Записки научных
семинаров ПОМИ
Том 385, 2010 г.

G. Seregin, V. Sverak

ON A BOUNDED SHEAR FLOW IN HALF–SPACE

Abstract. In this paper we describe a simple shear flow in half-space which has interesting properties from the point of view of boundary regularity. It is a solution with bounded velocity field to both the homogeneous Stokes system and the Navier–Stokes equation, and satisfies the homogeneous initial and boundary conditions. The gradient of the solution can become unbounded near the boundary. The example significantly simplifies an earlier construction by K. Kang, and shows that the boundary estimates obtained in [3] are sharp.

1. Motivation

In the present paper, we address the typical question of the theory of partial differential equations. Assume that we have a "weak" solution to a system of PDE's of the parabolic type in a space-time domain. Under what conditions it is smooth (classical) in subdomains? The corresponding property is called *local regularity* or *local smoothing*. Typically, for classical linear parabolic systems, we have such smoothing in space-time for both *interior* and *boundary* cases if the coefficients of the system are sufficiently regular. For nonlinear systems, one has smoothing effect in a neighborhood of a "regular" point. The notion of regular points depends on nonlinearity. For quasi-linear systems, this usually means Hölder continuity of the spatial gradient of the solution in a neighborhood of such a point.

In contrast to classical parabolic equations, the Stokes and Navier–Stokes systems have special features. In particular, classical smoothing in space-time is not necessary to take place. To demonstrate that, let us start with well-understood case of interior regularity.

There are different ways of describing interior smoothing. We shall formulate a selected statement from [2] in order to show one of them. Its attractive feature is that the statement can be extended to the case

Key words and phrases: Boundary regularity, Stokes and Navier–Stokes systems.
The first author was partially supported by the RFFI grant 08-01-00372-a.

of boundary regularity as well, see [3]. In the space-time cylinder $\mathcal{Q}(2)$, consider the non-stationary linear Stokes system

$$\partial_t v - \Delta\, v = f - \nabla\, q, \quad \operatorname{div}\, v = 0. \tag{1.1}$$

Here and in what follows, the notation

$$\begin{gathered} x = (x', x_3), x' = (x_1, x_2), \\ \mathcal{Q}(r) = \mathcal{C}(r) \times]-r^2, 0[\subset \mathbb{R}^3 \times \mathbb{R}, \mathcal{Q}_+(r) = \mathcal{C}_+(r) \times]-r^2, 0[\subset \mathbb{R}^3 \times \mathbb{R}, \\ \mathcal{C}(r) = b(r) \times]-r, r[\in \mathbb{R}^3, \mathcal{C}_+(r) = b(r) \times]0, r[\in \mathbb{R}^3, \\ b(r) = \{x' \in \mathbb{R}^2 : |x'| < r\} \end{gathered}$$

is used, v and q stand for the velocity field and for the pressure field, respectively.

We assume that v and q are a weak solution to (1.1) with the properties

$$v \in W^{1,0}_{m,n}(\mathcal{Q}(2)), \quad q \in L_{m,n}(\mathcal{Q}(2)). \tag{1.2}$$

As to the external force f, it is supposed that two conditions are satisfied

$$f \in L_{m_1,n}(\mathcal{Q}(2)) \tag{1.3}$$

and

$$m_1 \geq m. \tag{1.4}$$

Here,

$$L_{m,n}(\mathcal{Q}(r)) = L_n(-r^2, 0; L_m(\mathcal{C}(r)))$$

is a mixed Lebesgue space equipped with the norm

$$\|v\|_{m,n,\mathcal{Q}(r)} = \Big(\int\limits_{-r^2}^{0} \Big(\int\limits_{\mathcal{C}(r)} |v(x,t)|^m dx \Big)^{\frac{n}{m}} dt \Big)^{\frac{1}{n}},$$

and

$$\begin{aligned} W^{1,0}_{m,n}(\mathcal{Q}(r)) &= \{v \in L_{m,n}(\mathcal{Q}(r)),\ \nabla v \in L_{m,n}(\mathcal{Q}(r))\}, \\ W^{2,1}_{m,n}(\mathcal{Q}(r)) &= \{v \in W^{1,0}_{m,n}(\mathcal{Q}(r)), \\ &\qquad \nabla^2 v \in L_{m,n}(\mathcal{Q}(r)), \partial_t v \in L_{m,n}(\mathcal{Q}(r))\}. \end{aligned}$$

Proposition 1.1. *Assume that functions v and q satisfy conditions* (1.1)–(1.4). *Then $v \in W^{2,1}_{m_1,n}(\mathcal{Q}(1))$ and $q \in W^{1,0}_{m_1,n}(\mathcal{Q}(1))$ and the following estimate is valid:*

$$\begin{aligned}&\|\partial_t v\|_{L_{m_1,n}(\mathcal{Q}(1))} + \|\nabla^2 v\|_{L_{m_1,n}(\mathcal{Q}(1))} + \|\nabla q\|_{L_{m_1,n}(\mathcal{Q}(1))}\\ &\leq c(\|v\|_{L_{m,n}(\mathcal{Q}(2))} + \|\nabla v\|_{L_{m,n}(\mathcal{Q}(2))} + \|q\|_{L_{m,n}(\mathcal{Q}(2))}\\ &+ \|f\|_{L_{m_1,n}(\mathcal{Q}(2))}).\end{aligned}$$

Proposition 1.1 shows that we have certain smoothing in space and, if m_1 is sufficiently large, the velocity field v becomes Hölder continuous in space-time. There are other ways, based on the vorticity equations, to catch spatial smoothing even with no assumption on the pressure but they say nothing about smoothing in time and do not work near the boundary. If $f = 0$, exploiting bootstrap arguments, one can show that the velocity field has spatial derivatives of any order being Hölder continuous in space-time. So, starting regularity for the pressure determines Hölder continuity in time of all the spatial derivatives of the velocity but does not provide further regularity in time. Simple example showing this phenomenon of losing infinite smoothing in time might be as follows: $v(x,t) = c(t)\nabla h(x)$ and $q(x,t) = -c'(t)h(x)$, where h is a harmonic function. The same effect takes place in non-linear case and was pointed out by J. Serrin in [4].

Keeping in mind the heat equation, for which there is no difference between smoothing effects for interior and for boundary cases, one can assume the same for the Stokes system. The proposition below shows that it is true for the velocity itself but not for its spatial derivatives. The complete analogue of Proposition 1.1 has been proved recently in [3] and here it is.

Proposition 1.2. *Assume that we are given functions $v \in W^{1,0}_{m,n}(\mathcal{Q}_+(2))$, $q \in L_{m,n}(\mathcal{Q}_+(2))$, and $f \in L_{m_1,n}(\mathcal{Q}_+(2))$ with $m_1 \geq m$ satisfying the system*

$$\partial_t v - \Delta\, v = f - \nabla\, q, \quad \operatorname{div}\, v = 0 \quad \textit{in} \quad \mathcal{Q}_+(2),$$

and the homogeneous Dirichlet boundary condition

$$v(x', 0, t) = 0.$$

Then $v \in W^{2,1}_{m_1,n}(\mathcal{Q}_+(1))$ and $q \in W^{1,0}_{m_1,n}(\mathcal{Q}_+(1))$ with the estimate

$$\begin{aligned}&\|\partial_t v\|_{L_{m_1,n}(\mathcal{Q}_+(1))} + \|\nabla^2 v\|_{L_{m_1,n}(\mathcal{Q}_+(1))} + \|\nabla q\|_{L_{m_1,n}(\mathcal{Q}_+(1))}\\ &\leq c(\|v\|_{L_{m,n}(\mathcal{Q}_+(2))} + \|\nabla v\|_{L_{m,n}(\mathcal{Q}_+(2))} + \|q\|_{L_{m,n}(\mathcal{Q}_+(2))}\\ &+ \|f\|_{L_{m_1,n}(\mathcal{Q}_+(2))}).\end{aligned}$$

If we assume $f = 0$ and $1 < n < 2$, then, by embedding theorem, v is Hölder continuous in the closure of the space-time cylinder $\mathcal{Q}_+(r)$ for some positive $r > 0$. Hölder continuity is defined with respect to the parabolic metrics and the corresponding exponent does not exceed $2-2/n$. Our aim is to construct a simple and transparent example of shear flow in half-space showing that in contrast to the interior case, in general, there is no further smoothing even in spatial variables. It is a significant simplification of K. Kang's example for the Stoke system published in [1].

2. Bounded shear flow in half-space

In this section, we are looking for non-trivial bounded solutions to the following homogeneous initial boundary value problem

$$\left.\begin{array}{l}\partial_t v - \Delta\, v = -\nabla\, q\\ \mathrm{div}\ v = 0\end{array}\right\} \quad \text{in} \quad \mathbb{R}^3_+ \times]-4,0[, \tag{2.1}$$

under the homogeneous Dirichlet boundary condition

$$v(x',0,t) = 0 \quad x' \in \mathbb{R}^2, \quad -4 < t < 0, \tag{2.2}$$

and homogeneous initial data

$$v(x,-4) = 0 \quad x \in \mathbb{R}^3_+. \tag{2.3}$$

Here $R^3_+ = \{x = (x', x_3)\,:\, x_3 > 0\,\}$.

Taking an arbitrary function $f(t)$, we seek a non-trivial solution to (2.1)–(2.3) in the form of shear flow, say, along x_1-axis:

$$v(x,t) = (u(x_3,t), 0, 0), \quad q(x,t) = -f(t)x_1.$$

Here, a scalar function u solves the following initial boundary value problem

$$\partial_t u(y,t) - u_{yy}(y,t) = f(t), \tag{2.4}$$

$$u(0,t) = 0, \tag{2.5}$$

$$u(y,-4) = 0, \tag{2.6}$$

where $0 < y < +\infty$ and $-4 < t < 0$ and $u_{yy} = \partial^2 u/\partial y^2$.

It is not so difficult to solve (2.4)–(2.6) explicitly. So,

$$u(y,t) = \frac{2}{\sqrt{\pi}} \int\limits_{-4}^{t} f(t-\tau-4)d\tau \int\limits_{0}^{\frac{y}{\sqrt{4(\tau+4)}}} e^{-\xi^2} d\xi. \tag{2.7}$$

Keeping in mind that our aim is to construct irregular but summable solution, we choose the function f as follows

$$f(t) = \frac{1}{|t|^{1-\alpha}}, \quad 0 < \alpha < 1/2. \tag{2.8}$$

Then, direct calculations give us:
(i) u is a bounded smooth function in the strip $]0,+\infty[\times]-4,0[$ satisfying boundary and initial conditions;
(ii) $u_y(y,t) \geq c(\alpha)\frac{1}{y^{1-2\alpha}}$ for y and t subject to the inequalities $y^2 \geq -4t$, $0 < y \leq 3$, and $-9/8 \leq t < 0$.
(iii) Let s, s_1, l, and l_1 be numbers greater than 1 and satisfy the condition

$$K = \max\Big\{\frac{1}{2}\Big(1-\frac{1}{s}\Big), 1-\frac{1}{l_1}\Big\} < \alpha < \frac{1}{2}. \tag{2.9}$$

Then

$$v \in W^{1,0}_{s,l}(\mathcal{C}_+(3)\times]-9/4,0[), \quad q \in L_{s_1,l_1}(\mathcal{C}_+(3)\times]-9/4,0[).$$

Now, assume we are given numbers $1 < m < +\infty$ and $1 < n < 2$, letting $s = s_1 = m$ and $l = l_1 = n$ and choosing α so that inequality (2.9) holds. The functions v and q constructed above for the chosen α meet all the conditions of Proposition 1.2 with $f = 0$. However, ∇v is unbounded in any neighborhood of the space-time point $z = (x,t) = 0$.

Moreover, since, for any solution to (2.1)–(2.3) having this special form, convective term $v \cdot \nabla v$ is zero, it is a solution to the full Navier–Stokes system as well. If we let $s = l = 2$, $s_1 = l_1 = 3/2$, then $K = 1/3$ and there is α satisfying condition (2.9). It is not so difficult to show that with this set of numbers, s, s_1, l, l_1, and α, the corresponding solution to the Navier–Stokes system

$$\partial_t v + v \cdot \nabla v - \Delta v = -\nabla q, \quad \operatorname{div} v = 0$$

satisfies to initial and boundary conditions (2.2) and (2.3) and is a suitable weak solution in $\mathcal{Q}_+(2)$. Moreover, the space-time $z = (x,t) = 0$ is a regular point of the velocity field v but the gradient of v is unbounded in any neighborhood of $z = 0$.

References

1. K. Kang, *Unbounded normal derivative for the Stokes system near boundary.* — Math. Ann. **331** (2005), 87–109.
2. G. Seregin, *Local regularity theory of the Navier–Stokes equations*, Handbook of Mathematical Fluid Mechanics, Vol. 4, Edited by Friedlander, D. Serre, pp. 159–200.
3. G. Seregin, *A note on local boundary regularity for the Stokes system*, to appear in Zap. Nauchn. Semin. POMI.
4. J. Serrin, *On the interior regularity of weak solutions of the Navier–Stokes equations*, Arch. Ration. Mech. Anal., **9** (1962), 187–195.

St. Petersburg Department of Steklov
Institute of Mathematics,
Fontanka 27, 191023 St. Petersburg, Russia;
Oxforrd University

E-mail: seregin@pdmi.ras.ru

University of Minnesota

Поступило 1 октября 2010 г.