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NECESSARY CONDITIONS OF POTENTIAL BLOW UP FOR NAVIER–STOKES EQUATIONS

ABSTRACT. Assuming that T is a potential blow up time, we show that $H^{\frac{1}{2}}$-norm of the velocity field goes to ∞ as time t approaches T.

1. MOTIVATION

In the present paper, we address the following question. Consider the Cauchy problem for the classical Navier–Stokes system

$$\partial_t v + v \cdot \nabla v - \Delta v = -\nabla q, \qquad \operatorname{div} v = 0 \qquad (1.1)$$

in $Q_+ = \mathbb{R}^3 \times]0, \infty[$ with the initial condition

$$v\big|_{t=0} = a \qquad (1.2)$$

in $\mathbb{R}^3$. Here, as usual, v and q stand for the velocity field and for the pressure field, respectively. For simplicity, let us assume

$$a \in C_{0,0}^\infty(\mathbb{R}^3) \equiv \{v \in C_0^\infty(\mathbb{R}^3) : \operatorname{div} v = 0\}. \qquad (1.3)$$

It is well known due to J. Leray, see [4], that this problem has at least one weak solution obeying the energy inequality

$$\frac{1}{2} \int\limits_{\mathbb{R}^3} |v(x,t)| \ dx + \int\limits_{Q_+} |\nabla v|^2 \ dz \le \frac{1}{2} \int\limits_{\mathbb{R}^3} |a|^2 \ dx \qquad (1.4)$$

for all positive t. This solution is smooth for sufficiently small values of t. The first instant of time T when singularities occur is called *blow up* time. By definition, $z_0 = (x_0, t_0)$ is called a singular point of v if it is not

a regular one. The point z_0 is called regular if v is essentially bounded in a nonempty parabolic ball of z_0.[1]

It is an open problem whether or not there exists an energy solution to the Cauchy problem (1.1)–(1.3) exhibiting a finite time blow up.

However, J. Leray proved some necessary conditions for T to be a blow up time. They are as follows. Assume that T is a blow up time. Then, for any $3 < m \le \infty$, there exists a constant c_m depending on m only such that

$$\|v(\cdot, t)\|_m \equiv \|v(\cdot, t)\|_{m, \mathbb{R}^3} \equiv \left(\int_{\mathbb{R}^3} |v(x,t)|^m dx \right)^{\frac{1}{m}} \ge \frac{c_m}{(T-t)^{\frac{m-3}{2m}}} \quad (1.5)$$

for all $0 < t < T$.

For the limit case $n = 3$, it has been proved in [1] that

$$\limsup_{t \to T-0} \|v(\cdot, t)\|_3 = \infty \quad (1.6)$$

provided T is a blow up time. The interesting and open question is whether or not the following holds true:

$$\lim_{t \to T-0} \|v(\cdot, t)\|_3 = \infty. \quad (1.7)$$

The same kind of question appears for Sobolev spaces. Coming back to the pioneer paper of J. Leray [4], let us mention the following fact proved there:

$$\|\nabla v(\cdot, t)\|_2 \ge \frac{C}{(T-t)^{\frac{1}{4}}} \quad (1.8)$$

for all $0 < t < T$ and for some positive constant C independent of v. This, together with the Galiardo–Nirenberg inequality and (1.5), can be extended to the following necessary conditions of a finite blow up time

$$\lim_{t \to T-0} \|v(\cdot, t)\|_{H^l} \ge \frac{\widetilde{c}_l}{(T-t)^{\frac{2l-1}{4}}}, \quad 1/2 < l < 1, \quad (1.9)$$

for all $0 < t < T$ and for some $\widetilde{c}_l$, where the semi-norm $\|\cdot\|_{H^{\frac{1}{2}}}$ is defined as

$$\|f\|_{H^l}^2 \equiv \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|f(x) - f(y)|^2}{|x-y|^{3+2l}} \, dx \, dy.$$

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[1]$Q(z_0, r) = B(x_0, r) \times]t_0 - r^2, t_0[$ is a parabolic ball of radius r centered at the point z_0.

What we are interested in here is whether or not

$$\lim_{t \to T-0} \|v(\cdot, t)\|_{H^{\frac{1}{2}}} = \infty. \tag{1.10}$$

This can be regarded as the limit case in (1.9) for $l = 1/2$.

Both norms $\| \cdot \|_3$ and $\| \cdot \|_{H^{\frac{1}{2}}}$ are very important in the mathematical theory of the Navier–Stokes equations since they are invariant with respect to their scaling:

$$v^\lambda(x, t) = \lambda v(\lambda x, \lambda^2 t), \qquad q^\lambda(x, t) = \lambda^2 q(\lambda x, \lambda^2 t) \tag{1.11}$$

for positive λ.

Actually, if the answer to the first question is positive, then (1.9) holds true by continuity of imbedding $\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)^2$ into $L_3(\mathbb{R}^3)$.

In the paper, we are going to explain why (1.10) is valid. Unfortunately, we do not know the complete answer to (1.7). Let us list some results in this direction. First of all, in [7], a weaker version of (1.7) has been proved

$$\lim_{t \to T-0} \frac{1}{T-t} \int_t^T \|v(\cdot, s)\|_3^3 ds = \infty.$$

In [8], statement (1.7) has been demonstrated under the additional assumption that our blow up time T is of type I. More precisely, (1.7) holds true if for some $m \in]3, \infty]$ there exists a positive constant C_m depending on m only such that

$$\|v(\cdot, t)\|_m \leq \frac{C_m}{(T-t)^{\frac{m-3}{2m}}}$$

for all $0 < t < T$. Actually, the latter condition implies the validity of it for $m = \infty$, which means that we are dealing with blowups of type I.

Let us state our main result.

Theorem 1.1. *Let v be an energy solution to the Cauchy problem (1.1) and (1.2) with the smooth initial data satisfying (1.3). Let $T > 0$ be a finite blow up time. Then (1.10) holds true.*

Unfortunately, we still cannot justify (1.7). However, one could show the following.

$^2 \dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$ is the completion of $C_0^\infty(\mathbb{R}^3)$ with respect to $\| \cdot \|_{H^{\frac{1}{2}}}$.

Proposition 1.2. *Let v be an energy solution to the Cauchy problem (1.1) and (1.2) with the smooth initial data satisfying (1.3). Let $T > 0$ be a finite blow up time. Assume that for some positive number $T_1 \leq T$ and for a sequence $t_k \to T_1 - 0$ as $k \to \infty$ the following conditions hold:*

$$\sup_k \|v(\cdot, t_k)\|_3 = M < \infty \tag{1.12}$$

and

$$0 < \alpha_1 < \frac{t_{k+n_k} - t_k}{T - t_k} < \alpha_1 < 1 \quad \forall k \in \mathbb{N} \tag{1.13}$$

for some nondecreasing sequence of integer n_k and for real numbers α_1 and α_2.

Then $T_1 < T$.

The spirit of these two statements is that if our norms are bounded at least along a sequence converging to a potential finite blow up time T, then actually T is not a blow up time. For $H^{\frac{1}{2}}$-norm, this is a rigorous statement while, for L_3-norm, it is still a plausible conjecture. Proposition 1.2 says that the conjecture is true if a sequence t_k converges to a potential blow up not too fast.

Now, let us shortly discuss a proof. The known way is to reduce the problem either to Liouville type theorems for bounded ancient solutions, see [2] and [9], or to backward uniqueness for the heat operator with lower order term, see [1]. So far, the experience shows us that working with scale-invariant norms it is preferable to utilize the second approach. Although the theory of backward uniqueness itself is relatively well understood, its realization is not an easy task and based on fine regularity results for solutions to the Navier–Stokes equations. Using the blow up technique, one construct a nontrivial solution that is equal to zero at the last moment of time and has a reasonable decay at infinity with respect to spatial variables. The first property easily follows from the fact that the original solution has a finite L_3-norm at the blow up time T. And for this, boundedness along a sequence is sufficient. What is much more complicated is to construct a "blow up" solution (produced by scaling and limiting procedure) with required decay at infinity. Without such a property, the backward uniqueness might be even wrong. The idea, how to provide such a decay, is as follows. Applying the scaling as in [7] and [8], we construct a local energy solution being zero at the last moment of time $t = 0$. Such kind of solutions has been introduced in [3]. Here, we proceed as in [5]. This solution has the correct decay if the initial data

possess a modest decay. And this is exactly the point where the difference between $H^{\frac{1}{2}}$-norm and L_3-norm appears. We need strong compactness of initial data in $L_{2,\mathrm{loc}}$ which is the case if one has boundedness in $H^{\frac{1}{2}}$-norm and is not the case if one has boundedness in L_3-norm. Assumption (1.13) provides required compactness at some later instance of time for the case of L_3-norm.

2. Estimates of scaled solution

There is a common part when proving Theorem 1.1 and Proposition 1.2 and it is as follows. Assume that an increasing sequence t_k converges to T as $k \to \infty$ and

$$\sup_{k \in \mathbb{N}} \|v(\cdot, t_k)\|_{\mathcal{B}} = M < \infty, \tag{2.1}$$

where $\mathcal{B}$ is either $H^{\frac{1}{2}}$ or L_3. Using continuous embedding of $\dot{H}^{\frac{1}{2}}$ into L_3 and the partial regularity theory for the Navier–Stokes equations, we can state, see s imilar arguments in [7] and [8], that:

$$\|v(\cdot, T)\|_{3,B} < \infty. \tag{2.2}$$

Here, B is the unit ball of $\mathbb{R}^3$ centered at the origin.

Since T is a blow up time, there exists at least one singular point at time T. Without loss of generality, we may assume that it is $(0, T)$.

Let us scale v so that

$$u^{(k)}(y, s) = \lambda_k v(x, t), \quad p^{(k)}(y, s) = \lambda_k^2 q(x, t),$$
$$(y, s) \in \mathbb{R}^3 \times\,]-\infty, 0[, \tag{2.3}$$

where

$$x = \lambda_k y, \qquad t = T + \lambda_k^2 s,$$

$$\lambda_k = \sqrt{\frac{T - t_k}{S}}$$

and a positive parameter $S < 10$ will be defined later.

Since spaces $\mathcal{B}$ are scale-invariant, scaled functions have the following property

$$\sup_{k \in \mathbb{N}} \|u^{(k)}(\cdot, -S)\|_{\mathcal{B}} = M < \infty. \tag{2.4}$$

We know that our solution is smooth on $[-S, 0[$ and we may use the uniform local estimate from [3]

$$\alpha(s) + \beta(s) \le c\left[\|u^{(k)}(\cdot, -S)\|_{2,\text{unif}} + \int_{-S}^{s} \left(\alpha(\tau) + \alpha^3(\tau)\right) d\tau\right], \qquad (2.5)$$

which is valid for any $s \in [-S, 0[$ and for some positive constant c independent of k, s, and S. Here, the following notation have been used

$$\|f\|_{2,\text{unif}} = \sup_{x \in R^3} \|f\|_{2, B(x,r)}, \qquad \alpha(s) = \|u^{(k)}(\cdot, s)\|_{2,\text{unif}}^2,$$

$$\beta(s) = \sup_{x \in R^3} \int_{-S}^{s} \int_{B(x,r)} |\nabla u^{(k)}|^2 dy d\tau,$$

and $B(x, r)$ is a ball of $\mathbb{R}^3$ centered at the point x with radius r.

In addition, we have the following estimate of scaled pressure, see for instance [5],

$$\delta(0) \le c\left[\gamma(0) + \int_{-S}^{0} \alpha^{\frac{3}{2}}(s) ds\right] \qquad (2.6)$$

with some function $c_x^{(k)} \in L_{\frac{3}{2}}(-S, 0)$ and some positive constant c independent of k and S. Here, γ and δ are defined as

$$\gamma(s) = \sup_{x \in R^3} \int_{-S}^{s} \int_{B(x,r)} |u^{(k)}|^3 \, dy \, d\tau$$

and

$$\delta(s) = \sup_{x \in R^3} \int_{-S}^{s} \int_{B(x,3/2)} |p^{(k)}(y, \tau) - c_x^{(k)}(\tau)|^{\frac{3}{2}} \, dy \, d\tau$$

and γ satisfies the multiplicative inequality

$$\gamma(s) \le c\left(\int_{-S}^{s} \alpha^3(\tau) d\tau\right)^{\frac{1}{4}} \left(\beta(s) + \int_{-S}^{s} \alpha(\tau) d\tau\right)^{\frac{3}{4}}. \qquad (2.7)$$

Now, from (2.5)–(2.7), it follows the existence of two positive constants S and A independent of k such that

$$\sup_{-S<s<0} \alpha(s) + \beta(0) + \gamma(0) + \delta(0) \le A < \infty. \qquad (2.8)$$

And this defines a parameter S of our scaling.

In addition to the above energy estimates, one can get bounds for higher derivatives $\partial_t u$, $\nabla^2 u$, and ∇p in $L_{\frac{9}{8},\frac{3}{2}}$ locally. They are a simple consequence of the local regularity theory for the Stokes system. Finally, using covering by the unit balls, we can write down the following estimates

$$\|\partial_t u\|_{L_{\frac{9}{8},\frac{3}{2}}(B(a)\times]-5S/6,0[)} + \|\nabla^2 u\|_{L_{\frac{9}{8},\frac{3}{2}}(B(a)\times]-5S/6,0[)}$$

$$+\|\nabla p\|_{L_{\frac{9}{8},\frac{3}{2}}(B(a)\times]-5S/6,0[)} \le C(M,a). \qquad (2.9)$$

It is worthy to note that the right-hand side in (2.9) is independent of k.

3. Limiting procedure

Now let us see what happens if $k \to \infty$. Using the diagonal Cantor procedure, one can selected a subsequence, still denoted by $u^{(k)}$, such that, for any $a > 0$,

$$u^{(k)} \to u \qquad (3.1)$$

weakly-star in $L_\infty(-S,0; L_2(B(a)))$ and strongly in $L_3(B(a)\times]-S,0[)$ and in $C([\tau,0]; L_{\frac{9}{8}}(B(a)))$ for any $-S < \tau < 0$;

$$\nabla u^{(k)} \to \nabla u \qquad (3.2)$$

weakly in $L_2(B(a)\times]-S,0[)$;

$$t \mapsto \int_{B(a)} u^{(k)}(x,t) \cdot w(x)dx \to t \mapsto \int_{B(a)} u(x,t) \cdot w(x)dx \qquad (3.3)$$

strongly in $C([-S,0])$ for any $w \in L_2(B(a))$;

$$p^{(k)} - c^{(k)} = \widetilde{p}^{(k)} \to p \qquad (3.4)$$

weakly in $L_{\frac{3}{2}}(B(a)\times]-S,0[)$ for some suitable sequence $c^{(k)} \in L_{\frac{3}{2}}(-S,0)$.

Now, our aim is to show that u is not identically zero solution to the Navier–Stokes equations and, moreover, it is the so-called local energy solution in the interval $]-S_1, 0[$ with some $S_1 \leq S$. Let us start with the first task.

Using the inverse scaling, we have the following identity

$$\frac{1}{a^2} \int\limits_{Q(a)} (|u^{(k)}|^3 + |\widetilde{p}^{(k)}|^{\frac{3}{2}}) \, dy \, ds = \frac{1}{(a\lambda_k)^2} \int\limits_{Q(z_T, a\lambda_k)} (|v|^3 + |q - b^{(k)}|^{\frac{3}{2}}) dx \, dt$$

for all $0 < a < a_* = \inf\{\sqrt{S/10}, \sqrt{T/10}\}$ and for all $\lambda_k \leq 1$. Here, $z_T = (0, T)$ and $b^{(k)}(t) = \lambda_k^2 c^{(k)}(s)$. Since the pair v and $q - b^{(k)}$ is a suitable weak solution to the Navier–Stokes equations in $Q(z_T, \lambda_k a_*)$, we find

$$\frac{1}{a^2} \int\limits_{Q(a)} (|u^{(k)}|^3 + |\widetilde{p}^{(k)}|^{\frac{3}{2}}) dy \, ds > \varepsilon \tag{3.5}$$

for all $0 < a < a_*$ with a positive universal constant ε.

Here, we follow arguments from [8]. Our first observation is that, by (3.1) and (3.4),

$$\frac{1}{a^2} \int\limits_{Q(a)} |u^{(k)}|^3 \, dy \, ds \to \frac{1}{a^2} \int\limits_{Q(a)} (|u|^3 \, dy \, ds \tag{3.6}$$

for all $0 < a < a_*$ and

$$\sup_{k \in \mathbb{N}} \frac{1}{a_*^2} \int\limits_{Q(a_*)} (|u^{(k)}|^3 + |\widetilde{p}^{(k)}|^{\frac{3}{2}}) \, dy \, ds = M_1 < \infty. \tag{3.7}$$

To treat the pressure $\widetilde{p}^{(k)}$, we do the usual decomposition of it into parts. The first one is completely controlled by the pressure while the second one is a harmonic function in $B(a_*)$ for all admissible t. In other words, we have

$$\widetilde{p}^{(k)} = p_1^{(k)} + p_2^{(k)},$$

where $p_1^{(k)}$ obeys the estimate

$$\|p_1^{(k)}(\cdot, s)\|_{\frac{3}{2}, B(a_*)} \leq c\|u^{(k)}(\cdot, s)\|_{3, B(a_*)}^2. \tag{3.8}$$

For the harmonic counterpart of the pressure, we have

$$\sup_{y\in B(a_*/2)} |p_2^{(k)}(y,s)|^{\frac{3}{2}} \le c(a_*) \int\limits_{B(a_*)} |p_2^{(k)}(y,s)|^{\frac{3}{2}}\, dy$$

$$\le c(a_*) \int\limits_{B(a_*)} (|\widetilde{p}^{(k)}(y,s)|^{\frac{3}{2}} + |u^{(k)}(y,s)|^3)\, dy \qquad (3.9)$$

for all $-a_*^2 < s < 0$.

For any $0 < a < a_*/2$,

$$\varepsilon \le \frac{1}{a^2} \int\limits_{Q(a)} (|\widetilde{p}^{(k)}|^{\frac{3}{2}} + |u^{(k)}|^3)\, dy\, ds$$

$$\le c\frac{1}{a^2} \int\limits_{Q(a)} (|p_1^{(k)}|^{\frac{3}{2}} + |p_2^{(k)}|^{\frac{3}{2}} + |u^{(k)}|^3)\, dy\, ds$$

$$\le c\frac{1}{a^2} \int\limits_{Q(a)} (|p_1^{(k)}|^{\frac{3}{2}} + |u^{(k)}|^3)\, dy\, ds + ca^3\frac{1}{a^2} \int\limits_{-a^2}^{0} \sup_{y\in B(a_*/2)} |p_2^{(k)}(y,s)|^{\frac{3}{2}} ds.$$

From (3.7)–(3.9), it follows that

$$\varepsilon \le c\frac{1}{a^2} \int\limits_{Q(a_*)} |u^{(k)}|^3\, dy\, ds + ca \int\limits_{-a^2}^{0} ds \int\limits_{B(a_*)} (|\widetilde{p}^{(k)}(y,s)|^{\frac{3}{2}} + |u^{(k)}(y,s)|^3)\, dy$$

$$\le c\frac{1}{a^2} \int\limits_{Q(a_*)} |u^{(k)}|^3\, dy\, ds + ca \int\limits_{Q(a_*)} (|\widetilde{p}^{(k)}|^{\frac{3}{2}} + |u^{(k)}|^3)\, dy\, ds$$

$$\le c\frac{1}{a^2} \int\limits_{Q(a_*)} |u^{(k)}|^3\, dy\, ds + cM_1 a$$

for all $0 < a < a_*/2$. After passing to the limit and picking up sufficiently small a, we find

$$0 < c\varepsilon a^2 \le \int\limits_{Q(a_*)} |u|^3\, dy\, ds \qquad (3.10)$$

for some positive $0 < a < a_*/2$. So, our limiting solution is nontrivial.

To carry on the second task, let us first recall the definition of local energy solutions

Definition 3.1. *A pair of functions* u *and* p *defined in the space-time cylinder* $\widetilde{Q} = \mathbb{R}^3 \times]{-S}, 0[$ *is called a local energy weak Leray–Hopf solution or simply local energy solution to the Cauchy problem*

$$\partial_t u + u \cdot \nabla u - \Delta u = -\nabla p, \quad \operatorname{div} u = 0 \qquad (3.11)$$

in $\widetilde{Q} = \mathbb{R}^3 \times]-S, 0[$ *with the initial condition*

$$u|_{t=0} = b \qquad (3.12)$$

in $\mathbb{R}^3$ *if the following conditions are satisfied:*

$$u \in L_\infty(-S, 0; L_{2,\mathrm{unif}}), \qquad \sup_{x_0 \in \mathbb{R}^3} \int\limits_{-S}^{0} \int\limits_{B(x_0,1)} |\nabla u|^2 dy \, ds < +\infty,$$

$$p \in L_{\frac{3}{2}}(-S, 0; L_{\frac{3}{2},\mathrm{loc}}(\mathbb{R}^3)); \qquad (3.12)$$

$$u \text{ and } p \text{ meet } (3.11) \text{ in the sense of distributions}; \qquad (3.14)$$

$$\text{the function } t \mapsto \int\limits_{\mathbb{R}^3} u(x,t) \cdot w(x) \, dx \text{ is continuous on } [-S, 0] \qquad (3.15)$$

for any compactly supported function $w \in L_2(\mathbb{R}^3)$;
  for any compact K,

$$\|u(\cdot, t) - b(\cdot)\|_{L_2(K)} \to 0 \quad \text{as} \quad t \to -S + 0; \qquad (3.16)$$

$$\int\limits_{\mathbb{R}^3} \varphi |u(x,t)|^2 \, dx + 2 \int\limits_{-S}^{t} \int\limits_{\mathbb{R}^3} \varphi |\nabla u|^2 \, dx \, dt$$

$$\leq \int\limits_{-S}^{t} \int\limits_{\mathbb{R}^3} \Big(|u|^2 (\partial_t \varphi + \Delta \varphi) + u \cdot \nabla \varphi (|u|^2 + 2p) \Big) dx \, dt \qquad (3.17)$$

for a.a. $t \in]-S, 0[$ *and for all nonnegative functions* $\varphi \in C_0^\infty(\mathbb{R}^3 \times]-S, 1[)$;
  for any $x_0 \in \mathbb{R}^3$, *there exists a function* $c_{x_0} \in L_{\frac{3}{2}}(-S, 0)$ *such that*

$$p_{x_0}(x,t) \equiv p(x,t) - c_{x_0}(t) = p_{x_0}^1(x,t) + p_{x_0}^2(x,t) \qquad (3.18)$$

for $(x, t) \in B(x_0, 3/2) \times] - S, 0[$, where

$$p^1_{x_0}(x, t) = -\frac{1}{3}|u(x, t)|^2 + \frac{1}{4\pi} \int\limits_{B(x_0, 2)} K(x - y) : u(y, t) \otimes u(y, t) \, dy,$$

$$p^2_{x_0}(x, t) = \frac{1}{4\pi} \int\limits_{\mathbb{R}^3 \setminus B(x_0, 2)} (K(x - y) - K(x_0 - y)) : u(y, t) \otimes u(y, t) \, dy$$

and $K(x) = \nabla^2(1/|x|)$.

Here, we have used the marginal Morrey space $L_{m, \text{unif}}$ with the following finite norm

$$\|u\|_{m, \text{unif}} = \sup\{\|u\|_{m, B(x, 1)} : \ x \in \mathbb{R}^3\}.$$

Repeating arguments from [5], we can claim that our limiting functions u and p satisfy all conditions of Definition 3.1 except condition (3.16). In [6], it has been shown that if $u^{(k)}(\cdot, -S)$ converges $u(\cdot, 0)$ in $L_{2, \text{loc}}$, then u and p is a local energy solution to Cauchy problem (3.11), (3.12) with $b(\cdot) = u(\cdot, -S)$.

Proof of Theorem 1.1. By (2.4) and by the known compactness imbedding, we have

$$u^{(k)}(\cdot, -S) \to u(\cdot, 0)$$

in $L_{2, \text{loc}}$. Moreover, since the limit function $u(\cdot, -S) \in L_3$,

$$\|u(\cdot, -S)\|_{2, B(x, 1)} \to 0$$

as $|x_0| \to \infty$. The latter, together with Theorem 1.4 from [5] and ε-regularity theory for the Navier–Stokes equations, gives required decay at infinity. The last remark is that our solution has the following important property:

$$u(\cdot, 0) = 0. \tag{3.19}$$

This follows from (2.2) and (3.1), see the last statement in (3.1). So, we get (3.19), for details we refer to papers [7] and [8]. According to backward uniqueness for the Navier–Stokes, $u(\cdot, s) = 0$ for any $-a_*^2 < s < 0$, which contradicts (3.10). So, z_T is not a singular point. Theorem 1.1 is proved.

Prof of Proposition 1.2. We still have (3.19) but because of lack of compactness we do not have required strong convergence in $L_{2, \text{loc}}$.

We define s_k in the following way

$$T + \lambda_k^2 (s_k) = t_{k+n_k} = T + \lambda_{k+n_k}^2 (-S). \tag{3.20}$$

Then, by condition (1.13) and by (3.20), we find that there exists a subsequence still denoted by s_k such that

$$\lim_{k \to \infty} s_k = -S_0 \in\,]-S, 0[. \tag{3.21}$$

Then as it follows from the last statement in (3.1)

$$u(\cdot, s_k) \to u(\cdot, -S_0) \tag{3.22}$$

in $L_{2,\text{loc}}$. Moreover, since

$$\|v(\cdot, t_k)\|_3 = \|u^{(k+n_k)}(\cdot, -S)\|_3 = \|u^{(k)}(\cdot, s_k)\|_3,$$

we show that

$$u(\cdot, -S_0) \in L_3. \tag{3.23}$$

So, we assume that $a_* < \sqrt{S_0/10}$ and come up with the same situation as in the proof of Theorem 1.1, replacing S with S_0, which means that our assumption is wrong and z_T is not a singular point. Proposition 1.2 is proved.

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