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THE ORDER OF CONVERGENCE IN THE STEFAN PROBLEM WITH VANISHING SPECIFIC HEAT

Abstract. The paper is concerned with a two-phase Stefan problem with a small parameter ϵ which coresponds to the specific heat of the material. We assume that the initial condition does not coincide with the value at $t = 0$ of the solution to the limit problem related to $\varepsilon = 0$. To remove this discrepancy, we introduce an auxiliary boundary layer type function. We prove that the solution to the two-phase Stefan problem with parameter ϵ differs from the sum of the solution to the limit Hele–Shaw problem and the boundary layer type function by quantities of the order $O(\epsilon)$. The estimates are obtained in Hölder norms.

1. Introduction

Stefan problem with vanishing specific heat was considered in [1, 2]. The unique solvability of the Stefan problem with a small multiplier $\varepsilon \in (0, \varepsilon_0]$ at time derivative in the heat equation was proved on a certain time interval, independent of ε. Solution to the Stefan problem was compared with the solution to the Hele–Shaw problem, which corresponds to the case $\varepsilon = 0$ and can be regarded as a quasi-stationary approximation for the Stefan problem. In [1], to justify the quasi-stationary approximation, we considered the difference of solutions to the one-phase Stefan problem and to the Hele–Show problem in the case when the solutions do not coincide at the initial moment of time. We have shown that this difference is of the order $\mathcal{O}(\varepsilon) + \mathcal{O}(e^{-\frac{at}{\varepsilon}})$, $a > 0$. This was achieved by investigation of the boundary layer type function. Unfortunately, the proof contains a gap: it exploits an estimate of the solution to the Dirichlet problem, which in general may not hold. In the present paper (which can be regarded as a continuation of [1, 2]), we explain why this estimate is true in our case and extend the result to the two-phase case.

Key words and phrases: free boundary, Stefan problem, small parameter, boundary layer, Hölder norms.

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2. Formulation of the result

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with the smooth boundary S, and let Ω be splitted into two subdomains Ω_1 and Ω_2 by a closed surface Γ, $\partial\Omega_1 = S \cup \Gamma$, $\partial\Omega_2 = \Gamma$. The two-phase Stefan problem with a small parameter ε (specific heat) consists of finding a family of closed surfaces $\Gamma^\varepsilon(t)$, $t > 0$, and functions $u_1^\varepsilon(x,t)$, $u_2^\varepsilon(x,t)$, defined in the domains $\Omega_1^\varepsilon(t)$ and $\Omega_2^\varepsilon(t)$ with the boundaries $\partial\Omega_1^\varepsilon(t) = S \cup \Gamma^\varepsilon(t)$, $\partial\Omega_2^\varepsilon(t) = \Gamma^\varepsilon(t)$, from the following relations

$$\begin{aligned}
&\varepsilon \frac{\partial u_m^\varepsilon}{\partial t} - a_m^2 \Delta u_m^\varepsilon = 0, && x \in \Omega_m^\varepsilon(t), \quad t > 0,\\
&u_1^\varepsilon\Big|_S = b(x,t), &&\\
&u_1^\varepsilon\Big|_{\Gamma^\varepsilon(t)} = u_2^\varepsilon\Big|_{\Gamma^\varepsilon(t)} = 0, && \\
&\mathbf{V}_n^\varepsilon = -c_0\left(\lambda_1 \frac{\partial u_1^\varepsilon}{\partial n} - \lambda_2 \frac{\partial u_2^\varepsilon}{\partial n}\right) && \text{on} \quad \Gamma^\varepsilon(t),\\
&u_m^\varepsilon(x,0) = u_m^0, && x \in \Omega_m, \qquad m = 1,2.
\end{aligned} \tag{1}$$

Here we denote by $\mathbf{V}_n^\varepsilon$ the velocity of motion of the free surface $\Gamma^\varepsilon(t)$ in the direction of the normal $\mathbf{n}(t)$ to $\Gamma^\varepsilon(t)$, which is interior with respect to the domain $\Omega_1^\varepsilon(t)$; a_1, a_2, λ_1, λ_2, c_0 are positive constants, $\Gamma = \Gamma^\varepsilon(0)$ is the initial position of the free surface. The given functions b, u_m^0 satisfy the conditions

$$b \geq \delta_1 > 0, \quad \frac{\partial u_m^0}{\partial n}\Big|_\Gamma \geq \delta_2 > 0, \quad m = 1,2, \tag{2}$$

and the compatibility conditions of order zero

$$u_1^0\Big|_\Gamma = 0, \quad u_2^0\Big|_\Gamma = 0, \quad u_1^0\Big|_S = b(x,0). \tag{3}$$

Let $\Gamma \in C^{2+\alpha}$, $\operatorname{dist}(\Gamma, S) \geq \delta_3 > 0$. Let N be a smooth unit vector field such that $N(\xi)\cdot n(\xi) > 0$, $\xi \in \Gamma$. For sufficiently small $\lambda_0 > 0$ the distance from the set $G_\Gamma = \{\xi + N(\xi)\lambda \mid |\lambda| < \lambda_0\}$ to the surface S remains positive. The equation $x = \xi + N(\xi)\lambda$, $x \in G_\Gamma$ determines functions $\xi(x)$, $\lambda(x)$ of the class $C^{2+\alpha}$ (see [7]). We assume that for small t the free boundaries $\Gamma^\varepsilon(t)$ lie in the neighborhood G_Γ and can be presented by the equation

$$x = \xi + \varrho^\varepsilon(\xi,t) N(\xi). \tag{4}$$

For a fixed positive ε, classical solvability of the two-phase Stefan problem is well-known (see the bibliography in [4, 7]). In 2000, G. I. Bizhanova and V. A. Solonnikov [7] proved local in time existence of a solution to this problem, which is at least as regular as at the initial moment of time. It was done under minimal order of compatibility of the initial data and the boundary conditions. The result was achieved by using weighted Hölder spaces $C_\kappa^{l+\alpha,(l+\alpha)/2}$, whose elements admit singularities at $t=0$ (see [5, 7]). In [2], it was was proved that for sufficiently small ε, the solution to problem (1) exists on a certain small time interval independent on ε. This result was obtained by comparing solutions to the two-phase Stefan problem and the "zero specific heat limit problem," which is the same as in the one-phase case. Indeed, putting in (1) $\varepsilon = 0$, we arrive at the Hele–Shaw problem

$$\Delta w = 0, \quad x \in \Omega_1^0(t), \quad t \geqslant 0,$$

$$w\Big|_S = b(x,t), \quad w\Big|_{\Gamma^0(t)} = 0, \quad \mathbf{V}_n^0 = -c_0\lambda_1 \frac{\partial w}{\partial n}\Big|_{x\in\Gamma^0(t)}. \tag{5}$$

Here $\Omega_1^0(t) \subset \mathbb{R}^n$ is an unknown domain with the boundary $\partial\Omega_1^0(t) = \Gamma^0(t)\cup S$ consisting of two nonintersecting surfaces, where $\Gamma^0(t)$ is free and S is known; $\mathbf{V}_n^0$ is a velocity of motion of the surface $\Gamma^0(t)$ in the direction of the normal $\mathbf{n}^0(t)$ to $\Gamma^0(t)$, interior with respect the the domain $\Omega_1^0(t)$. We assume that $\Gamma^0(0) = \Gamma$, the constants c_0, λ_1, and the given function $b(x,t)$ are the same as in problem (1).

By $\widetilde{\mathcal{C}}\left([0,T);\mathcal{C}^l(\Omega(t))\right)$, we denote the space of functions continuous with respect to $(x,t) \in \{(x,t)|t\in[0,T), x\in\Omega(t)\}$ with a finite norm

$$\sup_{t<T} |u(\cdot,t)|_{\Omega(t)}^{(l)},$$

where l is a positive noninteger,

$$|u|_\Omega^{(l)} = \sum_{|j|<l} \sup_\Omega |D^j u(x)| + [u]_\Omega^{(l)}, \quad [u]_\Omega^{(l)} = \sum_{|j|=[l]} \sup_{x,y\in\Omega} \frac{|D^j u(x) - D^j u(y)|}{|x-y|^{l-[l]}}.$$

The following solvability result for the problem (5) has been proved in [3].

Theorem 1. *Assume that the given surfaces S and Γ belong to the class $\mathcal{C}^{2+\alpha}$ with some $\alpha \in (0,1)$. For any positive function $b \in$*

$\widetilde{\mathcal{C}}\left([0,T);\mathcal{C}^{2+\alpha}(S)\right)$, *the problem* (5) *has a unique solution* $(\Gamma^0(t), w(x,t))$, *which is defined on the time interval* $[0,t^*]$, $t^* \leqslant T$. *The corresponding free boundary* $\Gamma^0(t)$ *can be described by the equation*

$$x = \xi + \rho^0(\xi,t)N(\xi),$$

where $\xi \in \Gamma$, $N \in C^\infty(\Gamma)$ *is the unit vector field defined above,*

$$\rho^0\Big|_{t=0} = 0, \quad \rho^0 \in \widetilde{\mathcal{C}}\big([0,t^*);\mathcal{C}^{2+\alpha}(\Gamma)\big), \quad \frac{\partial\rho^0}{\partial t} \in \widetilde{\mathcal{C}}\big([0,t^*);\mathcal{C}^{1+\alpha}(\Gamma)\big).$$

The function $w(x,t)$ *is defined in the domain* $\{(x,t)\big|t\in[0,t^*], x\in\Omega_1^0(t)\}$ *and belongs to the space* $\widetilde{\mathcal{C}}\big([0,t^*);\mathcal{C}^{2+\alpha}(\overline{\Omega}\,{}_1^0(t))\big)$. *Moreover, for* $t \leqslant t^*$ *the following estimate*

$$\sup_{\tau<t}\left|w(\cdot,\tau)\right|^{(2+\alpha)}_{\overline{\Omega}(\tau)} + \sup_{\tau<t}\left|\rho^0(\cdot,\tau)\right|^{(2+\alpha)}_{\Gamma} + \sup_{\tau<t}\left|\frac{\partial\rho^0}{\partial\tau}(\cdot,\tau)\right|^{(1+\alpha)}_{\Gamma} \\ \le c\sup_{\tau<t}\left|b(\cdot,\tau)\right|^{(2+\alpha)}_{S}$$

holds, where c *is a positive constant depending on data and independent on the solution.*

Let us consider (5) at $t=0$. We have

$$\Delta w_0 = 0, \quad x\in\Omega_1,$$
$$w_0\Big|_S = b(x,0), \quad w_0\Big|_\Gamma = 0.$$

We do not assume that $\Delta u_i^0 = 0$, consequently w_0 may not coincide with the initial condition in problem (1). To remove this discrepancy at $t=0$, we introduce the auxiliary boundary layer type functions v_1^ε, v_2^ε as solutions to the following problems:

$$\varepsilon\frac{\partial v_1^\varepsilon}{\partial t} - a_1^2\Delta v_1^\varepsilon = 0, \qquad x\in\Omega_1^0(t), \qquad t\geqslant 0,$$
$$v_1^\varepsilon\Big|_{x\in S} = 0, \qquad v_1^\varepsilon\Big|_{\Gamma^0(t)} = 0, \qquad v_1^\varepsilon(x,o) = u_1^0 - w_0; \tag{6}$$
$$\varepsilon\frac{\partial v_2^\varepsilon}{\partial t} - a_2^2\Delta v_2^\varepsilon = 0, \qquad x\in\Omega_2^0(t) = \Omega\setminus\Omega_1^0(t), \qquad t\geqslant 0,$$
$$v_2^\varepsilon\Big|_{\Gamma^0(t)} = 0, \qquad v_2^\varepsilon(x,o) = u_2^0, \tag{7}$$

where $\Gamma^0(t)$, $t \in [0, t^*]$ is the same moving boundary as in the Hele–Shaw problem (5). In the present paper, we prove (under some additional assumptions) that the difference between solution to the Stefan problem (1) and the sum of the solution to the Hele–Shaw problem and the auxiliary boundary layer-type function is of the order $\mathcal{O}(\varepsilon)$ as $\varepsilon \to 0$.

To compare solutions of the free boundary problems, we need to reduce them to problems in one and the same domain. For this purpose, we apply the modification of the Hansawa coordinate transform [6], suggested by V. A. Solonnikov [7, 11].

We extend the vector field N into the neighborhood G_Γ by the formula $N(x) = N(\xi(x))$, and then into the domain Ω with preservation of the class. For any function $\varrho(\xi, t)$ defined on $\Gamma \times [0, T]$, we denote by $\varrho^*_m(y, t)$, $m = 1, 2$, an extension of this function into the domain $\Omega_m \times [0, T]$ with preservation of class and such that $\varrho^*_1\big|_S = 0$.

We introduce the coordinate transform

$$x = y + N(\xi)\varrho^{\varepsilon *}(y, t) \equiv e_{\varrho^{\varepsilon *}}(y, t). \tag{8}$$

The inverse transform maps the domains $\Omega^\varepsilon_1(t)$, $\Omega^\varepsilon_2(t)$ onto Ω_1 and Ω_2, respectively. Following the notation used in [1], we put $\varrho^{0*}_1 \equiv \Phi_1 = \Phi$, where Φ is the same extension which was constructed in [3], and denote ϱ^{0*}_2 by Φ_2. In [1], we proved that under some additional regularity assumptions on the data (which are also satisfied in Theorem 2 below)

$$D^2_x\Phi, \quad \frac{\partial \Phi}{\partial t} \in C^{\alpha, \alpha/2}(\overline{\Omega} \times [0, t]).$$

The coordinate transform e^{-1}_Φ, inverse to

$$x = y + N(y)\Phi(y, t) \equiv e_\Phi(y, t) \tag{9}$$

maps $\Omega^0_1(t)$ onto Ω_1, $\Omega \setminus \Omega^0_1(t)$ onto Ω_2 for $t \in [0, t^*]$. The Jacoby matrix of transform (9) has the entries

$$J_{\Phi, ik} = \delta_{ik} + N_i \frac{\partial \Phi}{\partial y_k} + \frac{\partial N_i}{\partial y_k}\Phi.$$

By J^{ik}_Φ we denote the elements of the inverse matrix J^{-1}_Φ. Clearly, the Jacobi matrix of transform (8) has a similar form.

With the help of the transform $e^{-1}_{\varrho^{\varepsilon *}}$, we pass from problem (1), to a problem in the domains Ω_1, Ω_2 for unknown functions $\varrho^{\varepsilon *}_m$ and

$\mathcal{V}_m(y,t) = u_m(e_{\varrho_m^{\varepsilon*}}(y,t),t)$. To simplify the notation in what follows, we omit the index ε in the unknown functions. We are looking for a solution in the form

$$\mathcal{V}_1 = \mathcal{W} + V_1 + \mathcal{U}_1, \quad \mathcal{V}_2 = V_2 + \mathcal{U}_2, \quad \varrho_1^* = \Phi_1 + \sigma_1, \quad \varrho_2^* = \Phi_2 + \sigma_2,$$

where $\mathcal{W} = w(e_{\Phi_1},t)$, $V_m = v(e_{\Phi_m},t)$, $m = 1,2, \mathcal{U}_m, \sigma_m$ are new unknown functions. Since up to now we have required only that the extensions ϱ_1^*, ϱ_2^* preserve the class and $\varrho_1^*\big|_S = 0$, we can assume additionally, to simplify the calculations, that σ_1, σ_2 are solutions to certain parabolic equations. Then, we arrive at the following relations:

$$\begin{aligned}
&\mathcal{L}_{\Phi_1+\sigma_1,\varepsilon}\mathcal{U}_1 = (\mathcal{L}_{\Phi_1,\varepsilon} - \mathcal{L}_{\Phi_1+\sigma,\varepsilon})(\mathcal{W}_1 + V_1)\\
&-\varepsilon\mathcal{W}_t + \varepsilon\frac{\partial\Phi_1}{\partial t}\left(N\cdot\left(J_\Phi^{-1}\right)^T\nabla\mathcal{W}\right) \quad \text{in} \quad \Omega_1, \quad t>0,\\
&\mathcal{U}_1\Big|_S = 0, \quad \mathcal{U}_1\Big|_\Gamma = 0, \quad \mathcal{U}_1\Big|_{t=0} = 0,\\
&L_{\Phi_m,\varepsilon}\sigma_m = 0, \quad \text{in} \quad \Omega_m, \quad m=1,2, \quad t>0,\\
&\sigma_1\Big|_S = 0, \quad \sigma_m\Big|_{t=0} = 0, \quad \sigma_1\Big|_\Gamma = \sigma_2\Big|_\Gamma,\\
&\mathcal{L}_{\Phi_2+\sigma_2,\varepsilon}\mathcal{U}_2 = (\mathcal{L}_{\Phi_2,\varepsilon} - \mathcal{L}_{\Phi_2+\sigma_2,\varepsilon})V_2 \quad \text{in} \quad \Omega_2, \quad t>0, \qquad (10)\\
&\mathcal{U}_2\Big|_\Gamma = 0, \quad \mathcal{U}_2\Big|_{t=0} = 0,\\
&\frac{\partial\sigma}{\partial t} + c_0\left(\lambda_1 B_{\Phi_1+\sigma_1}\frac{\partial\mathcal{U}_1}{\partial n} - \lambda_2 B_{\Phi_2+\sigma_2}\frac{\partial\mathcal{U}_2}{\partial n}\right)\\
&= -c_0\left(\lambda_1 B_{\Phi_1}\frac{\partial V_1}{\partial n} - \lambda_2 B_{\Phi_2}\frac{\partial V_2}{\partial n}\right)\\
&+c_0\left(\lambda_1(B_{\Phi_1} - B_{\Phi_1+\sigma_1})\frac{\partial(\mathcal{W}+V_1)}{\partial n} - \lambda_2(B_{\Phi_2} - B_{\Phi_2+\sigma_2})\frac{\partial V_2}{\partial n}\right) \quad \text{on} \quad \Gamma.
\end{aligned}$$

Here, we use the notation

$$\begin{aligned}
&\mathcal{L}_{\Phi_m,\varepsilon}\mathcal{U}_m = L_{\Phi_m,\varepsilon}\mathcal{U}_m - \left(N\cdot(J_\Phi^{-1})^T\nabla\mathcal{U}\right)_{\Phi_m,\varepsilon}\Phi_m - K(\Phi_m)\cdot\nabla\mathcal{U}_m,\\
&L_{\Phi_m,\varepsilon}\mathcal{U}_m = \varepsilon\frac{\partial\mathcal{U}}{\partial t} - L_{\Phi_m,0}\mathcal{U}_m,\\
&L_{\Phi_m,0}\mathcal{U}_m = \sum_{i,j=1}^n\sum_{k=1}^n J_{\Phi_m}^{ik}J_{\Phi_m}^{jk}\frac{\partial\mathcal{U}_m}{\partial y_i\partial y_j},
\end{aligned}$$

$$(K(\Phi_m))_k = a_m^2 \sum_{i,j,l,q=1}^{n} J_{\Phi_m}^{li} J_{\Phi_m}^{kj} J_{\Phi_m}^{qi} \times \left(\frac{\partial N_j}{\partial y_q} \frac{\partial \Phi_m}{\partial y_l} + \frac{\partial N_j}{\partial y_l} \frac{\partial \Phi_m}{\partial y_q} + \Phi_m \frac{\partial^2 N_j}{\partial y_l \partial y_q} \right),$$

$$B_\Phi = \big(N \cdot (J_\Phi^{-1})^T\big)^{-1} \big(n \cdot (J_\Phi^{-1} (J_\Phi^{-1})^T) n\big).$$

In [2], we have proved the existence of a solution to problem (10) on a certain finite time interval independent of ε for $\varepsilon \in (0, \varepsilon_0)$. To avoid compatibility conditions of the first order, we used the weighted Hölder spaces, whose elements admit singularities at $t = 0$. Following the notation of [1, 2], we denote by $X^\varepsilon[u, \Omega]$ the weighted norm

$$X^\varepsilon[u,\Omega](t) = X_\kappa[u,\Omega](t), \quad t \in (0, \varepsilon],$$
$$X^\varepsilon[u,\Omega](t) = X_\kappa[u,\Omega](\varepsilon) + X_0[u,\Omega,[\varepsilon,t]], \quad t \in (\varepsilon, T],$$

where

$$X_\kappa[u,\Omega](t) = \sup_{\tau<t} |u(\cdot,\tau)|_{\mathcal{C}^{1+\kappa}(\overline{\Omega})} + \varepsilon^{\frac{1+\kappa}{2}} [u]_{t,\overline{\Omega}\times[0,t]}^{(\frac{1+\kappa}{2})}$$
$$+ \sup_{\tau<t} \left(\frac{\tau}{\varepsilon}\right)^{\frac{1-\kappa}{2}} \Big(\sup_{\overline{\Omega}} |D^2 u(x,\tau)| + \varepsilon \sup_{\overline{\Omega}} |u_\tau(x,\tau)| \Big)$$
$$+ \sup_{\tau<t} \left(\frac{\tau}{\varepsilon}\right)^{\frac{1+\alpha-\kappa}{2}} \Big([D^2 u(\cdot,\tau)]_{\overline{\Omega}}^{(\alpha)} + \varepsilon [u_\tau(\cdot,\tau)]^{(\alpha)}$$
$$+ \varepsilon^{\frac{\alpha}{2}} [D^2 u]_{\tau,\overline{\Omega}\times[\tau/2,\tau]}^{(\alpha/2)} + \varepsilon^{1+\alpha/2} [u_\tau]_{\tau,\overline{\Omega}\times[\tau/2,\tau]}^{(\alpha/2)} \Big)$$

and

$$X_0[u,\Omega,[a,b]] = \sup_{\tau\in[a,b]} |u(\cdot,\tau)|_{C^{2+\alpha}(\overline{\Omega})} + \varepsilon \sup_{\tau\in[a,b]} |u_\tau(\cdot,\tau)|_{C^2(\overline{\Omega})}$$
$$+ \varepsilon^{\alpha/2} [D^2 u]_{t,\overline{\Omega}\times[a,b]}^{(\alpha/2)} + \varepsilon^{1+\alpha/2} [u_t]_{t,\overline{\Omega}\times[a,b]}^{(\alpha/2)}.$$

We use the usual Hölder norm in $[\varepsilon, t]$ is order to avoid growth of the coefficients of the type $\left(\frac{\tau}{\varepsilon}\right)^\gamma$, $\gamma > 0$, for $\tau > \varepsilon$ as $\varepsilon \to 0$. For fixed positive ε, the norm $X^\varepsilon[u,\Omega]$ is equivalent to the norm of the weighted space $\mathcal{C}_{1+\kappa}^{2+\alpha,1+\alpha/2}$. Now, we are in a position to formulate the main result of the present paper.

Theorem 2. *Let* $\Gamma, S \in C^{3+\alpha}$, $\alpha \in (0,1)$, $b \in C^{3+\alpha,\frac{3+\alpha}{2}}(S \times [0,t^*])$, $b_t \in \mathcal{C}([0,t^*]; C^{2+\alpha}(S))$. *If* $u_1^0 - w_0 \in C^{3+\alpha}(\Omega_1)$, $u_2^0 \in C^{3+\alpha}(\Omega_2)$, *and*

$$\Delta u_1^0 = \Delta w_0 \quad on \quad \Gamma \cup S, \quad \Delta u_2^0\big|_\Gamma = 0, \tag{11}$$

then, for the sufficiently small ε, *the estimate*

$$\sum_{m=1,2} \big(X^\varepsilon[\mathcal{U}_m, \Omega_m](t) + X^\varepsilon[\sigma_m, \Omega_m](t) \big) \le C\varepsilon, \quad t < t_0, \tag{12}$$

holds with the constant $C = C(b, u_m^0)$ *independent of* ε.

This implies that the solution to problem (1) differs from the sum of the solution to the Hele–Shaw problem and boundary layer type functions V_1, V_2 by quantities of order $\mathcal{O}(\varepsilon)$ as ε tends to zero.

Remark 1. Assumption (11) is not necessary. We make it to have in (27) the compatibility conditions of the first order which simplifies the calculations.

3. Estimates of the boundary layer type functions

In this section, we consider the functions $V_m = v(e_{\Phi_m}, t)$, which we call boundary layer type functions. Making use of the coordinate transform inverse to (9), we pass from (6) and (7) to problems

$$\mathcal{L}_{\Phi_1,\varepsilon} V_1 = 0, \quad x \in \Omega_1, \quad t > 0, \tag{13}$$

$$V_1\Big|_S = 0, \quad V_1\Big|_\Gamma = 0, \quad V_1\Big|_{t=0} = u_1^0 - w_0 \equiv V_1^0;$$

$$\mathcal{L}_{\Phi_2,\varepsilon} V_2 = 0, \quad x \in \Omega_2, \quad t > 0, \tag{14}$$

$$V_2\Big|_\Gamma = 0, \quad V_2\Big|_{t=0} = u_2^0 \equiv V_2^0.$$

We note that Dirichlet problems similar to (13), (14) has been studied in [1, Sec. 3] by the method suggested in [13]. Lemma 5 [1] implies that the solutions admit the exponential decay in time, namely

$$\| V_m(\cdot, t) \|_{L_2(\Omega)} \le c \| V_m^0 \|_{L_2(\Omega)} e^{-at/\varepsilon}, \quad a > 0, \quad m = 1, 2. \tag{15}$$

In accordance with (11), we have

$$\Delta V_m^0\Big|_\Gamma = 0, \quad \Delta V_1^0\Big|_S = 0.$$

It means that compatibility conditions of the first order are fulfilled in (13), (14); for $V_m^0 \in C^{2+\alpha}$, solutions to problems (13), (14) satisfy the estimates

$$\begin{aligned} &\varepsilon \sup_{[0,t]} \left| \frac{\partial V_m}{\partial \tau}(\cdot,\tau)\right|^{(\alpha)}_{\overline{\Omega}_m} + \sup_{[0,t]} |V_m(\cdot,\tau)|^{(2+\alpha)}_{\overline{\Omega}_m} \\ &+ \varepsilon^{\alpha/2}\left(\sum_{|\beta|\le 2} [D_x^\beta V_m]_t^{(\alpha/2)} + \varepsilon \left[\frac{\partial V_m}{\partial t}\right]_t^{(\alpha/2)}\right) \le c\left|V_m^0\right|_{C^{2+\alpha}}. \end{aligned} \tag{16}$$

Lemma (Proposition 1 [1]). *A solution to problem* (13) *is subject to the estimate*

$$\begin{aligned} &\varepsilon \sup_{t-\delta<\tau<t} |\frac{\partial V_1}{\partial \tau}(\cdot,\tau)|_{C^\alpha(\overline{\Omega}_1)} + \sup_{t-\delta<\tau<t} |V_1(\cdot,\tau)|_{C^{2+\alpha}(\overline{\Omega}_1)} \\ &\le c_1 \varepsilon \delta^{-1}\left(1+\varepsilon\delta^{-1}\right)^\beta \sup_{t-2\delta<\tau<t} \| V_1(\cdot,\tau) \|_{L_2(\Omega_1)}, \end{aligned} \tag{17}$$

where $t \in (\varepsilon, t_0)$, $\delta = t/4$, $\beta > 0$, *the constant* c_1 *is independent of* ε.

In the proof of Proposition 1 in [1], we have used the estimate of the solution to the Dirichlet problem in the norms of the space $\widetilde{C}([0,t_0], C^{2+\alpha}(\overline{\Omega}))$. We note that in the general case [8, 10] this estimate may be not true. Below we explain why in the case important for our analysis such an estimate can be indeed exploited. The right-hand sides of the equations satisfy the boundary conditions in all the problems to whose solutions we apply this estimate. Here we prove that in this case Hölder estimates with respect to spatial variables take place.

Let $\Omega \in \mathbb{R}^n$, $\partial\Omega \in C^{2+\alpha}$. We consider the Dirichlet problem

$$\begin{aligned} &\frac{\partial u}{\partial t} - Au = f, \qquad x \in \Omega, \qquad t > 0, \\ &u\Big|_{\partial\Omega} = 0, \qquad u\Big|_{t=0} = 0, \qquad x \in \Omega. \end{aligned} \tag{18}$$

Here

$$Au = \sum_{i,j=1}^n a_{ij}\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^n a_i \frac{\partial u}{\partial x_i} + au$$

is a uniformly elliptic operator with the coefficients from the space $C^{\alpha,\alpha/2}$.

Lemma 2. *Let* $f \in \widetilde{C}([0,T], C^\alpha(\overline{\Omega}))$*, and let the condition*

$$f\Big|_{\partial\Omega} = 0 \tag{19}$$

be satisfied. Then the solution to problem (18) *satisfies the inequality*

$$\sup_{\tau\le t} |u_\tau(\cdot,\tau)|_\Omega^{(\alpha)} + \sup_{\tau\le t} |u(\cdot,\tau)|_\Omega^{(2+\alpha)} \le c \sup_{\tau\le t} |f(\cdot,\tau)|_\Omega^{(\alpha)}, \quad t \le T. \tag{20}$$

Remark 2. This result for the case when the coefficients of the operator A are independent of t was established in [10, 8] by methods of semigroups theory.

Proof. We extend f and coefficients of the operator A into the whole space $\mathbb{R}^n$ with preservation of class and denote these extensions by $\widetilde{f}$, $\widetilde{a_{i,j}}$, $\widetilde{a}_i$, $\widetilde{a}$. We can also assume that after the extension the coefficients of the operator still satisfy the ellipticity condition. We consider the Cauchy problem

$$\begin{aligned} &\frac{\partial \widetilde{u}}{\partial t} - \widetilde{A}\widetilde{u} = \widetilde{f}, \quad x \in \mathbb{R}^n, \quad t > 0, \\ &\widetilde{u}\Big|_{t=0} = 0, \quad x \in \mathbb{R}^n. \end{aligned} \tag{21}$$

It is known [8, Theorem 5.1.9], [9], that the Hölder estimate with respect to the spatial variables for the solution to the Cauchy problem holds also in the case of time dependent coefficients. Namely, the solution to problem (21) satisfies the estimate

$$\begin{aligned} &\sup_{\tau\le t} |\widetilde{u}_\tau(\cdot,\tau)|_{C^\alpha(\mathbb{R}^n)} + \sup_{\tau\le t} |\widetilde{u}(\cdot,\tau)|_{C^{2+\alpha}(\mathbb{R}^n)} \\ &\le c \sup_{\tau\le t} |\widetilde{f}(\cdot,\tau)|_{C^\alpha(\mathbb{R}^n)} \le c \sup_{\tau\le t} |f(\cdot,\tau)|_{C^\alpha(\Omega)}. \end{aligned} \tag{22}$$

It is clear that $v = u - \widetilde{u}$ can be found from the following relations:

$$\begin{aligned} &\frac{\partial v}{\partial t} - Av = 0, \quad x \in \Omega, \quad t > 0, \\ &v\Big|_{\partial\Omega} = \widetilde{u}\Big|_{\partial\Omega}, \quad v\Big|_{t=0} = 0, \quad x \in \Omega. \end{aligned} \tag{23}$$

By the well-known Hölder estimate, we have

$$\begin{aligned}
&\sup_{\tau\le t}\big|v_\tau(\cdot,\tau)\big|_{C^\alpha(\overline{\Omega})}+\sup_{\tau\le t}|v(\cdot,\tau)|_{C^{2+\alpha}(\overline{\Omega})}\\
&\le c\big|\widetilde{u}\big|_{C^{2+\alpha,1+\alpha/2}(\partial\Omega\times[0,t])}\\
&\le c\Big(\sup_{\tau\le t}\big|\widetilde{u}_\tau(\cdot,\tau)\big|_{C^\alpha(\overline{\Omega})}+\sup_{\tau\le t}\big|\widetilde{u}(\cdot,\tau)\big|_{C^{2+\alpha}(\overline{\Omega})}\\
&+\sup_{\Omega}\big|\widetilde{u}_{xx}(x,\cdot)\big|_t^{(\alpha/2)}+\sup_{\Omega}\big|\widetilde{u}_t(x,\cdot)\big|_t^{(\alpha/2)}\Big).
\end{aligned}\tag{24}$$

In view of the assumption (19), we observe that $\widetilde{f}\Big|_{\partial\Omega}=0$. Consequently,

$$\widetilde{u}_t\Big|_{\partial\Omega}=A\widetilde{u}\Big|_{\partial\Omega}.$$

Thus, to estimate the right-hand side of (24) by the left-hand side of (22), we need only to consider the term $\sup_{\Omega}|\widetilde{u}_{xx}(x,\cdot)|_t^{(\alpha/2)}$. To this term we apply the inequality (see [12])

$$\sup_{\Omega}|D^2u(x,\cdot)|^{(\alpha/2)}\le c\Big(\sup_{\tau\le t}|u_\tau(\cdot,\tau)|_{C^\alpha(\Omega)}+\sup_{\tau\le t}|u(\cdot,\tau)|_{C^{2+\alpha}(\Omega)}\Big).\tag{25}$$

The inequalities (22), (24), and (25) imply (20). □

Now we prove a result similar to Lemma 6 in [1].

Lemma 3. *Under the assumptions of Theorem* 2*, solutions to the problems* (13), (14) *satisfy the estimate*

$$\int_0^T\big|V_m(\cdot,t)\big|_{C^{3+\alpha}(\Gamma)}\,dt\le c_2\varepsilon\big|V_m^0\big|_{C^{3+\alpha}(\overline{\Omega}_m)},\quad T\le t_0,\tag{26}$$

where the constant c_2 is independent of ε.

Proof. Let

$$\partial_k=\frac{\partial}{\partial y_k}-M_k(y)\sum_{i=1}^n M_i(y)\frac{\partial}{\partial y_i},$$

where $M_k\in C^{3+\alpha}(\Omega)$ is an extension of the normal components n_k from $\Gamma\cup S$ into Ω. As $\partial_k f\Big|_{\Gamma\cup S}$ depends only on $f\Big|_{\Gamma\cup S}$, this operator can be applied to the functions defined only on the boundary. We apply operator ∂_k to (13), (14) and arrive at

$$\varepsilon\frac{\partial}{\partial t}(\partial_k V_m) - \mathcal{L}_{\Phi_m,0}\partial_k V_m = Q^k(V_m), \quad y \in \Omega_m, \quad t \geqslant 0,$$

$$\partial_k V_m\Big|_\Gamma = 0, \quad \partial_k V_1\Big|_S = 0, \quad \partial_k V_m\Big|_{t=0} = \partial_k V_m^0, \tag{27}$$

where the commutator

$$Q^k(V_m) = \partial_k\left(\mathcal{L}_{\Phi_m,0}V_m\right) - \mathcal{L}_{\Phi_m,0}\partial_k V_m$$

depends on $D_y^2 V_m$, $D_y V_m$, V_m. It contains the derivatives not only with respect to tangential directions and may not satisfy the homogeneous Dirichlet boundary condition. Hence, we have to use usual Hölder estimate. As $\Phi_m\Big|_{t=0} = 0$, we conclude (by the assumption (11)) that $Q^k(V_m^0) = 0$. It means that the compatibility conditions of the first order are fulfilled. With the help of (16), we obtain

$$\begin{aligned}
&\sup_{\tau<t}|\partial_k V_m(\cdot,\tau)|^{(2+\alpha)} \le c\Big(\sup_{\tau<t}|Q^k(V_m)|_{C^\alpha(\overline{\Omega}_m)} \\
&+ \varepsilon^{\alpha/2}\sup_{\Omega}|Q^k(V_m)|^{(\alpha/2)}_{[0,t]} + |\partial_k V_m^0|^{(2+\alpha)}\Big) \\
&\le c\Big(\sup_{\tau<t}|V_m(\cdot,\tau)|^{(2+\alpha)} + \varepsilon^{\alpha/2}\sum_{|\beta|\le 2}|D_y^\beta V_m)|^{(\alpha/2)} + |V_m^0|^{(3+\alpha)}\Big) \\
&\le c|V_m^0|^{(3+\alpha)}.
\end{aligned} \tag{28}$$

Let ω_δ be a smooth monotone cut-off function, which is equal to zero for $\tau < t-\delta$, equals 1 for $\tau > t-\delta/2$, and satisfies the conditions

$$|\omega_\delta'(t)| \le \frac{c}{\delta}, \quad |\omega_\delta|^{(1+\alpha/2)} \le \frac{c}{\delta^{1+\alpha/2}}.$$

For $\partial_k V_m \omega_\delta = U_m^k$, we have the following problem:

$$\varepsilon\frac{\partial}{\partial t}U_m^k - \mathcal{L}_{\Phi_m,0}U_m^k = Q^k(\omega_\delta V_m) + \varepsilon\partial_k V_m\omega_\delta', \quad x \in \Omega_m, \quad t \geqslant 0,$$

$$U_m^k\Big|_\Gamma = 0, \quad U_1^k\Big|_S = 0, \quad U_m^k\Big|_{t=0} = 0,$$

$$m = 1, 2; \quad k = 1, 2, \dots, n. \tag{29}$$

Taking into account properties of the cut-off function, we deduce

$$\sup_{t-\delta/2<\tau<t} |\partial_k V_m(\cdot,\tau)|^{(2+\alpha)} \le \sup_{t-\delta<\tau<t} |U_m^k(\cdot,\tau)|^{(2+\alpha)}$$

$$\le c\Big(\sup_{t-\delta<\tau<t} |Q^k(V_m)|^{(\alpha)} + \varepsilon^{\alpha/2} \sup_{\Omega} \big[Q^k(V_m)\big]^{(\alpha/2)}_{[t-\delta,t]} \tag{30}$$

$$+\varepsilon\delta^{-1} \sup_{t-\delta<\tau<t} |\partial_k V_m|_{C^\alpha(\overline{\Omega}_i)} + \varepsilon\delta^{-1} \sup_{\overline{\Omega}_i} \big[\partial_k V_m\big]^{(\alpha/2)}_{[t-\delta,t]}$$

$$+(\varepsilon\delta^{-1})^{1+\alpha/2} \sup_{\overline{\Omega}_m \times [t-\delta,t]} |\partial_k V_m|\Big).$$

The first term on the right-hand side of (30) is not lesser than the quantity

$$c \sup_{t-\delta<\tau<t} |V_m(\cdot,\tau)|_{C^{2+\alpha}(\overline{\Omega}_i)}.$$

To estimate the second term, we use the same method as in [12]. We use the inequality

$$\sup_{\Omega} |D_x^2 u| \le c\bigg(\Big[\frac{\partial u}{\partial t}\Big]^{(\alpha)} \bigg)^{\alpha/2} \big([u]_x^{(2+\alpha)}\big)^{1-\alpha/2}$$

in order to estimate the difference

$$\varepsilon^{\alpha/2} \frac{|D_x^2 V_m(x,t) - D_x^2 V_m(x,\tau)|}{|t-\tau|^{\alpha/2}}$$

by the Young inequality. We obtain

$$\varepsilon^{\alpha/2} \sup_{\Omega} |D_y^2 V_m|^{(\alpha/2)}_{[t-\delta,t]}$$

$$\le c\bigg(\varepsilon \sup_{t-\delta<\tau<t} \Big| \frac{\partial}{\partial\tau} V_m(\cdot,\tau)\Big|^{(\alpha)}_{\overline{\Omega}_m} + \sup_{t-\delta<\tau<t} |V_m(\cdot,\tau)|^{(2+\alpha)}_{\overline{\Omega}_m} \bigg). \tag{31}$$

It is clear that $\sup_{\Omega} \big|D_y^1 V_m\big|^{(\alpha/2)}_{[t-\delta,t]}$, $\sup_{\Omega} \big|V_m\big|^{(\alpha/2)}_{[t-\delta,t]}$ can be estimated by the right-hand side of (31) with the help of (31) and interpolation inequalities.

Making use of (30), (31), and Lemma 1 (which is evidently valid also for V_2), we have:

$$\sup_{t-\delta/2<\tau<t} |\partial_k V_m(\cdot,\tau)|^{(2+\alpha)} \le c\varepsilon\delta^{-1}(1+\varepsilon\delta^{-1})^{\beta}$$
$$\times\Big(1+\varepsilon\delta^{-1}+(\varepsilon\delta^{-1})^{1+\alpha/2}\Big)\sup_{t-2\delta<\tau<t} \| V_m(\cdot,\tau)\|_{L_2(\Omega_i)}\,. \quad (32)$$

To obtain (26), we use (28) on $[0,\varepsilon]$ and (32) together with (15) on $[\varepsilon,T]$, taking into account that $\varepsilon\delta^{-1}(t)\le 4$, since $\delta(t)=t/4$, We have

$$\int_0^T |V_m(\cdot,t)|_{C^{3+\alpha}_\Gamma} dt \le c\int_0^T \sum_{k=1}^n |\partial_k V_m(\cdot,t)|_{C^{2+\alpha}_{\overline{\Omega}_m}}\, dt$$
$$\le c\Big(\int_0^\varepsilon |V^0_m|_{C^{3+\alpha}} dt + \int_\varepsilon^T \sum_{k=1}^n |\partial_k V_m(\cdot,t)|_{C^{2+\alpha}_{\overline{\Omega}_m}}\, dt\Big)$$
$$\le c\varepsilon|V^0_m|_{C^{3+\alpha}} + c\varepsilon \,\| V^0_m \|_{L_2(\Omega_i)} \int_\varepsilon^T \delta^{-1}(t) \quad (33)$$
$$\times\Big(1+\varepsilon\delta^{-1}(t)\Big)^{\beta}\Big(1+\varepsilon\delta^{-1}(t)+(\varepsilon\delta^{-1}(t))^{1+\alpha/2}\Big)e^{-\frac{at}{2\varepsilon}}dt \le c\varepsilon|V^0_m|_{C^{3+\alpha}}.$$

□

4. Proof of Theorem 2

To show that the solution to problem (10) is of the order $\mathcal{O}(\varepsilon)$, we must estimate the term

$$\psi \equiv c_0\Big(\lambda_2 B_{\Phi_2}\frac{\partial V_2}{\partial n} - \lambda_1 B_{\Phi_1}\frac{\partial V_1}{\partial n}\Big)\Big|_\Gamma$$

on the right-hand side of the dynamic boundary condition on Γ. The term ψ is not of the order $\mathcal{O}(\varepsilon)$, but this difficulty can be avoided with the help of Lemma 3. We introduce the auxiliary functions η_m, satisfying the following relations

$$\begin{aligned} &L_{\Phi_m,\varepsilon}\eta_m = 0 \quad \text{in} \quad \Omega_m, \quad t>0,\\ &\eta_1\Big|_S = 0, \qquad \eta_m\Big|_{t=0} = 0 \quad \text{in} \quad \Omega_m, \\ &\eta_m\Big|_\Gamma = \int_0^t \psi(\tau)d\tau \equiv g, \qquad m=1,2. \end{aligned} \quad (34)$$

We estimate the following Hölder norm of the function g:

$$\varepsilon \sup_{\tau<t} \left|\frac{\partial}{\partial \tau} g(\cdot,\tau)\right|_{\Gamma}^{(\alpha)} + \sup_{\tau<t} |g(\cdot,\tau)|_{\Gamma}^{(2+\alpha)} + \varepsilon^{\alpha/2} \sum_{|\beta|\le 2} \left|D_y^{\beta} g\right|_t^{(\alpha/2)} + \varepsilon^{1+\alpha/2} \left|\frac{\partial g}{\partial t}\right|_t^{(\alpha/2)} = I_1 + I_2 + I_3 + I_4 \tag{35}$$

and prove that it has the order $\mathcal{O}(\varepsilon)$. Indeed, making use of (16), and taking into account the fact that $\frac{\partial}{\partial t} g(\cdot,t) = \psi(\cdot,t)$, we obtain

$$\begin{aligned} I_1 &\le \varepsilon \sup_{\tau<t} |\psi(\cdot,\tau)|_{\Gamma}^{(\alpha)} \le c\varepsilon \sup_{\tau<t} \sum_{m=1,2} \left|\frac{\partial V_m}{\partial n}\right|_{\Gamma}^{(\alpha)} \\ &\le c\varepsilon \sup_{\tau<t} \sum_{m=1,2} \left|V_m\right|_{\Omega_m}^{(1+\alpha)} \le c\varepsilon \left|V_m^0\right|_{C^{2+\alpha}}, \end{aligned}$$

$$I_4 \le c\varepsilon\varepsilon^{\alpha/2} \sum_{m=1,2} \left[\frac{\partial V_m}{\partial n}\right]_t^{(\alpha/2)} \le c\varepsilon \sum_{m=1,2} \left|V_m^0\right|_{C^{2+\alpha}}.$$

To estimate I_2 and I_3, we apply inequalities (26) and (28), which gives

$$\begin{aligned} I_2 &\le \sup_{\tau<t} \int_0^{\tau} \left|\psi(\cdot,\xi)\right|_{\Gamma}^{(2+\alpha)} d\xi \le c \sum_{m=1,2} \int_0^t \left|\frac{\partial V_m}{\partial n}\right|_{\Gamma}^{(2+\alpha)} d\xi \\ &\le c \sum_{m=1,2} \int_0^t \left|V_m(\cdot,\xi)\right|_{C_{\Gamma}^{3+\alpha}} d\xi \le c\varepsilon \sum_{m=1,2} \left|V_m^0\right|_{C^{3+\alpha}}, \end{aligned}$$

$$\begin{aligned} I_3 &\le c\left(\varepsilon^{\alpha/2} \sum_{|\beta|\le 2} \sup_{\substack{\tau_1,\tau_2<t,\\ |\tau_1-\tau_2|\le\varepsilon}} \frac{\left|\int_{\tau_1}^{\tau_2} D_x^{\beta}\psi(x,\xi)d\xi\right|}{|\tau_1-\tau_2|^{\alpha/2}} + \sum_{m=1,2} \int_0^t |V_m(\cdot,\xi)|_{\Gamma}^{(3+\alpha)} d\xi\right) \\ &\le c\left(\varepsilon^{\alpha/2} \sum_{|\beta|\le 2} \varepsilon^{1-\alpha/2} \sup_{\tau<t} \sup_{\Gamma} \left|D_x^{\beta}\psi\right| + \varepsilon \sum_{m=1,2} \left|V_m^0\right|_{C^{3+\alpha}}\right) \\ &\le c\varepsilon \sum_{m=1,2} \left|V_m^0\right|_{C^{3+\alpha}}. \end{aligned}$$

By the Hölder estimate for the solution of the Dirichlet problem, we have

$$\sum_{m=1,2} X^{\varepsilon}[\eta_m,\Omega_m](t) \leq cX^{\varepsilon}[g,\Gamma](t) \leq c\,(I_1+I_2+I_3+I_4) \\ \leq c\varepsilon \sum_{m=1,2} \left|V_m^0\right|^{(3+\alpha)}, \quad m=1,2. \tag{36}$$

We separate the linear part in (10) in the same way as it was done in [2]. Then for $\mathcal{U}_m$, $s_m=\sigma_m-\eta_m$ we have

$$\begin{aligned}
&\mathcal{L}_{\Phi_1,\varepsilon}\mathcal{U}_1 + h_1\cdot\nabla s_1 + g_1 s_1 \\
&= \varepsilon f - h_1\cdot\nabla\eta_1 - g_1\eta_1 + \mathcal{F}_1(\mathcal{U}_1, s_1+\eta_1) \quad \text{in} \quad \Omega_1, \quad t>0,\\
&\mathcal{U}_1\Big|_S = 0, \quad \mathcal{U}_1\Big|_\Gamma = 0, \quad \mathcal{U}_1\Big|_{t=0} = 0,\\
&L_{\Phi_m,\varepsilon} s_m = 0 \quad \text{in} \quad \Omega_m, \quad m=1,2, \quad t>0,\\
&s_1\Big|_S = 0, \quad s_m\Big|_{t=0} = 0, \quad s_1\Big|_\Gamma = s_2\Big|_\Gamma,\\
&\mathcal{L}_{\Phi_2,\varepsilon}\mathcal{U}_2 + h_2\cdot\nabla s_2 + g_2 s_2\\
&= -h_2\cdot\nabla\eta_2 - g_2\eta_2 + \mathcal{F}_2(\mathcal{U}_2, s_2+\eta_2) \quad \text{in} \quad \Omega_2, \quad t>0,\\
&\mathcal{U}_2\Big|_\Gamma = 0, \quad \mathcal{U}_2\Big|_{t=0} = 0,\\
&\frac{\partial s}{\partial t} + c_0\left(\lambda_1 B_{\Phi_1}\frac{\partial \mathcal{U}_1}{\partial n} - \lambda_2 B_{\Phi_2}\frac{\partial \mathcal{U}_2}{\partial n}\right)\\
&- Q_1\frac{\partial(\mathcal{W}+V_1)}{\partial n}\frac{\partial s_1}{\partial n} + Q_2\frac{\partial V_2}{\partial n}\frac{\partial s_2}{\partial n} + q\cdot\nabla' s + ps\\
&= Q_1\frac{\partial(\mathcal{W}+V_1)}{\partial n}\frac{\partial \eta_1}{\partial n} - Q_2\frac{\partial V_2}{\partial n}\frac{\partial \eta_2}{\partial n} - q\cdot\nabla'\eta\\
&- p\eta + \mathcal{G}(\mathcal{U}_1,\mathcal{U}_2, s_1+\eta_1, s_2+\eta_2), \quad \text{on} \quad \Gamma,
\end{aligned} \tag{37}$$

where

$$f = -\frac{\partial \mathcal{W}}{\partial t} + \frac{\partial \Phi_1}{\partial t}\left(N\cdot(J_{\Phi_1}^{-1})^T\nabla\mathcal{W}\right),$$

the coefficients Q_1, Q_2, q, and p are continuous functions of N, Φ_m, $\frac{\partial \Phi_m}{\partial y_j}$, $j=1,\dots,n$. The coefficients h_m and g_m are continuous functions of the vector field N, of the functions Φ_m, $\mathcal{W}$, V_m, and of their spatial derivatives of the first and the second order. $\mathcal{F}_1$, $\mathcal{F}_2$, $\mathcal{G}$ are the nonlinear terms which are linear with respect to the derivatives $\frac{\partial s_m}{\partial t}$, $\frac{\partial^2 s_m}{\partial y_i \partial y_j}$, $\frac{\partial^2 \mathcal{U}}{\partial y_i \partial y_j}$, and have the multiplier ε at time derivative (see [1, 2]).

In [2], we proved the solvability of the problem similar to (37) by the method of successive approximations and obtained the estimates of the solution in weighted norms X^ε. In accordance with these results, we conclude that the estimate for the solution to problem (37) can be written in the form

$$\begin{aligned}\mathcal{R}(t) &= \sum_{m=1,2} (X^\varepsilon[\mathcal{U}_m, \Omega_m](t) + X^\varepsilon[s_m, \Omega_m](t)) \\ &\le c\Big(\varepsilon \sup_{\tau<t} |f(\cdot,\tau)|^{(\alpha)} + \varepsilon^{1+\alpha/2}\,[f]_t^{(\alpha/2)} \\ &+ \sum_{m=1,2} X^\varepsilon[\eta_m, \Omega_m](t)\Big)(1+\mathcal{R}(t)) \le \varepsilon C\left(\mathcal{W}, \Phi, V_m^0\right)(1+\mathcal{R}(t)).\end{aligned} \tag{38}$$

For the sufficiently small ε, (38) gives us

$$\mathcal{R}(t) \le \varepsilon C^*, \tag{39}$$

where the constant C^* depends on $\left|u_1^0 - w_0\right|^{(3+\alpha)}$, $\left|u_2^0\right|^{(3+\alpha)}$, and $\left|\mathcal{W}_t\right|_{C^{\alpha,\alpha/2}}$, $\left|\Phi_t\right|_{C^{\alpha,\alpha/2}}$ (which are estimated in [1, Sec. 5] by the Hölder norms of the given function $b \in C^{3+\alpha,\frac{3+\alpha}{2}}(S[0,t^*])$, $b_t \in \mathcal{C}([0,t^*]; C^{2+\alpha}(S))$). Inequalities (36), (39) imply (12).

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