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ON LOCAL REGULARITY FOR SUITABLE WEAK SOLUTIONS OF THE NAVIER—STOKES EQUATIONS NEAR THE BOUNDARY

ABSTRACT. A class of sufficient conditions for local boundary regularity of suitable weak solutions of the non-stationary three-dimensional Navier–Stokes equations is discussed. The corresponding results are formulated in terms of functionals which are invariant with respect to the scaling of the Navier–Stokes equations.

Dedicated to Grigory Alexandrovich Seregin

1. INTRODUCTION AND MAIN RESULTS

Let $\Omega \subset \mathbb{R}^3$ be a domain of class C^2 and $Q_T = \Omega \times (0,T)$. Assume that $\Gamma \subset \partial\Omega$ is an open subset of the boundary of Ω. We consider the nonstationary 3D-Navier–Stokes system (NSE) near Γ:

$$\left.\begin{aligned} \partial_t v + (v\cdot\nabla)v - \Delta v + \nabla p = 0 \\ \nabla\cdot v = 0 \end{aligned}\right\} \quad \text{in } Q_T, \qquad v|_{\Gamma\times(0,T)} = 0. \tag{1.1}$$

In this paper we continue investigation of boundary regularity for the *boundary suitable weak solutions* of the system (1.1) started in [19, 12, 11]. The main goal of the present paper is to extend all the results of [21] to the boundary case. This work was started in [11] and the present paper is a continuation of it.

Our main restriction on the boundary of the domain is the same as in [19]. Namely, we assume that Γ is C^2-uniform. This means that any point of Γ has some neighborhood of a fixed radius (which is the same for

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all points of Γ) in which $\partial\Omega$ can be represented as a graph of some C^2-function, and C^2-norms of these functions are uniformly bounded with respect to the points of Γ. Let us remind the formal definition for this property:

Main Condition on Γ: there exist positive numbers M and R_0 depending only on Γ such that for each point $x_0 \in \Gamma$ we can choose a Cartesian coordinate system $\{y_i\}_{i=1}^3$ associated to the origin x_0, and some function $\varphi_{x_0} \in C^2(\bar{K}_{R_0})$ satisfying the relations

$$\Omega(x_0, R_0) \equiv \Omega \cap B(x_0, R_0) = \{\ y \in B(R_0) : y_3 > \varphi_{x_0}(y_1, y_2)\ \}, \quad (1.2)$$

$$\varphi_{x_0}(0) = 0, \quad \nabla\varphi_{x_0}(0) = 0, \quad \sup_{|y'| \leqslant R_0} |\nabla^2 \varphi_{x_0}| \leqslant N. \quad (1.3)$$

We emphasize that Main Condition on Γ provides the uniform estimate of C^2- norms of functions φ_{x_0}:

$$\sup_{x_0 \in \Gamma} \|\varphi_{x_0}\|_{C^2(\bar{K}(R_0))} \leqslant 3N. \quad (1.4)$$

Here and later on, we accept the notations of [19 ,12, 11] for the sets, algebraic quantities, and functional spaces. For the convenience of the reader, we remind it below.

Notation for Sets: for $x_0 \in \mathbb{R}^3$, $z_0 = (x_0, t_0)$, $y'_0 \in \mathbb{R}^2$, $\rho > 0$ we introduce the sets

$$\begin{aligned}
\mathbb{R}^3_+ &= \{x \in \mathbb{R}^3 : x_3 > 0\},\\
B(x_0,\rho) &= \{x \in \mathbb{R}^3 : |x - x_0| < \rho\}, \quad B_\rho = B(\rho) = B(0,\rho),\\
B^+(x_0,\rho) &= \{x \in B(x_0,\rho) : x_3 > 0\}, \quad B^+_\rho = B^+(\rho) = B^+(0,\rho),\\
Q(z_0,\rho) &= B(x_0,\rho) \times (t_0 - \rho^2, t_0), \quad Q(\rho) = Q(0,\rho),\\
Q^+(z_0,\rho) &= B^+(x_0,\rho) \times (t_0 - \rho^2, t_0), \quad Q^+(\rho) = Q^+(0,\rho),\\
\Omega(x_0,\rho) &= \Omega \cap B(x_0,\rho), \quad \omega(z_0,\rho) = \Omega(x_0,\rho) \times (t_0 - \rho^2, t_0).
\end{aligned}$$

Other Notation: we use a convention on summation over repeated indexes. For u, $v \in \mathbb{R}^3$, A, $B \in \mathbb{M}^{3\times 3}$ we denote

$u \cdot v = u_i v_i \equiv \sum\limits_{i=1}^{3} u_i v_i, \quad v_{,k} = \frac{\partial v}{\partial x_k}, \quad \nabla v = (v_{i,j}),\ |\Omega| = \text{meas}\ \Omega,$

$\rightharpoonup$ and $\to$ are the weak and strong convergence, respectively.

For $\Omega \subset \mathbb{R}^3$, $[p]_\Omega$ denotes the spatial averagirs, i.e.,

$$[p]_\Omega = \frac{1}{|\Omega|} \int\limits_\Omega p(x,t)\ dx.$$

Notation for Functional Spaces:

- $L_q(\Omega)$, $L_q(Q_T)$, $W^k_q(\Omega)$, $\overset{\circ}{W}{}_q^k(\Omega)$, $W_q^{-k}(\Omega)$ are the usual Lebesgue and Sobolev spaces, $L_q(\Omega, \mathbb{R}^k)$ is the Lebesgue space of functions on Ω with values in $\mathbb{R}^k$ etc, but (when it is clear from the context) we shall often omit the tangent space in notation for the spaces of vector-valued functions.
- $L_{s,r}(Q_T) \equiv L_r(0,T; L_s(\Omega))$, $\quad L_{s,\infty}(Q_T) \equiv L_\infty(0,T; L_s(\Omega))$,
  $\|f\|_{L_{s,r}(Q_T)} \equiv \Big(\int\limits_0^T \|f(\cdot,t)\|^r_{s,\Omega}\ dt\Big)^{1/r}$,
  $\|f\|_{L_{s,\infty}(Q_T)} \equiv \operatorname*{esssup}_{t\in(0,T)} \|f(\cdot,t)\|_{L_s(\Omega)}$.
- $W^{1,0}_{s,r}(Q_T) \equiv L_r(0,T; W^1_s(\Omega)) = \{u \in L_{s,r}(Q_T) : \nabla u \in L_{s,r}(Q_T)\}$, $\|u\|_{W^{1,0}_{s,r}(Q_T)} \equiv \|u\|_{L_{s,r}(Q_T)} + \|\nabla u\|_{L_{s,r}(Q_T)}$.
- $W^{2,1}_{s,r}(Q_T) = \{u \in W^{1,0}_{s,r}(Q_T) : \nabla^2 u, \partial_t u \in L_{s,r}(Q_T)\}$, $[u]_{W^{2,1}_{s,r}(Q_T)} \equiv \|\nabla^2 u\|_{L_{s,r}(Q_T)} + \|\partial_t u\|_{L_{s,r}(Q_T)}$, $\|u\|_{W^{2,1}_{s,r}(Q_T)} \equiv \|u\|_{W^{1,0}_{s,r}(Q_T)} + [u]_{W^{2,1}_{s,r}(Q_T)}$.

Under appropriate conditions on Ω (see [6, 7]) existence of weak solutions of the initial-boundary value problem to the system (1.1) is known. In this paper, we study regularity of the so-called *boundary suitable weak solutions*. The definition of which is the following:

A pair of functions (v,p) is called *a boundary suitable weak solution for the NSE near* Γ, iff

i)
$$v \in L^{2,\infty} \cap W_2^{1,0}(Q_T; R^3), \quad p \in L^{\frac{3}{2}}(Q_T), \tag{1.5}$$

ii) the functions (v,p) satisfy (1.1) in the sense of distributions,

iii) the the following local energy inequality (LEI) holds near Γ:

$$\int\limits_\Omega \zeta(y,t)|v(y,t)|^2\, dy + 2\int\limits_0^t \int\limits_\Omega \zeta|\nabla v|^2\, dy\, d\tau$$
$$\le \int\limits_0^t \int\limits_\Omega \Big\{|v|^2(\partial_t\zeta + \Delta\zeta) + v\cdot\nabla\zeta(|v|^2 + 2p)\Big\}\, dy\, d\tau \tag{1.7}$$

for a.e. $t \in (0,T)$ and all nonnegative functions $\zeta \in C_0^\infty(\mathbb{R}^3 \times (0,T))$ vanishing near $(\partial\Omega\backslash\Gamma) \times (0,T)$.

Due to result of G. Seregin [22] we know that from i), ii), iii) immediately follows that $v \in W^{2,1}_{\frac{9}{8},\frac{3}{2}}(\theta_{\frac{T}{2}})$, $p \in W^{1,0}_{\frac{9}{8},\frac{3}{2}}(\theta_{\frac{T}{2}})$. And, therefore, our pair u and p is a suitable weak solution to the Navier–Stokes equations in a sense of definition from [11, 12, 14, 19].

It is well known that any function $v \in L_\infty(0,T;L_q(\Omega))$ possessing the property (1.5) can be redefined on a set of moments of time of measure zero in such a way that v become continuous in time with values in $L_q(\Omega)$ equipped with the weak topology, i.e for any $w \in L_{q'}(\Omega)$ the function

$$t \mapsto \int\limits_\Omega v(x,t) \cdot w(x)\ dx \quad \text{is continuous.}$$

In particular, this means that suitable weak solutions belonging to the class $L_\infty(0,T;L_q(\Omega))$ have values in $L_q(\Omega)$ for every moment of time. Below we always assume that our suitable weak solutions have this property from the very beginning.

It is known that the Navier–Stokes equations are invariant with respect to the scaling.

$$v^R(z,s) = Rv(Rz,R^2s), \quad p^R(z,s) = R^2p(Rz,R^2s),$$

We call this the scaling of the Navier–Stokes equations or, simply, the natural scaling. In the local regularity theory, functionals that are invariant under the natural scaling play a very important role. Some of them are listed below

$$\begin{aligned}
C(R) &\equiv \Big(\frac{1}{R^2}\int\limits_{\omega(z_0,R)} |v|^3\ dxdt\Big)^{1/3}, \\
D(R) &\equiv \Big(\frac{1}{R^2}\int\limits_{\omega(z_0,R)} |p-[p]_{\widehat{B}^+(R)}|^{3/2}\ dxdt\Big)^{2/3}, \\
E(R) &\equiv \Big(\frac{1}{R}\int\limits_{\omega(z_0,R)} |\nabla v|^2\ dxdt\Big)^{1/2}, \\
A(R) &\equiv \Big(\frac{1}{R}\sup_{t\in(-R^2+t_0,t_0)}\int\limits_{\widehat{B}^+(R)} |v|^2\ dx\Big)^{1/2}, \\
H(R) &\equiv \Big(\frac{1}{R^3}\int\limits_{\omega(z_0,R)} |v|^2\ dxdt\Big)^{1/2},
\end{aligned} \tag{1.8}$$

where $z_0 = (x_0, t_0)$.

Finally, we introduce the additional notation

$$G = \min\left\{\limsup_{r\to 0} A(r), \limsup_{r\to 0} E(r), \limsup_{r\to 0} C(r)\right\},$$

$$g = \min\left\{\liminf_{r\to 0} H(r), \liminf_{r\to 0} D(r)\right\}.$$

The main result of the present paper is the following theorem.

Theorem 1.1. *Let the pair v, p be a boundary suitable weak solution of the Navier–Stokes equations in $\omega(z_0, r)$. For any $M > 0$ there exists a positive number $\varepsilon(M)$ with the property that if $G < M$ and $g < \varepsilon(M)$, then the function v is Hölder continuous in $\omega(z_0, r_*)$ for some $r_* > 0$.*

Remark 1.1 A similar results was proved in [11], where the number g was equal to

$$\min\left\{\liminf_{r\to 0} A(r), \liminf_{r\to 0} E(r), \liminf_{r\to 0} C(r)\right\}.$$

2. Flattering of the boundary and the perturbed Navier–Stokes equations

We begin with recalling some mathematical technicalities required for analysis of problems involving curved boundaries (for more details see [11, 12]).

Let us fix a point $x_0 \in \Gamma$ and consider the function $\varphi = \varphi_{x_0}$, given by (1.2), (1.3), (1.4). We consider the new variables defined by formulas

$$x = \psi(y) \equiv \begin{pmatrix} y_1 \\ y_2 \\ y_3 - \varphi(y_1, y_2) \end{pmatrix}. \tag{2.1}$$

The diffeomorphism (2.1) transforms the set $\Omega(x_0, R_0)$ onto some subdomain $\psi(\Omega(x_0, R_0))$ of $\mathbb{R}^3_+ \equiv \{x \in \mathbb{R}^3 : x_3 > 0\}$. Note that our assumptions of Γ allow to choose R_0 sufficiently small, so that

$$B^+(R) \subset \psi(\Omega(x_0, \frac{3R}{2})) \subset B^+(2R) \quad \text{for all} \quad 2R \leqslant R_0, \tag{2.2}$$

and, vice verse,

$$\psi^{-1}(B^+(R)) \subset \Omega(x_0, \frac{3R}{2}) \subset \psi^{-1}(B^+(2R)) \quad \text{for all} \quad 2R \leqslant R_0, \tag{2.3}$$

see [19] for details.

The system (1.1) in $\Omega(x_0, R_0) \times (0,T)$ after the change of variables (2.1) transforms into the system which we call *the perturbed Navier–Stokes system*:

$$\left.\begin{aligned} \partial_t \widehat{v} + (\widehat{v} \cdot \widehat{\nabla}_\varphi)\widehat{v} - \widehat{\Delta}_\varphi \widehat{v} + \widehat{\nabla}_\varphi \widehat{p} = 0 \\ \widehat{\nabla}_\varphi \cdot \widehat{v} = 0 \end{aligned}\right\} \quad \text{in } \psi(\Omega(x_0, R_0)) \times (0,T) \qquad (2.4)$$
$$\widehat{v}|_{x_3=0} = 0.$$

Here $\widehat{v} = v \circ \psi^{-1}$, $\widehat{p} = p \circ \psi^{-1}$ and $\widehat{\nabla}_\varphi$ and $\widehat{\Delta}_\varphi$ are the differential operators with variable coefficients defined by formulas

$$\begin{gathered} \widehat{\nabla}_\varphi = \left(\frac{\partial}{\partial x_1} - \frac{\partial \varphi}{\partial y_1} \frac{\partial}{\partial x_3}, \frac{\partial}{\partial x_2} - \frac{\partial \varphi}{\partial y_2} \frac{\partial}{\partial x_3}, \frac{\partial}{\partial x_3} \right), \\ \widehat{\Delta}_\varphi = a_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j} + b_i(x) \frac{\partial}{\partial x_i}, \end{gathered} \qquad (2.5)$$

where

$$\begin{gathered} a_{11}(x) = a_{22}(x) = 1, \quad a_{33}(x) = 1 + (\varphi_{,1})^2 + (\varphi_{,2})^2, \\ a_{12}(x) = a_{21}(x) = 0, \\ a_{13}(x) = a_{31}(x) = -\varphi_{,1}, \\ a_{23}(x) = a_{32}(x) = -\varphi_{,2}, \\ b_1(x) = b_2(x) = 0, \quad b_3(x) = -\varphi_{,11} - \varphi_{,22}. \end{gathered} \qquad (2.6)$$

We remark that the coefficients of system (2.4) depend only on the first and second derivatives of φ but not on the function itself.

After the change of variables in (1.7) we obtain the inequality

$$\begin{gathered} \int\limits_{B+} \zeta(x,t)|\widehat{v}(x,t)|^2 \, dx + 2 \int\limits_0^t \int\limits_{B+} \zeta |\widehat{\nabla}_\varphi \widehat{v}|^2 \, dx d\tau \\ \leqslant \int\limits_0^t \int\limits_{B+} \left\{ |\widehat{v}|^2 \left(\partial_t \zeta + \widehat{\Delta}_\varphi \zeta \right) + \widehat{v} \cdot \widehat{\nabla}_\varphi \zeta \left(|\widehat{v}|^2 + 2\widehat{p} \right) \right\} \, dx d\tau. \end{gathered} \qquad (2.7)$$

for a.e. $t \in (0,T)$ and all nonnegative functions $\zeta \in C_0^\infty(B(\frac{2}{3}R_0) \times (0,T))$.

In the next paragraph, we introduce some notations and results from [11, 12]

3. Estimates of solutions to the perturbed Navier–Stokes equations

First results concerning perturbed Navier–Stokes equations were established in [19] and [12]. Here we collect all necessary results for proving theorem 1.1. So, first we consider a boundary suitable weak solution $(\widehat{v}, \widehat{p}, \varphi)$ of the perturbed Navier–Stokes system in a half-cylinder Q^+. We assume this solution satisfies relations (1.3), (1.4), (1.5), (2.4), (2.7). For $R \leqslant 1$ we introduce five principal functionals

$$\begin{aligned} C(R) &\equiv \Big(\frac{1}{R^2}\int\limits_{Q^+(R)} |\widehat{v}|^3 \, dxdt\Big)^{1/3} \\ D(R) &\equiv \Big(\frac{1}{R^2}\int\limits_{Q^+(R)} |\widehat{p} - [\widehat{p}]_{B^+(R)}|^{3/2} \, dxdt\Big)^{2/3}, \\ E(R) &\equiv \Big(\frac{1}{R}\int\limits_{Q^+(R)} |\nabla \widehat{v}|^2 \, dxdt\Big)^{1/2}, \\ A(R) &\equiv \Big(\frac{1}{R}\sup_{t\in(-R^2,0)}\int\limits_{B^+(R)} |\widehat{v}|^2 \, dx\Big)^{1/2}, \\ H(R) &\equiv \Big(\frac{1}{R^3}\int\limits_{Q^+(R)} |\widehat{v}|^2 \, dxdt\Big)^{1/2}. \end{aligned} \tag{3.1}$$

Choosing in (2.7) the cut-off function ζ in the appropriate way we obtain the following inequality:

$$A(\frac{3\rho}{4}) + E(\frac{3\rho}{4}) \le c \, \Big\{ C(\rho) + C^{\frac{3}{2}}(\rho) + C^{\frac{1}{2}}(\rho) D^{\frac{1}{2}}(\rho) \Big\}. \tag{3.2}$$

The interpolation inequality provides the estimates:

$$C(\theta\rho) \le c A^{\frac{1}{2}}(\theta\rho) E^{\frac{1}{2}}(\theta\rho). \tag{3.3}$$

In what follows, we will refer to results concerning estimates of solutions to Perturbed Stokes system (see [19] Lemma 3.1, Lemma 3.2 or [12] Proposition 2.1, Proposition 2.2). The question is that these estimates were derived under some smallness condition of norms of the function φ

$$\|\nabla \varphi\|_{C(\bar{K})} + \|\nabla^2 \varphi\|_{C(\bar{K})} \leqslant \frac{\mu_*}{2}. \tag{3.4}$$

To satisfy the last condition, we perform the trick, introduced in [19] (see [12] section 2 for details). Namely, instead of the functions $(\widehat{v},\widehat{p},\varphi)$ for any $R\leqslant R_0$ we can consider three scaled function

$$\widehat{v}^R(z,s)=R\widehat{v}(Rz,R^2s),\quad \widehat{p}^R(z,s)=R^2\widehat{p}(Rz,R^2s),$$

$$\varphi^R(z_1,z_2)=\frac{1}{R}\varphi(Rz_1,Rz_2).$$

If $(\widehat{v},\widehat{p},\varphi)$ is a boundary suitable weak solution of the Perturbed NSE in $Q^+(R)$ then $(\widehat{v}^R,\widehat{p}^R,\varphi^R)$ is a boundary suitable weak solution of the same system in Q^+, i.e. these functions satisfy the system

$$\left.\begin{aligned}\partial_s\widehat{v}^R+(\widehat{v}^R\cdot\widehat{\nabla}_R)\widehat{v}^R-\widehat{\Delta}_R\widehat{v}^R+\widehat{\nabla}_R\widehat{p}^R=0,\\ \widehat{\nabla}_R\cdot\widehat{v}^R=0\end{aligned}\right\}\quad \text{in}\quad Q^+, \tag{3.5}$$
$$\widehat{v}^R|_{z_3=0}=0,$$

where operators $\widehat{\Delta}_R$, $\widehat{\nabla}_R$ are defined via relations (2.5), (2.6) with functions φ^R instead of φ. From (1.3) and relations $\nabla_z\varphi^R(z')=\nabla_x\varphi(x')$, $\nabla_z^2\varphi^R(z')=R\nabla_x^2\varphi(x')$, where $x'=Rz'$, and also using Taylor formula we obtain for any $R\leqslant R_0$

$$\|\varphi^R\|_{C^2(\bar{K})}\leqslant R\|\varphi\|_{C^2(\bar{K}(R))}\leqslant 3NR.$$

Therefore, if we choose R satisfying the inequality

$$3NR\leqslant\mu_*$$

then the functions $(\widehat{v}^R,\widehat{p}^R,\varphi^R)$ satisfy all the required conditions to use results from [12], [19]. In the sequel of the paper, we for the sake of simplicity write $(\widehat{v},\widehat{p},\varphi)$ instead of $(\widehat{v}^R,\widehat{p}^R,\varphi^R)$.

The estimates exposed below are the so-called decay estimates for the pressure, which has been proved in [12] (see (3.7)) and [11] (see(3.6))

$$D(\theta\rho)\le c\theta^{-1}A^{\frac{2}{3}}(\rho)E^{\frac{4}{3}}(\rho)+c\theta^{\frac{4}{3}}\left[E(\rho)+D(\rho)+E^{\frac{4}{3}}(\rho)A^{\frac{2}{3}}(\rho)\right]. \tag{3.6}$$

$$\begin{aligned}D(\theta\rho)\le c\theta^{-\frac{7}{6}}E^{\frac{7}{6}}(\rho)A^{\frac{7}{12}}(\rho)C^{\frac{1}{4}}(\rho)\\ +\theta^{\frac{4}{3}}[E^{\frac{7}{6}}(\rho)A^{\frac{7}{12}}(\rho)C^{\frac{1}{4}}(\rho)+E(\rho)+D(\rho)].\end{aligned} \tag{3.7}$$

The next lemma ([11] Lemma 3.2) shows that if one of the numbers $\sup\limits_{0<r<1}E(r)$, $\sup\limits_{0<r<1}C(r)$, or $\sup\limits_{0<r<1}A(r)$ is finite, then so are the others.

Lemma 3.1. *Let the pair v, p be a boundary suitable weak solution of the perturbed Navier–Stokes equations in Q^+. Then the following estimates are hold:*

(1) *If $\sup\limits_{0<r<1} E(r) = E_0 < +\infty$, then there exists a positive constant d depending only on E_0 such that*

$$C^3(r) + A^3(r) + D^3(r) \leq d(E_0)\left[1 + r^{\frac{1}{2}}(A^3(1) + D^3(1))\right]. \quad (3.8)$$

(2) *If $\sup\limits_{0<r<1} C(r) = C_0 < +\infty$, then there exists a positive constant c such that*

$$A^2(r) + E^2(r) + D^{\frac{3}{2}}(r) \leq c\left[c(C_0) + rD^{\frac{3}{2}}(1)\right]. \quad (3.9)$$

(3) *If $\sup\limits_{0<r<1} A(r) = A_0 < +\infty$, then there exists a positive constant e depending only on A_0 such that*

$$C^{\frac{4}{3}}(r) + E^2(r) + D^{\frac{3}{2}}(r) \leq e(A_0)\left[1 + r(E^2(1) + D^{\frac{3}{2}}(1))\right]. \quad (3.10)$$

4. Proof of Theorem 1.1

The proof is based upon the scheme presented in [21] and repeated (with some modifications) in [11]. The key ingredient of it is the following proposition.

Proposition 4.1. *Let the v, p, φ be a suitable weak solution of the perturbed Navier–Stokes equations in Q^+. For any $M > 0$ there exists a positive number $\varepsilon_1 = \varepsilon_1(M)$ such that if*

$$\sup_{0<r<1} E(r) = E_0 \leq M \quad (4.1)$$

and

$$g_{r_*} = \min\{H(r_*), D(r_*)\} < \varepsilon_1(M) \quad (4.2)$$

for some $r_* \in (0, \min\{1/4, (A^3(1) + D^3(1))^{-2}\})$, then $z = 0$ is a regular point of v (i.e. function v is a Hölder continuous in a small parabolic neighborhood of $z = 0$).

Proof. Assume that the statement of the proposition is false. Then there exist a positive number M and a sequence v_n, p_n, φ_n of suitable weak solutions of the perturbed Navier–Stokes equations in Q^+ such that for any $n \in N$

$$E(v_n, r) \equiv \Big(\frac{1}{r} \int\limits_{\widehat{Q}^+(r)} |\nabla v_n|^2 \, dxdt\Big)^{1/2} \leq M \tag{4.3}$$

for all $r \in (0, 1]$ and

$$g_{r_n}(v_n, p_n) = \min\{H(v_n, r_n), D(v_n, r_n)\} \leq \frac{1}{n}. \tag{4.4}$$

for some

$$r_n \in (0, \min\{1/4, (A^3(v_n, 1) + D^3(v_n, 1))^{-2}\}), \tag{4.5}$$

but $z = 0$ is a singular point of v_n. Here we have used the notation

$$H(v_n, r) \equiv \Big(\frac{1}{r^3} \int\limits_{\widehat{Q}^+(r)} |v_n|^2 \, dxdt\Big)^{1/2},$$

$$D(p_n, r) \equiv \Big(\frac{1}{r^2} \int\limits_{\widehat{Q}^+(r)} |p_n - [p_n]_{\widehat{B}^+(r)}|^{3/2} \, dxdt\Big)^{2/3},$$

$$A(v_n, r) \equiv \Big(\frac{1}{r} \sup_{t \in (-r^2, 0)} \int\limits_{\widehat{B}^+(r)} |v_n|^2 \, dx\Big)^{1/2}.$$

On the other hand, since $z = 0$ is a singular point of v_n, there exists a universal positive number ε such that

$$C(v_n, r) + D(p_n, r) > \varepsilon > 0 \tag{4.6}$$

for all $0 < r \leq 1$ (see, for example, [19]). We emphasize that (4.6) is valid for any natural number n.

Not begging generality we can assume that $r_n \to 0$ as $n \to \infty$. Indeed, by (4.5) r_n are bounded. We can consider numbers $\widetilde{r}_n = r_n(1/n)^{\alpha}$. Then, due to scaling inequalities

$$D(p_n, \widetilde{r}_n) \leq D(p_n, r_n)(\frac{r_n}{\widetilde{r}_n})^{4/3}(1 + \widetilde{r}_n^3),$$

$$H(v_n, \widetilde{r}_n) \leq H(v_n, r_n)(\frac{r_n}{\widetilde{r}_n})^{3/2}$$

and (4.4) we could find that (if $\alpha < 2/3$) $g_{\widetilde{r}_n}(v_n, p_n) \le (1/n)^{(1-3/2\alpha)}$.

By (3.8) from Lemma 3.1 and (4.5) we find the estimate

$$C^3(v_n, r) + A^3(v_n, r) + D^3(p_n, r) \le d(M)\left[1 + \left(\frac{r}{r_n}\right)^{\frac{1}{2}} r_n^{\frac{1}{2}}(A^3(v_n, 1) + D^3(p_n, 1))\right] \le d_0(M), \quad (4.7)$$

is valid for all $r \in (0, r_n)$.

Let us now scale our functions v_n, p_n, φ_n so that

$$u_n(y, s) = r_n v_n(r_n y, r_n^2 s), \ q_n(y, s) = r_n^2 p_n(r_n y, r_n^2 s), \ \varphi_n(y) = \frac{1}{r_n}\varphi(r_n y).$$

By the invariance of the functionals and equations with respect to the natural scaling, we have the following: u_n, q_n, φ_n is a suitable weak solution of the perturbed Navier–Stokes equations in Q^+ for each $n \in N$

$$E(u_n, r) \le M \quad (4.8)$$

for all $0 < r \le 1$ and for each $n \in N$

$$g_{r_n}(v_n, p_n) = g_1(u_n, q_n) \to 0 \quad \text{as } n \to \infty \quad (4.9)$$

$$C(u_n, r) + D(q_n, r) > \varepsilon > 0 \quad (4.10)$$

for all $0 < r \le 1$ and for each $n \in N$

$$C^3(u_n, r) + A^3(u_n, r) + D^3(q_n, r) \le d_0(M) \quad (4.11)$$

for all $0 < r \le 1$ and for each $n \in N$.

Now let n tend to ∞. First of all, in order to pass to the limit in non-linear terms, we need to prove strong compactness. To this end, we estimate the weak derivative of v with respect to t in the standard way using the peturbed Navier–Stokes equations.

Further, the limiting functions u, q, φ is a suitable weak solution of the perturbed Navier–Stokes equations in Q^+, and

$$C^3(u, r) + A^3(u, r) + D^3(q, r) \le d_0(M). \quad (4.12)$$

From (4.9) it can be involved that there is a subsequence $\{n_k\}_{k=1}^{\infty}$ such that one of the following

$$H(u_{n_k},1)\to 0, \tag{4.13}$$

$$D(u_{n_k},1)\to 0 \tag{4.14}$$

is valid as $k\to\infty$.

We discuss each case separately, starting with (4.13).

In this case the limit velocity is $u=0$. Therefore $C(u_{n_k},r)\to C(u,r)$ as $k\to+\infty$ for all $0<r\le 1$ and we can apply Proposition 4.1 from [M] and say that (4.13) cannot occur.

Assume now that (4.14) holds. Here we have $q_{n_k}-[q_{n_k}]\to 0$ in $L_{\frac{3}{2}}(Q^+)$ and we find that

$$C(u,r)\ge\varepsilon>0 \tag{4.15}$$

for all $0<r\le 1$. Let us describe the properties of the limit function u: u is a boundary suitable weak solution to NSE in Q^+ vanishing on the set $x_3=0$ with pressure $q=0$. We consider an odd continuation of vector-function u to the hole cylinder Q and vector-function v which is equal to u in Q^+ and having odd continuation of third component and even of first and second ones. Then these functions satisfy the equation

$$\partial_t u+v\cdot\nabla u-\Delta u=0,\quad \operatorname{div} v=0 \tag{4.16}$$

in Q in distribution sense. $u\in L_{2,\infty}(Q)\cap W_2^{1,0}(Q)$, $v\in L_{\frac{10}{3}}(Q)$. As it was shown in [21, Appendix (§4) Proposition 4.4] $u\in L_{3,\infty}(Q(3/4))$. Then by [25] we can conclude that u is a Holder continuous in, say, $Q^+(1/4)$ and in particular,

$$\sup_{z\in Q^+(1/4)}|u(z)|\le d_1(M).$$

Then, it follows from (4.15) that

$$cd_2(M)r\ge\varepsilon$$

for all $0<r\le 1/4$, which is not true, so the case (4.14) excluded. □

Proposition 4.2. *Let v, p, φ be a suitable weak solution of the perturbed Navier–Stokes equations in Q^+. If*

$$\limsup_{r\to 0} E(r) < \frac{1}{2}m = M \tag{4.17}$$

and

$$g < \frac{1}{2}\varepsilon_1(m) = \varepsilon_{1'}(M), \tag{4.18}$$

then $z = 0$ is a regular point of v.

Proof. By the condition (4.19), we can find a number $r_1 \in (0, 1)$ such that

$$\sup_{0<r\leq r_1} E(r) \leq m.$$

and then we can scale v, p and φ so that

$$u(x,t) = r_1 v(r_1 x, r_1^2 t), \quad q(x,t) = r_1^2 p(r_1 x, r_1^2 t), \quad \psi(x,t) = \frac{1}{r}\varphi(r_1 x, r_1^2 t).$$

Using invariance property of perturbed Navier–Stokes equations under natural scaling we conclude that functions u, q, ψ is then a suitable weak solution of the perturbed Navier–Stokes equations in Q^+, and the following two inequalities hold:

$$\sup_{0<r\leq 1} E(r,u) \leq m$$

and

$$g(u,q) \leq \frac{1}{2}\varepsilon_1(m)$$

From the last inequality one can involve that there exist a number $r_* \in (0, \min\{1/4, (A^3(1) + D^3(1))^{-2}\})$ such that

$$g_{r_*}(u,q) \leq \varepsilon_1(m).$$

By Proposition 4.1 the point $z = 0$ is a regular point of u, therefore $z = 0$ is a regular point of v. Proposition 4.2 is proved. □

In the same way one can prove the following statements.

Proposition 4.3. *Let v, p, φ be a suitable weak solution of the perturbed Navier–Stokes equations in Q^+. If*

$$\limsup_{r\to 0} A(r) < M \tag{4.19}$$

and

$$g < \varepsilon_2(M), \tag{4.20}$$

then $z = 0$ is a regular point of v.

Proposition 4.4. *Let v, p, φ be a suitable weak solution of the perturbed Navier–Stokes equations in Q^+. If*

$$\limsup_{r\to 0} C(r) < M \tag{4.21}$$

and

$$g < \varepsilon_3(M), \tag{4.22}$$

then $z = 0$ is a regular point of v.

Proof of Theorem 1.1. The proof is a direct consequence of Propositions 4.2–4.4 and the fact that functionals (1.8) are bilaterally equivalent to functionals (3.1) with respect to change of variables (2.1) and $\tau = t - t_0$. □

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