functions u from the space $W^1_1(\Omega;\mathbb{R}^2)$ has been given in [5], and from this work the BD-variant follows by approximation. The higher dimensional case will be a consequence of the arguments needed for the proof of Theorem 1.

For proving Theorem 1 we first consider the case $u \in C^\infty_0(B;\mathbb{R}^n)$, B denoting the open unit ball. From Dain's paper [3] we quote the identity ($i = 1, \dots, n$)

$$\frac{1}{2}\Delta u^i = \sum_{j=1}^n \partial_j \varepsilon^D_{ij}(u) - \left(\frac{1}{2} - \frac{1}{n}\right)\partial_i(\mathrm{div}\, u),$$

which gives in combination with Green's representation formula

$$u^i(x) = \int_B \Gamma(y-x) 2 \left\{ \sum_{j=1}^n \partial_j \varepsilon^D_{ij}(u)(y) - \left(\frac{1}{2} - \frac{1}{n}\right) \partial_i (\operatorname{div} u)(y) \right\} dy \quad (6)$$

valid for all $x \in B$. Here Γ denotes the normalized fundamental solution of the Laplace equation (see, e.g. [7, (2.12)]). Now, if $n = 2$, the right-hand side of (6) equals

$$-2 \int_B \sum_{j=1}^n \frac{\partial}{\partial y_j} \Gamma(y-x) \varepsilon^D_{ij}(u)(y)\, dy,$$

and we can apply the theory of Riesz potentials (compare [24] or [7]) to deduce our claims (1) and (2) for $\Omega = B$ and u as above. Unfortunately this argument does not work in case $n \geqslant 3$, since then the right-hand side of (6) not only consists of terms involving $\varepsilon^D(u)$. Instead of (6) we now use a different representation, which is due to Reshetnyak [11]. According to formula (2.43) of this paper it holds

$$u(x) = P_2 u(x) + R_2(Q_2 u)(x), \quad x \in B, \quad (7)$$

where the quantities on the right-hand side of (7) have the following meaning: $P_2 u$ denotes a suitable Killing vector, i.e., an element of the kernel of ε^D, $Q_2 u$ is just the tensor $\varepsilon^D(u)$ and R_2 is the potential operator being defined in (2.41) of [11]. According to the structure of R_2 and the representation of its kernel stated in (2.42) of [11], we can apply the theory of Riesz potentials (see, e.g. [24] or [7]) to deduce

$$\|R_2(Q_2 u)\|_{L^1(B)} \leqslant c(n) \|Q_2 u\|_{L^1(B)}. \quad (8)$$

In order to continue we need more information concerning the projection $P_2 u$. Again we benefit from Reshetnyak's work: we use formula (2.20) and pass to the mean value $⨍_B \dots dy$ with respect to the variable $y \in B$ on the right-hand side. According to the comment given after (2.22) the i^{th} component of $P_2 u(x)$ is the remaining expression on the right-hand side, in which no integration with respect to the variable $t \in [0,1]$ is performed, i.e., we have the identity $(i = 1, \dots, n,\ x \in B)$

$$
\begin{aligned}
(P_2 u)^i(x) = & \fint_B u^i dy + \fint_B \sum_{j=1}^n \frac{1}{2}(\partial_j u^i - \partial_i u^j)(y)(x_j - y_j)\ dy \\
& + \fint_B \frac{1}{n} \operatorname{div} u(y)(x_i - y_i)\ dy \\
& + \fint_B \sum_{j=1}^n (x_j - y_j) \frac{1}{n} \partial_j \operatorname{div} u(y)(x_i - y_i)\ dy \\
& - \fint_B \frac{1}{2}|x-y|^2 \frac{1}{n} \partial_i \operatorname{div} u(y)\ dy.
\end{aligned}
\tag{9}
$$

Since u has compact support in B, we may integrate by parts on the right-hand side of (9) to get

$$
P_2 u \equiv \alpha(n)\overline{u}, \quad \overline{u} := \fint_B u\ dy, \quad \alpha(n) := 1 + \frac{n-1}{2} + \frac{n+1}{n}. \tag{10}
$$

With (10) we return to (7) and take $\fint_B \dots dx$ on both sides with the result

$$
\overline{u} = \alpha(n)\overline{u} + \fint_B R_2(Q_2 u)\, dx;
$$

hence

$$
|\overline{u}| \leqslant \frac{1}{\alpha(n) - 1} \fint_B |R_2(Q_2 u)|\, dx,
$$

and we can apply (8) to get (with constant $c(n)$ being defined in inequality (8))

$$
|\overline{u}| \leqslant c(n) \frac{1}{\alpha(n)-1} \frac{1}{|B|} \|\varepsilon^D(u)\|_{L^1(B)},
$$

which on account of (10) implies

$$
\|P_2 u\|_{L^1(B)} \leqslant c(n) \frac{\alpha(n)}{\alpha(n)-1} \|\varepsilon^D(u)\|_{L^1(B)}. \tag{11}
$$

By combining (7), (8), and (11) we finally arrive at

$$\|u\|_{L^1(B)} \leqslant 2c(n)\Big(1+\frac{n}{n^2+n+2}\Big)\|\varepsilon^D(u)\|_{L^1(B)}. \tag{12}$$

Suppose next that $u \in C_0^\infty(\Omega;\mathbb{R}^n)$ for a bounded Lipschitz domain Ω. Then we have $u \in C_0^\infty(B_R(x_0);\mathbb{R}^n)$ for a suitable ball $B_R(x_0) \supset \Omega$ with radius R equal to diam (Ω), and by using (12) our claim (1) for smooth u just follows by scaling. Moreover, we see that the constant $c(n)$ occurring in inequality (1) can be chosen in a more explicit form. Finally, if u is from the space $\overset{\circ}{W}{}_1^1(\Omega;\mathbb{R}^n)$ we apply (1) to a sequence $u_m \in C_0^\infty(\Omega;\mathbb{R}^n)$ such that $\|u_m - u\|_{W_1^1(\Omega)} \to 0$ as $m \to \infty$. In order to verify (2) we only observe that for $1 \leqslant p < \frac{n}{n-1}$, inequality (8) can be replaced by

$$\|R_2(Q_2u)\|_{L^p(B)} \leqslant c(n,p)\|Q_2u\|_{L^1(B)},$$

which is a well-known property of Riesz potentials. □

Next we prove Theorem 3 for the case $n \geqslant 3$: from [11], we deduce as before (compare (7) and (8))

$$\|u-\kappa\|_{L^1(\Omega)} \leqslant c(n)\text{diam }(\Omega)\|\varepsilon^D(u)\|_{L^1(\Omega)} \tag{13}$$

at least for smooth fields u with a suitable Killing vector $\kappa = \kappa(u)$. For $u \in \mathrm{BD}(\Omega)$ we can use the approximation argument of [2] stated after Theorem 2 (of course (ii)) now reads $u_m|_{\partial\Omega} = u|_{\partial\Omega}$) with the result that (13) is valid for the sequence u_m with corresponding Killing vectors κ_m. At the same time it holds (see (7))

$$u_m = \kappa_m + R_2(Q_2u_m),$$

which gives

$$\|\kappa_m\|_{L^1(\Omega)} \leqslant \|u_m\|_{L^1(\Omega)} + c(n,\Omega)\|\varepsilon^D(u_m)\|_{L^1(\Omega)};$$

hence

$$\sup_m \|\kappa_m\|_{L^1(\Omega)} < \infty.$$

Since the vectors κ_m belong to a space of finite dimension, this bound is enough to deduce that $\kappa_m \to: \kappa$ in $L^1(\Omega;\mathbb{R}^n)$ at least for a subsequence and a Killing vector κ. This proves our claim. □

We finish our discussion by mentioning an open problem: suppose that Γ is a subset of $\partial\Omega$ having positive $(n-1)$-dimensional measure. Do we have the validity of the inequality

$$\|u\|_{L^1(\Omega)} \leqslant c(n, \Gamma, \Omega) \int\limits_{\Omega} |\varepsilon^D(u)| \tag{14}$$

for all $u \in \mathrm{BD}(\Omega)$ such that $u|_\Gamma = 0$? A positive answer would provide a stronger result as stated in Corollary 1.11 of [2], but if we try to prove (14) by contradiction we do not have enough information to use the continuity of the trace operator (cf. the comments given in [2] after Theorem 1.4), which would lead to the desired contradiction. So, this open problem is in some sense related to the question if there is a reasonable concept of a trace for fields $u \in L^1(\Omega; \mathbb{R}^n)$ whose distributional deviator $\varepsilon^D(u)$ is a tensor-valued measure of finite total variation. However, a meaningful definition of boundary values for fields in this class seems to be impossible: let B denote the open unit disc centered at the origin and let

$$u : B \to \mathbb{C}, \quad u(z) := \frac{1}{z-1}.$$

Then u is in the space $L^1(B; \mathbb{C})$ and $\varepsilon^D(u) = 0$ on B holds, since u is holomorphic on B. If a trace $u|_{\partial B}$ of u in the space $L^1(\partial B; \mathbb{C})$ would exist, then it should hold

$$u(z) = u|_{\partial B}(z) \ \mathcal{H}^1 \text{ — a.e. on } \quad \partial B,$$

but this contradicts the fact that

$$\int\limits_{\partial B} \frac{1}{|z-1|} d\mathcal{H}^1(z) = +\infty.$$

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