Записки научных
семинаров ПОМИ
Том 385, 2010 г.

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A REGULARITY CRITERION FOR AXIALLY SYMMETRIC SOLUTIONS TO THE NAVIER–STOKES EQUATIONS

Abstract. We study the axially-symmetric solutions to the Navier–Stokes equations. Assume that the radial component of velocity (v_r) belongs either to $L_\infty(0,T;L_3(\Omega_0))$ or to v_r/r to $L_\infty(0,T;L_{3/2}(\Omega_0))$, where Ω_0 is some neighbourhood of the axis of symmetry. Assume additionally that there exist subdomains Ω_k, $k=1,\dots,N$, such that $\Omega_0 \subset \bigcup_{k=1}^N \Omega_k$ and assume that there exist constants α_1, α_2 such that either $\|v_r\|_{L_\infty(0,T;L_3(\Omega_k))} \leqslant \alpha_1$ or $\left\|\frac{v_r}{r}\right\|_{L_\infty(0,T;L_{3/2}(\Omega_k))} \leqslant \alpha_2$ for $k=1,\dots,N$. Then the weak solution becomes strong ($v \in W_2^{2,1}(\Omega\times(0,T))$, $\nabla p \in L_2(\Omega\times(0,T))$).

1. Introduction

Let $\Omega \subset \mathbb{R}^3$ be a cylindrical domain with boundary S. The aim of this paper is to investigate existence of axially symmetric solutions to the initial-boundary value problem

$$
\begin{aligned}
&v_t + v\cdot\nabla v - \operatorname{div}\mathbb{T}(v,p) = f && \text{in } \ \Omega^T = \Omega\times(0,T),\\
&\operatorname{div} v = 0 && \text{in } \ \Omega^T,\\
&\nu\overline{n}\cdot\mathbb{D}(v)\cdot\overline{\tau}_\alpha + \gamma v\cdot\overline{\tau}_\alpha = 0, \quad \alpha=1,2, && \text{on } \ S^T = S\times(0,T),\\
&v\cdot\overline{n} = 0 && \text{on } \ S^T,\\
&v|_{t=0} = v_0 && \text{in } \ \Omega,
\end{aligned}
\tag{1.1}
$$

under the assumption that the radial component of velocity belongs to $L_\infty(0,;L_3(\Omega))$.

Here, $v = v(x,t) \in \mathbb{R}^3$ is the velocity vector function $p = p(x,t) \in \mathbb{R}$ is the pressure functuin $x = (x_1,x_2,x_3)$ denotes coordinates in the global

Key words and phrases: Navier–Stokes equations, axially symmetric solutions, regularity criterions.

Cartesian system. By $\mathbb{T}(v,p) = \nu\mathbb{D}(v) - pI$ we denote the stress tensor with the unit matrix I and dilatation (shear) tensor

$$\mathbb{D}(v) = \nabla v + (\nabla v)^{\mathbb{T}},$$

where $\nu > 0$ and $\gamma > 0$ are constant viscosity and slip coefficients, respectively.

By $\overline{n}$ we denote the unit outward normal vector to S and $\overline{\tau}_\alpha$, $\alpha = 1, 2$, is a tangent vector to S. Ω is a cylinder with x_3-axis as the axis of symmetry and with the boundary $S = S_1 \cup S_2$, where S_1 is parallel to the axis of symmetry and S_2 is perpendicular.

Let us introduce the cylindrical coordinates (r, φ, z) by the relations: $x_1 = r\cos\varphi$, $x_2 = r\sin\varphi$, $x_3 = z$. Then

$$\begin{aligned} S_1 &= \big\{x \in \mathbb{R}^3 :\ r = R,\ -a < z < a\big\}, \\ S_2 &= \{x \in \mathbb{R}^3 :\ r < R,\ z \ \text{ equals either } \ -a \ \text{ or } \ a\}, \end{aligned}$$

where R and a are given numbers. Also, let

$$\overline{e}_r = (\cos\varphi, \sin\varphi, 0), \quad \overline{e}_\varphi = (-\sin\varphi, \cos\varphi, 0), \quad \overline{e}_z = (0, 0, 1)$$

be the corresponding coordinate vector. In this system, we define the velocity by the component, as follows: $v_r = v \cdot \overline{e}_r$, $v_\varphi = v \cdot \overline{e}_\varphi$, $v_z = v \cdot \overline{e}_z$.

Definition 1.1. *By the axially symmetric solutions to the Navier–Stokes equations we mean such solutions that*

$$v_{r,\varphi} = v_{\varphi,\varphi} = v_{z,\varphi} = p_{,\varphi} = 0, \quad f_{r,\varphi} = f_{\varphi,\varphi} = f_{z,\varphi} = 0.$$

Lemma 1.2. *Assume that $v_0 \in L_2(\Omega)$ and $f \in L_2(\Omega^T)$. Then, there exists an axially symmetric weak solution to problem* (1.1) *such that $v \in V_2^0(\Omega^T) = L_\infty(0,T; L_2(\Omega)) \cap L_2(0,T; H^1(\Omega))$ and the following estimate holds*

$$\|v(t)\|^2_{L_2(\Omega)} + \int_0^t \int_\Omega \left(|\nabla v|^2 + \left|\frac{v_r}{r}\right|^2 + \left|\frac{v_\varphi}{r}\right|^2\right) r\,dr\,dz\,dt \leqslant c\big(\|v_0\|^2_{L_2(\Omega)} + \|f\|^2_{L_2(\Omega^t)}\big) \equiv d_1^2, \quad t \leqslant T. \quad (1.2)$$

Assumption M. *The assumptions either*

$$v_r \in L_\infty(0,T;L_3(\Omega)) \tag{M$_1$}$$

or

$$\frac{v_r}{r} \in L_\infty(0,T;L_{3/2}(\Omega)) \tag{M$_2$}$$

are not sufficient to prove regularity of weak solutions. Therefore, we assume that $\xi^{(k)}$, $k=1,\dots,N$, *are points of the axis of symmetry crossing* Ω. *Let* $\xi^{(1)}$ *and* $\xi^{(N)}$ *belong to* S. *Let* $Q^{(k)}_{r_0,a}=\{(r,z)\,:\,r<r_0,\ |\xi^{(k)}-z|<a\}$, $k=1,\dots,N$, *where* r_0 *and* a *are given numbers. We assume that either*

$$\|v_r\|_{L_\infty(0,T,L_3(Q^{(k)}_{r_0,a}))} \leqslant \frac{\nu}{s}\sqrt{2(s-1)c_0} \equiv \alpha_1(s) \tag{M$_3$}$$

or

$$\left\|\frac{v_r}{r}\right\|_{L_\infty(0,T;L_{3/2}(Q^{(k)}_{r_0,a}))} \leqslant \frac{\sqrt{2}\,\nu(s-1)c_0}{s^2} \equiv \alpha_2(s), \tag{M$_4$}$$

where c_0 *is the constant from imbedding* (2.6). *Assume that* $\Omega_{r_0}=\{(r,z)\in\Omega:\ r<r_0\}$ *and* $\Omega_{r_0}\subset\bigcup_{k=1}^N Q^{(k)}_{r_0,a}$.

The main results of this paper are as follows:

Theorem A. *Assume that* M *with* (M$_3$) *for* $s=4$. *Moreover, let*

$$v_r\in L_\infty(0,T;L_3(\Omega_{r_0})) \quad \textit{and} \quad \|v_r\|_{L_\infty(0,T;L_3(\Omega_{r_0}))}\leqslant N\alpha_1(4) \tag{1.3}$$

Assume that $v_0\in H^1(\Omega)$, $f\in L_2(\Omega^T)$. *Then*

$$v\in W^{2,1}_2(\Omega^T), \quad \nabla p\in L_2(\Omega^T). \tag{1.4}$$

We prove regularity (1.4) locally (near the axis of symmetry) and globally (in all domain Ω). In [16, 17], the regularity of axially symmetric weak solutions was proved under the assumption that

$$v_r\in L_q(0,T;L_p(\Omega)), \quad \frac{3}{p}+\frac{2}{q}=1, \quad q<\infty. \tag{1.5}$$

To prove Theorem A we use the equation

$$v_{\varphi t}+v\cdot\nabla v_\varphi-\frac{v_r}{r}v_\varphi-\nu\Delta v_\varphi+\nu\frac{v_\varphi}{r^2}=f_\varphi. \tag{1.6}$$

Theorem B. *Assume that M with* (M_4) *for $s = 4$. Moreover, let*

$$\frac{v_r}{r} \in L_\infty(0,T;L_{\frac{3}{2}}(\Omega_{r_0})) \quad \textit{and} \quad \left\|\frac{v_r}{r}\right\|_{L_\infty(0,T;L_{3/2}(\Omega_{r_0}))} \leqslant N\alpha_2(4). \tag{1.7}$$

Assume also that $v_0 \in H^1(\Omega)$, $f \in L_2(\Omega^T)$. Then (1.4) holds.

While the existence of weak solutions to the system $(1.1)_{1,2,3}$ for $f \in L_2(0,T;W_2^{-1}(\mathbb{R}^3))$, $v_0 \in L_2(\mathbb{R}^3)$ is proved by J. Leray in 1933 (see [12, 13]) and to the system $(1.1)_{1,2,3}$, $v|_S = 0$ for $f \in L_2(0,T;\overset{\circ}{W}{}_2^{-1}(\Omega))$, $v_0 \in L_2(\Omega)$ by E. Hopf in 1952 (see [4]) their regularity and uniqueness properties remain fundamental open questions.

Up to now the uniqueness has been proved only for weak solutions additionally belonging to the class $L_{r,s}(\Omega^T) = L_s(0,T;L_r(\Omega))$ with $\frac{3}{r} + \frac{2}{s} \leqslant 1$, $s \in [2,\infty]$, $r \in [3,\infty]$ (see [5, 7–9, 18, 22]).

Moreover, a weak solution in $L_{r,s}(\Omega^T)$ with $\frac{3}{r} + \frac{2}{s} \leqslant 1$, $r \in [3,\infty]$, $s \in [2,\infty]$ becomes a regular solution under sufficiently smooth data (see [3, 8, 9] for $s < \infty$ and [20, 21] for $s = \infty$).

In the two-dimensional case existence and uniqueness of strong solutions is a well established fact (see [12] and [11]).

In the axially symmetric case, the situation is much more complicated. In the case, where the considered domain does not contain the axis of symmetry the axially symmetric Navier–Stokes system can be treated as two-dimensional and the result from [11] can be applied. In the case of slip boundary conditions and also in domains without axis of symmetry the existence of long time regular solutions is proved in [24].

For domains with the axis of symmetry the existence of regular solutions is proved in the case $v_\varphi = 0$, $f_\varphi = 0$ in [10] (for a bounded) domain Ω and in [14, 23] (for $\mathbb{R}^3$).

For nonvanishing swirl ($v_\varphi \neq 0$), the existence of regular axially symmetric solutions can up to now be proved under additional assumptions only. In [1], the authors assumed that $\frac{v_r}{r} \in L_q(0,T;L_p(\Omega_\delta))$, $q \in (1,\infty]$, $p \in \left(\frac{3}{2},\infty\right)$, $\frac{2}{q} + \frac{3}{p} \leqslant 2$, $v_r \in L_q(0,T;L_p(\Omega_\delta))$, $\frac{2}{q} + \frac{3}{p} \leqslant 1$, $p \in (3,\infty)$, $q \in (2,\infty]$, $\Omega_\delta = \{x \in \mathbb{R}^3 :\ r < \delta\}$.

In [16, 17], it is assumed that $v_r \in L_{q,s}(\Omega^T)$, with $q \in (3,\infty]$, $s \in [2,\infty]$, $\frac{3}{q} + \frac{2}{s} \leqslant 1$. In [19] it is considered the case where $v_\varphi \in L_{q,s}(\Omega^T)$ with $\frac{3}{q} + \frac{2}{s} < 1$, $s \in (2,\infty]$, $q \in (4,\infty]$.

The result presented in [19] was improved in [6], where v_φ must belong to $L_{q,s}(\Omega^T)$ with $\frac{3}{q} + \frac{2}{s} < \frac{7}{4} - \frac{3}{q}$, $q \in \left(\frac{24}{7},4\right]$, $s \in \left(\frac{8q}{7q-24},\infty\right]$. In the above cases a sufficient regularity of data functions must be additionally prescribed.

There are many regularity criteria for weak solutions to the Navier–Stokes equations such as some components either velocity or vorticity have prescribed additional regularity. The cases are described in [26, 27].

There is also one other approach to analysis of regularity for weak solutions to the Navier–Stokes equations. In this case, some smallness conditions are imposed either on the initial velocity and the external force or on their derivatives. Then by some, fixed point arguments the existence of long time regular solutions can be proved (see [25–28]).

2. Local estimate

First, we derive obtain some estimate for v_φ in a neighbourhood of an interior point of the axis of symmetry. Let $\widetilde{\Omega}$ be such neighbourhood that $\widetilde{\Omega}\cap S=\phi$. Let $\zeta=\zeta(x)$ be a smooth function with $\Omega_\zeta=\operatorname{supp}\zeta\subset\widetilde{\Omega}$ and $\zeta(x)=1$ for $x\in\omega_\zeta\subset\Omega_\zeta$. Then, the function $\widetilde{v}_\varphi=v_\varphi\zeta$ is a solution to the problem

$$
\begin{aligned}
&\widetilde{v}_{\varphi,t}+v\cdot\nabla\widetilde{v}_\varphi-\nu\Delta\widetilde{v}_\varphi+\frac{\widetilde{v}_\varphi}{r^2}\\
&=v\cdot\nabla\zeta v_\varphi+\frac{v_r}{r}\widetilde{v}_\varphi-2\nu\nabla v_\varphi\nabla\zeta-\nu v_\varphi\Delta\zeta+\widetilde{f}_\varphi && \text{in}\quad \widetilde{\Omega}^T,\\
&\widetilde{v}_\varphi=0 && \text{on}\quad \partial\widetilde{\Omega}^T, \quad (2.1)\\
&\widetilde{v}_\varphi|_{t=0}=\widetilde{v}_\varphi(0) && \text{in}\quad \widetilde{\Omega}.
\end{aligned}
$$

Lemma 2.1. *Assume that v is a sufficiently regular weak solution to the problem (1.1), $f_\varphi\in L_s(\Omega^T)$, $v_\varphi(0)\in L_s(\Omega)$, and $v_\varphi\in L_s(\Omega^T)$, $s>1$.*

Let ξ_0 be an interior point of the axis of symmetry. Let

$$
Q(r_0,a,\xi_0)=\big\{(r,z)\in\Omega:\ r<r_0,\ |z-\xi_0|<a\big\}.
$$

Assume that

$$
\|v_r\|_{L_\infty(0,T;L_3(Q(r_0,a,\xi_0)))}\leqslant\frac{\nu}{s}\sqrt{2(s-1)c_0}, \tag{2.2}
$$

where c_0 is the constant from imbedding (2.6). Then, the following inequality holds

$$
\|\,|v_\varphi(t)|^{s/2}\|_{L_2(Q(r_0/2,a/2,\xi_0))}+\|\,|v_\varphi|^{s/2}\|_{V_2^0(Q(r_0/2,a/2,\xi_0)\times(0,t))}
$$

$$+\left\|\frac{|v_\varphi|^{s/2}}{r}\right\|_{L_2(Q(r_0/2,a/2,\xi_0)\times(0,t))} \leqslant c\big(\|\,|v_\varphi|^{s/2}\|_{L_2(Q(r_0,a,\xi_0)\times(0,t))} \tag{2.3}$$

$$+\|\,|f_\varphi|^{s/2}\|_{L_2(Q(r_0,a,\xi_0)\times(0,t))}+\|\,|v_\varphi(0)|^{s/2}\|_{L_2(Q(r_0,a,\xi_0))}\big),$$

where

$$V_2^0(Q\times(0,T))=L_\infty(0,T;L_2(Q))\cap L_2(0,T;W_2^1(Q)).$$

Proof. Multiplying (2.1) by $\widetilde{v}_\varphi|\widetilde{v}_\varphi|^{s-2}$ and integrating over Ω yields

$$\frac{1}{s}\frac{d}{dt}\|\widetilde{v}_\varphi\|^s_{L_s(\Omega)}+\frac{4\nu(s-1)}{s^2}\int\limits_\Omega|\nabla|\widetilde{v}_\varphi|^{s/2}|^2\,dx+\nu\int\limits_\Omega\frac{|\widetilde{v}_\varphi|^s}{r^2}\,dx$$

$$=\int\limits_\Omega v\cdot\nabla\zeta v_\varphi\widetilde{v}_\varphi|\widetilde{v}_\varphi|^{s-2}\,dx+\int\limits_\Omega\frac{v_r}{r}|\widetilde{v}_\varphi|^s\,dx-2\nu\int\limits_\Omega\nabla v_\varphi\nabla\zeta\widetilde{v}_\varphi|\widetilde{v}_\varphi|^{s-2}\,dx \tag{2.4}$$

$$-\nu\int\limits_\Omega v_\varphi\Delta\zeta\widetilde{v}_\varphi|\widetilde{v}_\varphi|^{s-2}\,dx+\int\limits_\Omega\widetilde{f}_\varphi\cdot\widetilde{v}_\varphi|\widetilde{v}_\varphi|^{s-2}\,dx.$$

The second term on the r.h.s. of (2.4) can be expressed in the form

$$\int\limits_\Omega\frac{v_r}{r}|\widetilde{v}_\varphi|^s\,dx=\int\limits_\Omega v_r\frac{|\widetilde{v}_\varphi|^{s/2}}{r}|\widetilde{v}_\varphi|^{s/2}\,dx\equiv I_1.$$

Hence,

$$|I_1|\leqslant\frac{\varepsilon_1}{2}\int\limits_\Omega\frac{|\widetilde{v}_\varphi|^s}{r^2}\,dx+\frac{1}{2\varepsilon_1}\int\limits_\Omega v_r^2|\widetilde{v}_\varphi|^s\,dx\equiv I_2,$$

where the second term in I_2 is esitmated by

$$\frac{1}{2\varepsilon_1}\|v_r\|^2_{L_3(\Omega_\zeta)}\bigg(\int\limits_\Omega|\widetilde{v}_\varphi|^{3s}\,dx\bigg)^{1/3}=\frac{1}{2\varepsilon_1}\|v_r\|^2_{L_3(\Omega_\zeta)}\|\widetilde{v}_\varphi\|^s_{L_{3s}(\Omega)}.$$

Since

$$|\nabla\zeta|\leqslant\frac{c_1}{\lambda},$$

the first term on the r.h.s. of (2.4) is bounded by

$$\frac{c_1}{\lambda}\int_{\Omega_\zeta\setminus\omega_\zeta}|v|\,|v_\varphi|^s\,dx,$$

where we have used the fact that diam $\Omega_\zeta\leqslant\lambda$.

Now, we consider the sum of the third and the fourth terms on the r.h.s. of (2.4). First we examine

$$\begin{aligned}
&-2\int_\Omega\nabla v_\varphi\nabla\zeta\widetilde{v}_\varphi|\widetilde{v}_\varphi|^{s-2}\,dx=-2\int_\Omega\nabla v_\varphi v_\varphi\nabla\zeta\zeta|\widetilde{v}_\varphi|^{s-2}\,dx\\
&=-\int_\Omega\nabla v_\varphi^2\nabla\zeta\zeta|\widetilde{v}_\varphi|^{s-2}\,dx=\int_\Omega v_\varphi^2\Delta\zeta\zeta|\widetilde{v}_\varphi|^{s-2}\,dx\\
&\quad+\int_\Omega v_\varphi^2|\nabla\zeta|^2|\widetilde{v}_\varphi|^{s-2}\,dx+\int_\Omega v_\varphi^2\nabla\zeta\zeta\nabla|\widetilde{v}_\varphi|^{s-2}\,dx\equiv I_3.
\end{aligned}$$

Let us examine the last term in I_3. We present it in the form

$$\begin{aligned}
&\int_\Omega v_\varphi^2\nabla\zeta\zeta\nabla(|\widetilde{v}_\varphi|^2)^{\frac{s-2}{2}}\,dx=\frac{s-2}{2}\int_\Omega v_\varphi^2\nabla\zeta\zeta(|\widetilde{v}_\varphi|^2)^{\frac{s-2}{2}-1}2\widetilde{v}_\varphi\nabla\widetilde{v}_\varphi\,dx\\
&\qquad=(s-2)\int_\Omega v_\varphi^2\nabla\zeta\zeta|\widetilde{v}_\varphi|^{s-4}\widetilde{v}_\varphi(\nabla v_\varphi\zeta+v_\varphi\nabla\zeta)\,dx\\
&=(s-2)\int_\Omega\nabla v_\varphi\nabla\zeta\widetilde{v}_\varphi|\widetilde{v}_\varphi|^{s-2}\,dx+(s-2)\int_\Omega v_\varphi^2|\nabla\zeta|^2|\widetilde{v}_\varphi|^{s-2}\,dx.
\end{aligned}$$

Using the expression in I_3, we obtain the equation

$$\begin{aligned}
-2\int_\Omega\nabla v_\varphi\nabla\zeta\widetilde{v}_\varphi|\widetilde{v}_\varphi|^{s-2}\,dx&=\int_\Omega v_\varphi^2\Delta\zeta\zeta|\widetilde{v}_\varphi|^{s-2}\,dx+\int_\Omega v_\varphi^2|\nabla\zeta|^2|\widetilde{v}_\varphi|^{s-2}\,dx\\
&+(s-2)\int_\Omega\nabla v_\varphi\nabla\zeta\widetilde{v}_\varphi|\widetilde{v}_\varphi|^{s-2}\,dx+(s-2)\int_\Omega v_\varphi^2|\nabla\zeta|^2|\widetilde{v}_\varphi|^{s-2}\,dx.
\end{aligned}$$

Hence, we get

$$\int_\Omega \nabla v_\varphi \nabla\zeta \widetilde{v}_\varphi |\widetilde{v}_\varphi|^{s-2}\, dx = -\frac{1}{s}\int_\Omega v_\varphi^2 \Delta\zeta\zeta |\widetilde{v}_\varphi|^{s-2}\, dx - \frac{s-1}{s}\int_\Omega v_\varphi^2 |\nabla\zeta|^2 |\widetilde{v}_\varphi|^{s-2}\, dx.$$

Then the sum of the third and the fourth terms on the r.h.s. of (2.4) equals

$$\frac{2-s}{s}\nu\int_\Omega v_\varphi^2 \Delta\zeta\zeta |\widetilde{v}_\varphi|^{s-2}\, dx + 2\frac{s-1}{s}\nu\int_\Omega v_\varphi^2 |\nabla\zeta|^2 |\widetilde{v}_\varphi|^{s-2}\, dx. \tag{2.5}$$

Setting $\varepsilon_1 = \nu$, using the estimate

$$c_0 \|\widetilde{v}_\varphi\|^s_{L_{3s}(\Omega)} \leqslant \int_\Omega |\nabla |\widetilde{v}_\varphi|^{s/2}|^2\, dx, \tag{2.6}$$

and the expression (2.5) in (2.4), we obtain

$$\begin{aligned} \frac{1}{s}\frac{d}{dt}\|\widetilde{v}_\varphi\|^s_{L_s(\Omega)} &+ \frac{2\nu(s-1)}{s^2} c_0 \|\widetilde{v}_\varphi\|^s_{L_{3s}(\Omega)} + \frac{2\nu(s-1)}{s^2}\int_\Omega |\nabla|\widetilde{v}_\varphi|^{s/2}|^2\, dx \\ &+\frac{\nu}{2}\int_\Omega \frac{|\widetilde{v}_\varphi|^s}{r^2}\, dx \leqslant \frac{c_1}{\lambda}\int_{\Omega_\zeta\setminus\omega_\zeta} |v|\,|v_\varphi|^s\, dx \\ &+\frac{1}{2\nu}\|v_r\|^2_{L_3(\Omega_\zeta)}\|\widetilde{v}_\varphi\|^s_{L_{3s}(\Omega)} + \frac{c_1^2}{\lambda^2}\frac{s-2}{s}\nu\int_{\Omega_\zeta\setminus\omega_\zeta} v_\varphi^2 |\widetilde{v}_\varphi|^{s-2}\, dx \\ &+2\frac{c_1^2}{\lambda^2}\frac{s-1}{s}\nu\int_{\Omega_\zeta\setminus\omega_\zeta} v_\varphi^2|\widetilde{v}_\varphi|^{s-2} dx + \int_\Omega \widetilde{f}_\varphi \widetilde{v}_\varphi |\widetilde{v}_\varphi|^{s-2}\, dx. \end{aligned} \tag{2.7}$$

Assuming that the diameter of Ω_ζ is so small that

$$\|v_r\|_{L_3(\Omega_\zeta)} \leqslant \frac{\nu}{s}\sqrt{2(s-1)c_0},$$

we obtain from (2.7) the inequality

$$\frac{1}{s}\ \frac{d}{dt}\|\widetilde{v}_\varphi\|^s_{L_s(\Omega)} + \frac{\nu(s-1)}{s^2}c_0\|\widetilde{v}_\varphi\|^s_{L_{3s}(\Omega)} + \frac{2\nu(s-1)}{s^2}\int\limits_\Omega |\nabla|\widetilde{v}_\varphi|^{s/2}|^2\ dx$$

$$+\frac{\nu}{2}\int\limits_\Omega \frac{|\widetilde{v}_\varphi|^s}{r^2}\ dx \leqslant \frac{c_1}{\sqrt{\lambda}}\int\limits_{\Omega_\zeta\setminus\omega_\zeta} |v|\,|v_\varphi|^s\ dx \tag{2.8}$$

$$+\frac{c_1^2}{\lambda^2}\nu\frac{3s-4}{s}\|v_\varphi\|^s_{L_s(\Omega_\zeta\setminus\omega_\zeta)} + \int\limits_\Omega |\widetilde{f}_\varphi|\,|\widetilde{v}_\varphi|^{s-1}\ dx.$$

Introducing the quantity

$$u = |v_\varphi|^{s/2} \tag{2.9}$$

using (2.7), nothing that

$$\|\widetilde{v}_\varphi\|^s_{L_{3s}(\Omega)} = \|\,|\widetilde{v}_\varphi|^{s/2}\|^2_{L_6(\Omega)} = \|\widetilde{u}\|^2_{L_6(\Omega)},$$

and

$$\|\widetilde{u}\|^2_{L_2(\Omega)} \leqslant \lambda^2\|\widetilde{u}\|^2_{L_6(\Omega)},$$

we obtain instead of (2.8) the inequality

$$\frac{1}{s}\frac{d}{dt}\|\widetilde{u}\|^2_{L_2(\Omega)} + \frac{\nu(s-1)}{s^2}\frac{c_0}{\lambda^2}\|\widetilde{u}\|^2_{L_2(\Omega)} + \frac{2\nu(s-1)}{s^2}\|\nabla\widetilde{u}\|^2_{L_2(\Omega)}$$

$$+\nu\int\limits_\Omega \frac{|\widetilde{u}|^2}{r^2}\ dx \leqslant \frac{c_1}{\lambda}\|v\|_{L_2(\Omega_\zeta\setminus\omega_\zeta)}\|u\|^2_{L_4(\Omega_\zeta\setminus\omega_\zeta)} \tag{2.10}$$

$$+\frac{(3s-4)c_1^2\nu}{\lambda^2 s}\|u\|^2_{L_2(\Omega_\zeta\setminus\omega_\zeta)} + \int\limits_\Omega |\widetilde{f}_\varphi|\,|\widetilde{u}|^{\frac{2(s-1)}{s}}\ dx.$$

By the Hölder and the Young inequalities the last term on the r.h.s. of (2.10) is estimated by the quantity

$$\frac{1}{2}\frac{\nu(s-1)}{s^2}\frac{c_0}{\lambda^2}\|\widetilde{u}\|^2_{L_2(\Omega)} + \left(\frac{2s\lambda^2}{\nu c_0}\right)^{s-1}\frac{1}{s}\|\widetilde{f}_\varphi\|^2_{L_s(\Omega)}.$$

Using the above estimate in (2.10), multiplying the result by s, integrating with respect to time and applying the energy estimate (1.2) yields

$$\begin{aligned}
&\|\widetilde{u}(t)\|^2_{L_2(\Omega)} + \frac{\nu(s-1)c_0}{2s\lambda^2}\int_0^t \|\widetilde{u}(t')\|^2_{L_2(\Omega)}dt' \\
&+\frac{2\nu(s-1)}{s}\int_0^t \|\nabla\widetilde{u}(t')\|^2_{L_2(\Omega)}dt' + \nu s\int_{\Omega^t}\frac{|\widetilde{u}|^2}{r^2}\,dx\,dt' \\
&\leqslant \frac{c_1 s}{\lambda}d_1\int_0^t \|u(t')\|^2_{L_4(\Omega_\zeta)}dt' + \frac{(3s-4)c_1^2\nu}{\lambda^2}\int_0^t \|u(t')\|^2_{L_2(\Omega_\zeta)}dt' \\
&+\left(\frac{2s\lambda^2}{\nu c_0}\right)^{s-1}\int_0^t \|\widetilde{g}_\varphi(t')\|^2_{L_2(\Omega)}dt' + \|\widetilde{u}(0)\|^2_{L_2(\Omega)},
\end{aligned} \tag{2.11}$$

where $\widetilde{g}_\varphi = |\widetilde{f}_\varphi|^{s/2}$.

In view of the interpolation inequality

$$\|u\|^2_{L_4(\Omega_\zeta)} \leqslant \varepsilon_0^{1/2}\|\nabla u\|^2_{L_2(\Omega_\zeta)} + c\varepsilon_0^{-3/2}\|u\|^2_{L_2(\Omega_\zeta)}, \tag{2.12}$$

we represent (2.11) in the form

$$\begin{aligned}
&\|\widetilde{u}\|^2_{L_2(\Omega)} + \frac{\nu(s-1)c_0}{2s\lambda^2}\int_0^t \|\widetilde{u}(t')\|^2_{L_2(\Omega)}dt' \\
&+\frac{2\nu(s-1)}{s}\int_0^t \|\nabla\widetilde{u}(t')\|^2_{L_2(\Omega)}dt' + s\nu\int_{\Omega^t}\frac{|\widetilde{u}(t')|^2}{r^2}\,dx\,dt' \\
&\leqslant \varepsilon c_1 s d_1\int_0^t \|\nabla u(t')\|^2_{L_2(\Omega_\zeta)}dt' + \frac{c_1 c s d_1}{\lambda^4}\varepsilon^{-3}\int_0^t \|u(t')\|^2_{L_2(\Omega_\zeta)}dt' \\
&+\frac{(3s-4)c_1^2\nu}{\lambda^2}\int_0^t \|u(t')\|^2_{L_2(\Omega_\zeta)}dt' + \left(\frac{2s\lambda^2}{\nu c_0}\right)^{s-1}\|\widetilde{g}_\varphi\|^2_{L_2(\Omega^t)} + \|\widetilde{u}(0)\|^2_{L_2(\Omega)}.
\end{aligned} \tag{2.13}$$

Omitting the first, the second and the fourth terms on the l.h.s. of (2.13), multiplying the result by λ^4, we obtain

$$\frac{2\nu(s-1)}{s}\lambda^4\int\limits_0^t\|\nabla u(t')\|^2_{L_2(\omega_\zeta)}dt'\leqslant c_1sd_1\lambda^4\varepsilon\int\limits_0^t\|\nabla u(t')\|^2_{L_2(\Omega_\zeta)}dt'$$
$$+(c_1scd_1\varepsilon^{-3}+(3s-4)c_1^2\nu\lambda^2)\int\limits_0^t\|u(t')\|^2_{L_2(\Omega_\zeta)}dt' \tag{2.14}$$
$$+\left(\frac{2s}{\nu c_0}\right)^{s-1}\lambda^{2(s+1)}\|\widetilde{g}_\varphi\|^2_{L_2(\Omega^t)}+\lambda^4\|\widetilde{u}(0)\|^2_{L_2(\Omega)}.$$

Let us assume that

$$\zeta^{(\lambda)}(x)=\zeta^{(\lambda)}(r,z)=\begin{cases}1 & r\leqslant r_0-\lambda \quad |z|\leqslant a-\lambda\\ 0 & r\geqslant r_0-\frac{\lambda}{2}\quad |z|\geqslant a-\lambda/2.\end{cases}$$

We define

$$\omega_\zeta(x)=\{(r,z)\in\mathbb{R}^2:\ |z|\leqslant a-\lambda,\ r\leqslant r_0-\lambda\},$$
$$\Omega_\zeta(\lambda)=\{(r,z)\in\mathbb{R}^2:\ |z|\leqslant a-\lambda/2,\ r\leqslant r_0-\lambda/2\}=\omega_{\zeta^{(\lambda/2)}}$$

and introduce the functions

$$f(\lambda)=\frac{2\nu(s-1)}{s}\lambda^4\int\limits_0^t\|\nabla u(t')\|^2_{L_2(\omega_{\zeta^{(\lambda)}})}dt'$$

and

$$K(\lambda)=[c_1csd_1\varepsilon^{-3}+(3s-4)c_1^2\nu\lambda^2]\int\limits_0^t\|u(t')\|^2_{L_2(\omega_{\zeta^{(0)}})}dt'$$
$$+\left(\frac{2s}{\nu c_0}\right)^{s-1}\lambda^{2(s+1)}\|\widetilde{g}_\varphi\|^2_{L_2(\Omega^t)}+\lambda^4\|\widetilde{u}(0)\|^2_{L_2(\Omega)}.$$

Setting

$$\varepsilon=\frac{\nu(s-1)}{c_1s^2d_1}$$

(2.14) implies the inequality

$$f(\lambda) \leqslant \frac{1}{2} f(\lambda/2) + K(\lambda), \quad \lambda \leqslant \min\{r_0, a\}. \tag{2.15}$$

As $K(\lambda)$ is an increasing function of λ and for local regular solutions $f(\lambda)$ is finite for $\lambda \leqslant \min\{a, r_0\}$, we obtain from (2.15) (after iterating the inequality, see [15, Chap. 4, Sec. 10])

$$f(\lambda) \leqslant \sum_{j=0}^{\infty} \frac{1}{2^j} K\left(\frac{\lambda}{2^j}\right) \leqslant \sum_{j=0}^{\infty} \frac{1}{2^j} K(\lambda) \leqslant 2\, K(\lambda). \tag{2.16}$$

Hence, (2.16) and (2.13) imply the inequality

$$\begin{aligned} &\| \, |v_\varphi|^{s/2} \|_{V_2^0(\omega_{\varsigma(\lambda)} \times (0,t))} + \left\| \frac{|v_\varphi|^{s/2}}{r} \right\|_{L_2(\omega_{\varsigma(\lambda)} \times (0,t))} \\ &\leqslant c(\| \, |v_\varphi|^{s/2} \|_{L_2(\omega_{\varsigma(0)} \times (0,t))} + \| \, |f_\varphi|^{s/2} \|_{L_2(\omega_{\varsigma(0)} \times (0,t))} \\ &\qquad + \| \, |v_\varphi(0)|^{s/2} \|_{L_2(\omega_{\varsigma(0)})}). \end{aligned} \tag{2.17}$$

This concludes the proof.

Remark 2.2. From (1.2) we have

$$\int_{\Omega^t} \frac{v_\varphi^2}{r^2} \, dx \, dt' \leqslant c_1^2, \quad t \leqslant T. \tag{2.18}$$

Multiplying (1.6) by $r(rv_\varphi)|rv_\varphi|^{\sigma-2}$, integrating over Ω^t, using initial and boundary conditions and passing to the limit as $\sigma \to \infty$, we obtain (see also [1])

$$\|rv_\varphi\|_{L_\infty(\Omega^t)} \leqslant c_2, \quad t \leqslant T. \tag{2.19}$$

From (2.18) and (2.19), we deduce

$$\|v_\varphi\|_{L_4(\Omega^t)} \leqslant \sqrt{c_1 c_2}, \quad t \leqslant T. \tag{2.20}$$

Then, (2.3) implies

$$\begin{aligned} &\|v_\varphi^2\|_{V_2^0(Q(r_0,a,\xi_0) \times (0,t))} \\ &\qquad \leqslant c(c_1 c_2 + \|f_\varphi\|^2_{L_4(\Omega^t)} + \|v_\varphi(0)\|^2_{L_4(\Omega)}) \equiv c_3^2, \quad t \leqslant T. \end{aligned} \tag{2.21}$$

Hence, (2.21) yields the estimate

$$\|v_\varphi\|_{L_{\frac{20}{3}}(Q(r_0/2,a/2,\xi_0)\times(0,T))} \leqslant c_3 \tag{2.22}$$

from the results of [6, 17, 19], it follows that

$$v \in W_2^{2,1}(Q(r_0/4, a/4, \xi_0) \times (0,T)), \quad \nabla p \in L_2(Q(r_0/4, a/4, \xi_0) \times (0,T)).$$

Remark 2.3. Let ξ_0 be a point where the axis of symmetry meets S, $S = \partial\Omega$. After flattening locally the boundary and reflecting solutions to (1.1) with respect to the flat part of S we obtain inequality (2.3) in this case.

Hence in the neighborhood of the axis of symmetry such that $r \leqslant r_0/4$ we show that a solution of (1.1) satisfying condition (2.2) is of class (1.4). Then repeating the arguments from [24] we obtain that (v,p) satisfy (1.4) for $r \geqslant r_0/4$. In this case we do not need (1.3). Hence Theorem A is proved.

Remark 2.4. In the proof of Lemma 2.1, we need regularity of weak solutions. To guarantee this we use the existence of local regular solutions such that

$$\int_0^t \|\nabla v_\varphi^2(t')\|_{L_2(Q(r_0))}^2 dt', \quad t' \leqslant T_*$$

is finite, where $Q(r_0) = \{(r,z) \in \Omega : \ r < r_0\}$.

Then, by Lemma 2.1, the local solution can be extended on the interval $(T_*, 2T_*)$. Hence the procedure can be continued step by step. However, we must mention, that the constant estimating the norms (1.4) will increase at each new step.

Proof of Theorem B. We estimate the second term on the r.h.s. of (2.4) by the quantity

$$\frac{\varepsilon}{2}\|\widetilde{v}_\varphi\|_{L_{3s}(\Omega)}^s + \frac{1}{2\varepsilon}\left\|\frac{v_r}{r}\right\|_{L_{\frac{3}{2}}(\Omega_\zeta)}^2 \|\widetilde{v}_\varphi\|_{L_{3s}(\Omega)}^s.$$

Setting $\varepsilon = \frac{2\nu(s-1)c_0}{s^2}$ and using the inequality $\left\|\frac{v_r}{r}\right\|_{L_\infty(0,T;L_{3/2}(\Omega_\zeta)} \leqslant \frac{\sqrt{2}\nu(s-1)c_0}{s^2}$, we prove (2.3).

References

1. D. Chae, J Lee, *On the regularity of axisymmetric solutions to the Navier–Stokes equations.* — Math. Z. **239** (2002), 645–671.
2. G. P. Galdi, *An introduction to the Navier–Stokes initial boundary value.* In: Topics in Mathematical Fluid Mechanics, G. P. Galdi, J. G. Heywood, R. Ranacher (ed.), Birkhäuser Verlag, Basel (2000).
3. Y. Giga, *Solutions for semilinear parabolic equations in L^p and regularity of weak solutions of the Navier–Stokes equations.* — J. Diff. Eqs. **61** (1986), 186–212.
4. E. Hopf, *Über die Anfangwertaufgabe für die Hydrodynamischen Grundgleichungen.* — Math. Nachr. **4** (1951), 213–231.
5. H. Kozono, *Uniqueness and regularity of weak solutions to the Navier–Stokes equations.* Lect. Notes Num. Appl. Anal. **16** (1998), 161–208.
6. O. Kreml, M. Pokorný, *A regularity criterion for the angular velocity component in axisymmetric Navier–Stokes equations.* — Electronic J. Diff. Eqs. **2007**, No. 08 (2007), 1–10.
7. H. Kozono, H. Sohr, *Remark on uniqueness of weak solutions to the Navier–Stokes equations.* — Analysis **16** (1996), 255–271.
8. O. A. Ladyzhenskaya, *The mathematical theory of viscous incompressible flow* [in Russian]. Moscow, Nauka, 1970.
9. O. A. Ladyzhenskaya, *On uniqueness and regularity of weak solutions to the Navier–Stokes equations.* — Zap. Nauchn. Semin. LOMI **5** (1967), 169–185.
10. O. A. Ladyzhenskaya, *On unique solvability of three-dimensional Cauchy problem for the Navier–Stokes equations under the axial symmetry.* — Zap. Nauchn. Semin. LOMI **7** (1968), 155–177.
11. O. A. Ladyzhenskaya, *Solutions "in the large" of the nonstationary boundary-value problem for the Navier–Stokes system with two space variables.* — Comm. Pure Appl. Math. **12** (1959), 427–433.
12. J. Leray, *Sur le mouvements d'un liquide visqueux emplissant l'espace.* — Acta Math. **63** (1934), 193–248.
13. J. Leray, *Etude de diverses équations intégrales non linéaires et de quelques problémes que pose l'hydrodynamique.* — J. Math. Pures Appl. **9** (12) (1933), 1–82.
14. S. Leonardi, J. Málek, J. Nečas, M. Pokorný, *On axially symmetric flows in $\mathbb{R}^3$.* — ZAA **18** (1999), 639–649.
15. O. A. Ladyzhenskaya, V. A. Solonnikov, N. N. Uraltseva, *Linear and quasilinear equations of parabolic type* [in Russian]. Nauka, Moscow, 1967.
16. J. Neustupa, M. Pokorný, *Axisymmetric flow of Navier–Stokes fluid in the whole space with nonzero angular velocity component.* — Math. Boh. **126** (2) (2001).
17. J. Neustupa, M. Pokorný, *An interior regularity criterion for an axially symmetric suitable weak solution to the Navier–Stokes equations.* Preprint, Charles Univ, Prague (2000).
18. G. Prodi, *Un teorema di unicitá per el equazioni di Navier–Stokes.* — Ann. Mat. Pura Appl. **48** (1959), 173–182.
19. M. Pokorný, *A regularity criterion for the angular velocity component in the case of axisymmetric Navier–Stokes equations.* — In: Proceedings of the 4th European

Congress on Elliptic and Parabolic Problems, Rolduc and Gaeta 2001, World Scientific (2002), pp. 233–242.
20. L. Escauriaza, G. Seregin, V. Šverák, *$L_{3,\infty}$-solutions of the Navier–Stokes equations and backward uniqueness.* — Russ. Math. Surv. **58** (2) (2003), 211–250.
21. G. Seregin, *On smoothness of $L_{3,\infty}$-solutions to the Navier–Stokes equations up to boundary.* — Math. Ann. **332** (2005), 219–238.
22. H. Sohr, W. Wahl, *On the singular set and the uniqueness of weak solutions to the Navier–Stokes equations.* — Manuscripta Math. **49** (1984), 27–59.
23. M. R. Ukhovskii, V. I. Yudovich, *Axially symmetric motions of ideal and viscous fluid filling all space.* — Prikl. Mat. Mekh. **32** (1968), 59–69.
24. W. M. Zajączkowski, *Global special regular solutions to the Navier–Stokes equations in a cylindrical domain without the axis of symmetry.* — Top. Meth. Nonlin. Anal. **24** (2004), 69–105.
25. W. M. Zajączkowski, *Global axially symmetric solutions with large swirl to the Navier–Stokes equations. Top. Meth. Nonlin. Anal.*, **29** (2007), 295–331.
26. W. M. Zajączkowski, *Global special regular solutions to the Navier–Stokes equations in a cylindrical domain under boundary slip conditions.* — Gakuto Series Math. **21** (2004), 1–188.
27. W. M. Zajączkowski, *Global special regular solutions to the Navier–Stokes equations in axially symmetric domains under boundary slip conditions.* — Diss. Math. **432** (2005), 1–138.
28. W. M. Zajączkowski, *Special regular solutions to the Navier–Stokes equations.* — Zap. Nauchn. Semin. POMI **362** (2008), 120–152; J. Math. Sc. **159** (4) (2009), 452–471.

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Поступило 20 ноября 2010 г.