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ON THE BOUNDARY REGULARITY OF WEAK SOLUTIONS TO THE MHD SYSTEM

Abstract. We prove the partial regularity of the boundary suitable weak solutions to the MHD system near the plane part of the boundary.

**Dedicated to the jubilee of
G. A. Seregin**

1. Introduction

Assume $\Omega\subset\mathbb{R}^3$ is a C^2 – smooth bounded domain and $Q_T=\Omega\times(0,T)$. In this paper, we investigate the boundary regularity of solutions to the principal system of magnetohydrodynamics (the MHD equations):

$$\left.\begin{aligned}\partial_t v+(v\cdot\nabla)v-\Delta v+\nabla p=\operatorname{rot}H\times H\\ \operatorname{div}v=0\end{aligned}\right\}\quad\text{in}\quad Q_T,\tag{1.1}$$

$$\left.\begin{aligned}\partial_t H+\operatorname{rot}\operatorname{rot}H=\operatorname{rot}(v\times H)\\ \operatorname{div}H=0\end{aligned}\right\}\quad\text{in}\quad Q_T.\tag{1.2}$$

Here unknowns are the velocity field $v:Q_T\to\mathbb{R}^3$, pressure $p:Q_T\to\mathbb{R}$, and the magnetic field $H:Q_T\to\mathbb{R}^3$. We impose on v and H the initial and boundary conditions:

$$v\big|_{\partial\Omega\times(0,T)}=0,\quad H_\nu\big|_{\partial\Omega\times(0,T)}=0,\quad(\operatorname{rot}H)_\tau\big|_{\partial\Omega\times(0,T)}=0,\tag{1.3}$$

$$v\big|_{t=0}=v_0,\quad H\big|_{t=0}=H_0.\tag{1.4}$$

Henceforth, we denote by ν the outer normal to $\partial\Omega$ and $H_\nu=H\cdot\nu$, $(\operatorname{rot}H)_\tau=\operatorname{rot}H-\nu(\operatorname{rot}H\cdot\nu)$. We introduce the following definition:

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Definition. *Let $\Gamma\subset\partial\Omega$. The functions (v,H,p) are called a boundary suitable weak solution to the system (1.1), (1.2) near $\Gamma_T\equiv\Gamma\times(0,T)$ if*

(1) $v\in L_{2,\infty}(Q_T)\cap W^{1,0}_2(Q_T)\cap W^{2,1}_{\frac98,\frac32}(Q_T)$,
$H\in L_{2,\infty}(Q_T)\cap W^{1,0}_2(Q_T)$,

(2) $p\in L_{\frac32}(Q_T)\cap W^{1,0}_{\frac98,\frac32}(Q_T)$,

(3) $\operatorname{div} v=0$, $\operatorname{div} H=0$ *a.e. in* Q_T,

(4) $v|_{\partial\Omega}=0$, $H_\nu|_{\partial\Omega}=0$ *in the sense of traces,*

(5) *for any $w\in L_2(\Omega)$ the functions*

$$t\mapsto\int\limits_\Omega v(x,t)\cdot w(x)\;dx\quad\text{and}\quad t\mapsto\int\limits_\Omega H(x,t)\cdot w(x)\;dx$$

are continuous,

(6) *(v,H) satisfy the following integral identities: for any $t\in[0,T]$*

$$\int\limits_\Omega v(x,t)\cdot\eta(x,t)\;dx-\int\limits_\Omega v_0(x)\cdot\eta(x,0)\;dx$$

$$+\int\limits_0^t\int\limits_\Omega\Big(-v\cdot\partial_t\eta+(\nabla v-v\otimes v+H\otimes H):\nabla\eta-(p+\frac12|H|^2\operatorname{div}\eta\Big)dxdt=0,$$

for all $\eta\in W^{1,1}_{\frac52}(Q_t)$ such that $\eta\big|_{\partial\Omega\times(0,t)}=0$,

$$\int\limits_\Omega H(x,t)\cdot\psi(x,t)\;dx-\int\limits_\Omega H_0(x)\cdot\psi(x,0)\;dx$$

$$+\int\limits_0^t\int\limits_\Omega\Big(-H\cdot\partial_t\psi+\operatorname{rot}H\cdot\operatorname{rot}\psi-(v\times H)\cdot\operatorname{rot}\psi\Big)\;dxdt=0,$$

for all $\psi\in W^{1,1}_{\frac52}(Q_t)$ such that $\psi_\nu|_{\partial\Omega\times(0,t)}=0$.

(7) *For every $z_0=(x_0,t_0)\in\Gamma_T$ such that $\Omega_R(x_0)\times(t_0-R^2,t_0)\subset Q_T$ and for any $\zeta\in C_0^\infty(B_R(x_0)\times(t_0-R^2,t_0])$ such that $\frac{\partial\zeta}{\partial\nu}\Big|_{\partial\Omega}=0$ the*

following "local energy inequality near Γ_T" holds:

$$\begin{aligned}
&\sup_{t\in(t_0-R^2,t_0)} \int\limits_{\Omega_R(x_0)} \zeta\Big(|v|^2+|H|^2\Big)\,dx \\
&+2\int\limits_{t_0-R^2}^{t_0}\int\limits_{\Omega_R(x_0)} \zeta\Big(|\nabla v|^2+|\operatorname{rot} H|^2\Big)\,dxdt \\
&\leqslant \int\limits_{t_0-R^2}^{t_0}\int\limits_{\Omega_R(x_0)} \Big(|v|^2+|H|^2\Big)(\partial_t\zeta+\Delta\zeta)\,dxdt \\
&+\int\limits_{t_0-R^2}^{t_0}\int\limits_{\Omega_R(x_0)} \Big(|v|^2+2\overline{p}\Big)v\cdot\nabla\zeta\,dxdt \\
&-2\int\limits_{t_0-R^2}^{t_0}\int\limits_{\Omega_R(x_0)} (H\otimes H):\nabla^2\zeta\,dxdt \\
&+2\int\limits_{t_0-R^2}^{t_0}\int\limits_{\Omega_R(x_0)} (v\times H)(\nabla\zeta\times H)\,dxdt.
\end{aligned} \tag{1.5}$$

We remark also that the following identity holds

$$\begin{aligned}
&\int\limits_{t_0-R^2}^{t_0}\int\limits_{\Omega_R(x_0)} (v\times H)(\nabla\zeta\times H)\,dxdt \\
&=\int\limits_{t_0-R^2}^{t_0}\int\limits_{\Omega_R(x_0)} (v\cdot\nabla\zeta)|H|^2\,dxdt-\int\limits_{t_0-R^2}^{t_0}\int\limits_{\Omega_R(x_0)} (v\cdot H)(H\cdot\nabla\zeta)\,dxdt.
\end{aligned}$$

Here $L_{s,l}(Q_T)$ is the anisotropic Lebesgue space equipped with the norm

$$\|f\|_{L_{s,l}(Q_T)}:=\Big(\int\limits_0^T\Big(\int\limits_\Omega |f(x,t)|^s\,dx\Big)^{l/s}dt\Big)^{1/l},$$

and we use the following notation for the functional spaces:

$$W^{1,0}_{s,l}(Q_T) \equiv L_l(0,T;W^1_s(\Omega)) = \{\ u \in L_{s,l}(Q_T):\ \nabla u \in L_{s,l}(Q_T)\ \},$$
$$W^{2,1}_{s,l}(Q_T) = \{\ u \in W^{1,0}_{s,l}(Q_T):\ \nabla^2 u,\ \partial_t u \in L_{s,l}(Q_T)\ \},$$
$$\overset{\circ}{W}{}^1_s(\Omega) = \{\ u \in W^1_s(\Omega):\ u|_{\partial\Omega} = 0\ \}.$$

The corresponding norms are defined as follows:

$$\|u\|_{W^{1,0}_{s,l}(Q_T)} = \|u\|_{L_{s,l}(Q_T)} + \|\nabla u\|_{L_{s,l}(Q_T)},$$
$$\|u\|_{W^{2,1}_{s,l}(Q_T)} = \|u\|_{W^{1,0}_{s,l}(Q_T)} + \|\nabla^2 u\|_{L_{s,l}(Q_T)} + \|\partial_t u\|_{L_{s,l}(Q_T)}.$$

Theorem 1.1. *For any sufficiently smooth (v_0, H_0) satisfying (1.3) there exists at least one boundary suitable weak solution near $\partial\Omega\times(0,T)$, which satisfies the initial conditions*

$$\|v(\cdot,t) - v_0(\cdot)\|_{L_2(\Omega)} \to 0, \quad \|H(\cdot,t) - H_0(\cdot)\|_{L_2(\Omega)} \to 0 \quad \text{as} \quad t \to +0,$$

and additionally satisfies the global energy inequality

$$\|v\|_{L_{2,\infty}(Q_T)} + \|H\|_{L_{2,\infty}(Q_T)} + \|\nabla v\|_{L_2(Q_T)} + \|\operatorname{rot} H\|_{L_2(Q_T)} \leqslant \|v_0\|_{L_2(\Omega)} + \|H_0\|_{L_2(\Omega)}.$$

The global existence of weak solutions to the MHD Eqs. (1.1)–(1.3) was originally established in [6]. We sketch the proof of Theorem 1.1 in Sec. 3.

2. Main results

Let $B(x_0, r)$ denote the open ball in $\mathbb{R}^3$ of radius r centered at x_0 and let $B^+(x_0, r)$ denote the half-ball $\{x \in B(x_0, r) \mid x_3 > 0\}$. For $z_0 = (x_0, t_0)$,

$$Q(z_0, r) = B(x_0, r) \times (t_0 - r^2, t_0), \quad Q^+(z_0, r) = B^+(x_0, r) \times (t_0 - r^2, t_0).$$

In this paper, we use the following abbreviations: $B(r) = B(0, r)$, $B^+(r) = B^+(0, r)$ etc, $B = B(1)$, $B^+ = B^+(1)$ etc.

Theorems below present the main result of the paper.

Theorem 2.1. *There exists an absolute constant $\varepsilon_* > 0$ with the following property. Assume (v, H, p) is a boundary suitable weak solution in Q_T and assume $z_0 = (x_0, t_0) \in \partial\Omega \times (0, T)$ is such that x_0 belongs to the plane part of $\partial\Omega$. If there exists $r_0 > 0$ such that $Q^+(z_0, r_0) \subset Q_T$ and*

$$\frac{1}{r_0^2} \int\limits_{Q^+(z_0,r_0)} \Big(|v|^3 + |H|^3 + |p|^{\frac{3}{2}} \Big) dxdt < \varepsilon_*,$$

then the functions v and H are Hölder continuous on the closure of $Q^+(z_0, \frac{r_0}{2})$.

Theorem 2.2. *For any $K > 0$ there exists $\varepsilon_0(K) > 0$ with the following property. Assume (v, H, p) is a boundary suitable weak solution in Q_T and assume $z_0 = (x_0, t_0) \in \partial\Omega \times (0, T)$ is such that x_0 belongs to the plane part of $\partial\Omega$. If*

$$\limsup_{r\to 0} \Big(\frac{1}{r} \int\limits_{Q(z_0,r)} |\nabla H|^2 \, dxdt \Big)^{1/2} < K \tag{2.6}$$

and

$$\limsup_{r\to 0} \Big(\frac{1}{r} \int\limits_{Q(z_0,r)} |\nabla v|^2 \, dxdt \Big)^{1/2} < \varepsilon_0, \tag{2.7}$$

then there exists $\rho_* > 0$ such that the functions v and H are Hölder continuous on the closure of $Q^+(z_0, \rho_*)$.

Theorem 2.3. *Assume that (v, H, p) is a boundary suitable weak solution in Q_T and denote by Γ the plane part of $\partial\Omega$. Then there exists a closed set $\Sigma \subset \Gamma$ such that for any $z_0 \in (\Gamma \setminus \Sigma) \times (0, T]$ the functions (v, H) are Hölder continuous in a certain neighborhood of z_0, and*

$$\mathcal{P}^1(\Sigma) = 0,$$

where $\mathcal{P}^1(\Sigma)$ is the one-dimensional parabolic Hausdorff measure of Σ.

For the MHD equations, Theorem 2.3 presents a result which is a boundary analog of the famous Caffarelli–Kohn–Nirenberg (CKN) theorem for the Navier-Stokes system (see [2] and [7]). The boundary regularity of solutions to the Navier–Stokes equations was originally investigated by G. Seregin in [8] and [9] in the case of a plane part of the boundary and by G. Seregin, T. Shilkin, and V. Solonnikov in [11] in the case of a curved boundary. The internal partial regularity of solutions to the MHD system was originally proved by C. He and Z. Xin in [4], see also [15, 16]. Note that thought using the methods of out paper one can prove various ε-regularity conditions involving various scale-invariant functionals (such as in [4]) in the present paper we concentrate on the condition (2.6), (2.7) as this condition provides the optimal estimate of the Hausdorff measure of thesingular set Σ in Theorem 2.3.

Our paper is organized as follows: in Sec. 3, we outline the proof of Theorem 1.1. In Sec. 4, we prove that the weak solutions of the linearized MHD equations are Hölder continuous up to the boundary. Section 5 contains the proof of the Decay Lemma and the sketch of the proof of Theorem 2.1. Section 6 is concerned with the estimates of some Morrey-type functionals of weak solutions to the heat equation near the boundary. These estimates together with the estimates of the scale invariant energy functionals obtained in Sec. 7 turn to be crucial for the key estimate (8.3). Finally, in Sec. 8, we present the proofs of Theorems 2.2 and 2.3.

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3. Local energy inequality and existence of boundary suitable weak solutions

Let ω denote the usual smooth Sobolev kernel and $\omega_\delta(x) = \delta^{-3}\omega(\delta x)$. By $\omega_\delta * H$ we denote the convolution of H with ω_δ. Assume that v_0^δ and H_0^δ are sooth divergent-free functions satisfying the boundary conditions (1.3) and additionally satisfying the properties

$$v_0^\delta \to v_0, \quad H_0^\delta \to H_0 \quad \text{in} \quad L_2(\Omega) \quad \text{as} \quad \delta \to 0.$$

Consider the problem

$$\left.\begin{array}{c} \partial_t v + ((\omega_\delta * v)\cdot\nabla)v - \Delta v + \nabla p = \operatorname{rot} H \times (\omega_\delta * H) \\ \operatorname{div} v = 0 \end{array}\right\} \quad \text{in} \quad Q_T, \tag{3.1}$$

$$\left.\begin{array}{c} \partial_t H + \operatorname{rot}\operatorname{rot} H = \operatorname{rot}(v \times (\omega_\delta * H)) \\ \operatorname{div} H = 0 \end{array}\right\} \quad \text{in} \quad Q_T, \tag{3.2}$$

$$v|_{\partial\Omega\times(0,T)} = 0, \quad H_\nu|_{\partial\Omega\times(0,T)} = 0, \quad (\operatorname{rot} H)_\tau|_{\partial\Omega\times(0,T)} = 0,$$

$$v\big|_{t=0} = v_0^\delta, \quad H\big|_{t=0} = H_0^\delta.$$

Using Galerkin approximations and energy estimates it is not difficult to prove existence of the unique smooth solution $(v^\delta, H^\delta, p^\delta)$ to this problem. Moreover, these functions satisfy the global energy inequality

$$\|v^\delta\|_{L_{2,\infty}(Q_T)} + \|H^\delta\|_{L_{2,\infty}(Q_T)} + \|v^\delta\|_{W_2^{1,0}(Q_T)} + \|H^\delta\|_{W_2^{1,0}(Q_T)} \leqslant c\,\Big(\|v_0\|_{L_2(\Omega)} + \|H_0\|_{L_2(\Omega)} \Big).$$

From this estimate using the interpolation inequality we obtain the estimate

$$\|((\omega_\delta * v^\delta)\cdot\nabla)v^\delta\|_{L_{\frac{9}{8},\frac{3}{2}}(Q_T)} + \|\operatorname{rot} H^\delta \times (\omega_\delta * H^\delta)\|_{L_{\frac{9}{8},\frac{3}{2}}(Q_T)} \leqslant c.$$

Applying the coercive estimates for the linear Stokes problem (see, for example, [14]), we obtain

$$\|v^\delta\|_{W^{2,1}_{\frac{9}{8},\frac{3}{2}}(Q_T)} + \|\nabla p^\delta\|_{L_{\frac{9}{8},\frac{3}{2}}(Q_T)} \leqslant c.$$

So, the only thing we need is to verify the local energy inequality (1.5). Then the result of Theorem 1.1 follows if we pass to the limit as $\delta \to 0$.

To simplify notations below we omit the index δ and denote by (v, H, p) the smooth functions $(v^\delta, H^\delta, p^\delta)$. Take $z_0 = (x_0, t_0) \in \partial\Omega \times (0, T)$ and choose R so that $\Omega_R(x_0) \times (t_0 - R^2, t_0) \subset Q_T$. Without a loss of generality we can put $x_0 = 0$, $t_0 = 0$. Assume $\zeta \in C_0^\infty(B_R \times (-R^2, 0])$ satisfies $\frac{\partial\zeta}{\partial\nu}\big|_{\partial\Omega} = 0$ and multiply equation (3.1) by the test-function $\eta = \zeta v$. Integrating the result over Ω, integrating by parts and taking into account the relation

$$\int\limits_\Omega (\omega_\delta * v_i) v_{j,i} \zeta v_j \; dx = -\frac{1}{2} \int\limits_\Omega |v|^2 (\omega_\delta * v)\cdot\nabla\zeta \; dx.$$

We obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt} \int\limits_\Omega \zeta|v|^2 + \int\limits_\Omega \zeta|\nabla v|^2 \; dx &= \frac{1}{2}\int\limits_\Omega |v|^2(\partial_t\zeta + \Delta\zeta)\; dx \\ &+ \int\limits_\Omega \overline{p}\; v\cdot\nabla\zeta \; dx + \frac{1}{2}\int\limits_\Omega |v|^2(\omega_\delta * v)\cdot\nabla\zeta\; dx \\ &+ \int\limits_\Omega (\operatorname{rot} H \times (\omega_\delta * H))\cdot\zeta v \; dx = 0. \qquad (3.3)\end{aligned}$$

Now we multiply the equation for the magnetic field by ζH. Integrating by parts and taking into account the boundary conditions (1.3) after routine calculations we arrive at the relation

$$\frac{1}{2}\frac{d}{dt}\int\limits_{\Omega}\zeta|H|^2\,dx+\int\limits_{\Omega}\zeta|\operatorname{rot}H|^2\,dx$$

$$=\frac{1}{2}\int\limits_{\Omega}|H|^2(\partial_t\zeta+\Delta\zeta)\,dx-\int\limits_{\Omega}H\otimes H:\nabla^2\zeta\,dx+\int\limits_{\Omega}\operatorname{rot}(v\times(\omega_\delta*H))\cdot\zeta H\,dx.$$

Integrating by parts, we obtain

$$\int\limits_{\Omega}\operatorname{rot}(v\times(\omega_\delta*H))\cdot\zeta H\,dx$$

$$=\int\limits_{\Omega}\zeta(v\times(\omega_\delta*H))\cdot\operatorname{rot}H\,dx+\int\limits_{\Omega}(v\times(\omega_\delta*H))\cdot(\nabla\zeta\times H)\,dx.$$

Note that

$$\int\limits_{\Omega}\zeta(v\times(\omega_\delta*H))\cdot\operatorname{rot}H\,dx=\int\limits_{\Omega}\zeta((\omega_\delta*H)\times\operatorname{rot}H)\cdot v\,dx.$$

Hence we obtain

$$\frac{1}{2}\frac{d}{dt}\int\limits_{\Omega}\zeta|H|^2\,dx+\int\limits_{\Omega}\zeta|\operatorname{rot}H|^2\,dx=\frac{1}{2}\int\limits_{\Omega}|H|^2(\partial_t\zeta+\Delta\zeta)\,dx$$

$$-\int\limits_{\Omega}H\otimes H:\nabla^2\zeta\,dx+\int\limits_{\Omega}((\omega_\delta*H)\times\operatorname{rot}H)\cdot\zeta v\,dx$$

$$+\int\limits_{\Omega}(v\times(\omega_\delta*H))\cdot(\nabla\zeta\times H)\,dx. \tag{3.4}$$

Adding identities (3.3) and (3.4) together, using the conciliation

$$\int\limits_{\Omega}(\operatorname{rot}H\times(\omega_\delta*H))\cdot\zeta v dx+\int\limits_{\Omega}((\omega_\delta*H)\times\operatorname{rot}H)\cdot\zeta v\,dx=0,$$

and passing to the limit as $\delta\to 0$ we arrive at (1.5). Theorem 1.1 is proved.

In conclusion of this section, we introduce the following version of the local energy inequality near the plane part of the boundary.

Theorem 3.1. *Assume (v, H, p) is a boundary suitable weak solution satisfied the MHD equations in a domain containing the set $Q^+ = B^+ \times (-1, 0)$ such that the plane part of Q^+ belongs to the boundary $\partial\Omega \times (-1, 0)$. Obviously, we have*

$$v\big|_{x_3=0} = 0, \quad H_3\big|_{x_3=0} = 0, \quad H_{1,3}\big|_{x_3=0} = H_{2,3}\big|_{x_3=0} = 0.$$

Let $\zeta \in C_0^\infty(B \times (-1, 0])$ be a cut-off function such that $\zeta_{,3}|_{x_3=0} = 0$. Assume $b \in \mathbb{R}^3$ is an arbitrary constant vector of the form $b = (b_1, b_2, 0)$. Then the following inequality holds

$$\sup_{t\in(-1,0)} \int\limits_{B^+} \zeta\Big(|v|^2 + |\overline{H}|^2\Big)\, dx + 2\int\limits_{Q^+} \zeta\Big(|\nabla v|^2 + |\operatorname{rot}\overline{H}|^2\Big)\, dz$$

$$\leqslant \int\limits_{Q^+} \Big(|v|^2 + |\overline{H}|^2\Big)(\partial_t\zeta + \Delta\zeta)\, dz + \int\limits_{Q^+} \Big(|v|^2 + 2\overline{p}\Big) v \cdot \nabla\zeta\, dz$$

$$-2\int\limits_{Q^+} (\overline{H} \otimes \overline{H}) : \nabla^2\zeta\, dz + 2\int\limits_{Q^+} (v \times H)(\nabla\zeta \times \overline{H})\, dz, \qquad (3.5)$$

where $\overline{H} = H - b$.

Proof. Relation (3.5) is a combination of (1.5) with he relation obtained from (1.2) multiplied by the test function $\psi = \zeta b$ where $b = (b_1, b_2, 0)$ is a constant vector. The proof is simple and we omit it.

4. Linear estimate

Theorem 4.1. *For any $M > 0$ there exists $C(M) > 0$ such that for any $a \in \mathbb{R}^3$ such that $a = (a_1, a_2, 0)^T$ and $|a| \leqslant M$, and for any (u, h, q) satisfying the linear system*

$$\begin{aligned} \partial_t u - \Delta u + \nabla q &= \operatorname{rot} h \times a \\ \operatorname{div} u &= 0, \end{aligned} \qquad (4.1)$$

$$\begin{aligned} \partial_t h - \Delta h &= \operatorname{rot}(u \times a) \\ \operatorname{div} h &= 0, \end{aligned} \qquad (4.2)$$

$$u\Big|_{x_3=0}=0,\quad h_3\Big|_{x_3=0}=0,\quad \frac{\partial h_\alpha}{\partial x_3}\Big|_{x_3=0}=0,\quad \alpha=1,2, \tag{4.3}$$

the following estimate holds:

$$\|u\|_{C^{\frac13,\frac16}(Q^+(\frac12))}+\|h\|_{C^{\frac13,\frac16}(Q^+(\frac12))}$$
$$\leqslant C(M)\Big(\|u\|_{L_3(Q^+)}+\|h-b\|_{L_3(Q^+)}+\|q-c\|_{L_{3/2}(Q^+)}\Big). \tag{4.4}$$

Here, $b\in\mathbb{R}^3$ is an arbitrary vector of the form $b=(b_1,b_2,0)^T$, and $c\in\mathbb{R}$ is an arbitrary constant.

Proof.
1. Similar to Theorem 3.1 we obtain the relation

$$\|u\|_{L_{2,\infty}(Q^+(\frac{9}{10}))}+\|h\|_{L_{2,\infty}(Q^+(\frac{9}{10}))}+\|\nabla u\|_{L_2(Q^+(\frac{9}{10}))}+\|\nabla h\|_{L_2(Q^+(\frac{9}{10}))}$$
$$\leqslant C(M)\ \Big(\|u\|_{L_2(Q^+)}+\|h-b\|_{L_2(Q^+)}+\|u\overline{q}\|_{L_1(Q^+)}\Big). \tag{4.5}$$

Here, $\overline{q}\equiv q-c$. Note that the right-hand side of the last inequality can be estimated by the right-hand side of (4.4) via Hölder inequality.

2. For the function u satisfying system (4.3), we have the following estimate with arbitrary s, $l\in(1,+\infty)$, and $0<\rho<r<\frac{9}{10}$ (see 9):

$$\|u\|_{W^{2,1}_{s,l}(Q^+(\rho))}$$
$$\leqslant C\ \Big(\|\operatorname{rot} h\|_{L_{s,l}(Q^+(r))}+\|u\|_{L_3(Q^+)}+\|\overline{q}\|_{L_{3/2}(Q^+)}\Big). \tag{4.6}$$

Here, C depends on M, r, ρ, s, and l.

3. Denote by h^* the following extension of the function h from Q^+ onto Q: we take $h^*(x,t)=h(x,t)$ for $x_3\geqslant 0$ and take

$$\begin{cases} h^*_\alpha(x_1,x_2,x_3,t)=\ \ h_\alpha(x_1,x_2,-x_3,t) \\ h^*_3(x_1,x_2,x_3,t)=-h_3(x_1,x_2,-x_3,t)\end{cases}\qquad \text{for}\quad x_3<0.$$

We denote also $g=\operatorname{rot}(u\times H)$ and let g^* be the extension of g from Q^+ onto Q obtained in the following way: for g_1 and g_2 we take the even extensions and for g_3 we consider the odd one. Then functions h^*, u^* satisfy the equation

$$\partial_t h^*-\Delta h^*=g^*\quad\text{in}\quad Q. \tag{4.7}$$

4. For the function h^* satisfying the heat equation (4.7), the estimate similar to (4.6) holds

$$\|h^*\|_{W^{2,1}_{s,l}(Q(\rho))} \leqslant C \, \Big(\|g^*\|_{L_{s,l}(Q(r))} + \|h^* - b\|_{L_3(Q)} \Big).$$

Here, s, $l \in (1, +\infty)$, and $0 < \rho < r \leqslant \frac{9}{10}$ are arbitrary and C depends on M, r, ρ, s, and l. This estimate provides the inequality

$$\|h\|_{W^{2,1}_{s,l}(Q^+(\rho))} \leqslant C \, \Big(\|\nabla u\|_{L_{s,l}(Q^+(r))} + \|h - b\|_{L_3(Q^+)} \Big). \tag{4.8}$$

5. First we apply (4.8) with $s = l = 2$ and $\rho = \frac{4}{5}$, $r = \frac{9}{10}$. Taking into account the energy estimate (4.5) we obtain the estimate of the norm $\|h\|_{W^{2,1}_2(Q^+(\frac{4}{5}))}$ by the right-hand side of (4.4). Then using the imbedding theorem $W^{2,1}_2(Q^+(\frac{4}{5})) \hookrightarrow W^{1,0}_{\frac{10}{3}}(Q^+(\frac{4}{5}))$ we obtain the estimate of the norm $\|\nabla h\|_{L_{\frac{10}{3}}(Q^+(\frac{4}{5}))}$.

6. Now applying (4.6) with $s = \frac{10}{3}$, $l = \frac{3}{2}$ and $\rho = \frac{7}{10}$, $r = \frac{4}{5}$, we obtain the estimate

$$\|u\|_{W^{2,1}_{\frac{10}{3},\frac{3}{2}}(Q^+(\frac{7}{10}))} \leqslant C \, \Big(\|\operatorname{rot} h\|_{L_{\frac{10}{3},\frac{3}{2}}(Q^+(\frac{4}{5}))} + \|u\|_{L_3(Q^+)} + \|\overline{q}\|_{L_{3/2}(Q^+)} \Big).$$

By Hölder inequality, we estimate $\|\operatorname{rot} h\|_{L_{\frac{10}{3},\frac{3}{2}}(Q^+(\frac{4}{5}))}$ by the norm $\|\nabla h\|_{L_{\frac{10}{3}}(Q^+(\frac{4}{5}))}$, which was already estimated on the previous step. On the other hand, by the imbedding theorem we obtain

$$\|\nabla u\|_{L_{\frac{3}{2}}(-(\frac{7}{10})^2,0;L_\infty(B^+(\frac{7}{10})))} \leqslant C\|u\|_{W^{2,1}_{\frac{10}{3},\frac{3}{2}}(Q^+(\frac{7}{10}))}. \tag{4.9}$$

7. Estimate (4.9) implies in particular that $\nabla u \in L_{9,\frac{3}{2}}(Q^+(\frac{7}{10}))$. Applying (4.8) with $s = 2$, $l = \frac{3}{2}$, we obtain the estimate

$$\|h\|_{W^{2,1}_{9,\frac{3}{2}}(Q^+(\frac{3}{5}))} \leqslant C \, \Big(\|\nabla u\|_{L_{9,\frac{3}{2}}(Q^+(\frac{7}{10}))} + \|h - b\|_{L_3(Q^+)} \Big).$$

8. Finally, we apply (4.6) with $s = 9$, $l = \frac{3}{2}$ and obtain

$$\|u\|_{W^{2,1}_{9,\frac{3}{2}}(Q^+(\frac{1}{2}))} \leqslant C \, \Big(\|\operatorname{rot} h\|_{L_{9,\frac{3}{2}}(Q^+(\frac{3}{5}))} + \|u\|_{L_3(Q^+)} + \|\overline{q}\|_{L_{3/2}(Q^+)} \Big).$$

9. Gathering all the estimates together we arrive at the estimate

$$\|u\|_{W^{2,1}_{9,\frac{3}{2}}(Q^+(\frac{1}{2}))} + \|h\|_{W^{2,1}_{9,\frac{3}{2}}(Q^+(\frac{1}{2}))}$$

$$\leqslant C(M)\Big(\|u\|_{L_3(Q^+)} + \|h-b\|_{L_3(Q^+)} + \|q-c\|_{L_{3/2}(Q^+)}\Big).$$

The statement of the theorem follows now from the imbedding $W^{2,1}_{s,l}(Q_T) \hookrightarrow C^{\alpha,\frac{\alpha}{2}}(\overline{Q}_T)$ as $\alpha = 2 - \frac{3}{s} - \frac{2}{l} > 0$, where $s = 9$, $l = \frac{3}{2}$, and $\alpha = \frac{1}{3}$. Theorem 4.1 is proved.

Corollary 4.1. *Let us introduce the functional*

$$\begin{aligned} Y_\tau(v,H,p) := \Big(\fint\limits_{Q^+(\tau)} |v|^3 \, dxdt \Big)^{1/3} \\ + \tau \Big(\fint\limits_{Q^+(\tau)} |p - [p]_{B^+(\tau)}|^{3/2} \, dxdt \Big)^{2/3} \\ + \Big(\fint\limits_{Q^+(\tau)} |H - b_\tau(H)|^3 \, dxdt \Big)^{1/3}, \end{aligned} \tag{4.10}$$

where $b_\tau(H) := ((H_1)_{Q^+(\tau)}, (H_2)_{Q^+(\tau)}, 0)$. *Then for any* $M > 0$ *there is a constant* $C_l(M) > 0$ *such that for any solution* (u,h,q) *of the linear system* (4.1)–(4.3), *the following estimate holds:*

$$Y_\tau(u,h,q) \leqslant C_l(M)\ \tau^{1/3}\ Y_1(u,h,q).$$

5, The decay estimate and the proof of Theorem 2.1

In this section, we consider the MHD system

$$\left.\begin{aligned} \partial_t v + (v\cdot\nabla)v - \Delta v + \nabla p = \operatorname{rot} H \times H \\ \operatorname{div} v = 0 \end{aligned}\right\} \text{ in } Q^+, \tag{5.1}$$

$$\left.\begin{aligned} \partial_t H - \Delta H = \operatorname{rot}(v \times H) \\ \operatorname{div} H = 0 \end{aligned}\right\} \text{ in } Q^+, \tag{5.2}$$

$$v|_{x_3=0} = 0, \quad H_3|_{x_3=0} = 0, \quad H_{1,3}\big|_{x_3=0} = H_{2,3}\big|_{x_3=0} = 0. \tag{5.3}$$

We denote by $Y_\tau(v,H,p)$ the functional introduced in (4.10). We also denote by $Y_\tau(v)$, $\widetilde{Y}_\tau(H)$, and $\widehat{Y}_\tau(p)$ the functionals

$$Y_\tau(v) := \Big(\fint_{Q^+(\tau)} |v|^3 \, dxdt \Big)^{1/3},$$

$$\widetilde{Y}_\tau(H) := \Big(\fint_{Q^+(\tau)} |H - b_\tau(H)|^3 \, dxdt \Big)^{1/3},$$

$$\widehat{Y}_\tau(p) := \tau \Big(\fint_{Q^+(\tau)} |p - [p]_{B^+(\tau)}|^{3/2} \, dxdt \Big)^{2/3}.$$

Theorem 5.1. *There exists an absolute constant $\varepsilon_0 > 0$ such that for any $M > 0$ there exists $C_* = C_*(M)$ with the following properties. For any boundary suitable weak solution (v,H,p) of the* MHD *system* (5.1)–(5.3), *the following implication holds: if*

$$Y_1(v,H,p) < \varepsilon_0$$

and

$$|(H_1)_{Q^+}| + |(H_2)_{Q^+}| \leqslant M,$$

then

$$Y_\tau(v,H,p) \leqslant C_* \, \tau^{1/3} \, Y_1(v,H,p). \tag{5.4}$$

Proof.

1. Arguing by contradiction we assume there exists a sequence of numbers $\varepsilon_m \to 0$, and a sequence of boundary suitable weak solutions (v^m, H^m, p^m) such that

$$Y_1(v^m, H^m, p^m) = \varepsilon_m \ \to \ 0$$

and

$$Y_\tau(v^m, H^m, p^m) \geqslant C_* \tau^{1/3} \varepsilon_m.$$

2. Let us introduce functions

$$u^m(x,t) = \frac{1}{\varepsilon_m} \, v^m(x,t),$$

$$q^m(x,t) = \frac{1}{\varepsilon_m} \Big(p^m(x,t) - [p^m]_{B^+}(t) \Big),$$

$$h^m(x,t) = \frac{1}{\varepsilon_m} \Big(H^m(x,t) - a^m \Big),$$

$$a^m = b_1(H^m).$$

Then

$$Y_1(u^m, h^m, q^m) = 1, \quad Y_\tau(u^m, h^m, q^m) \geqslant C_* \tau^{1/3}, \tag{5.5}$$

and (u^m, h^m, q^m) satisfy the following equations in $\mathcal{D}'(Q^+)$

$$\begin{aligned} \partial_t u^m + \varepsilon_m (u^m \cdot \nabla) u^m - \Delta u^m + \nabla q^m &= \operatorname{rot} h^m \times (\varepsilon_m h^m + a^m) \\ \operatorname{div} u^m &= 0, \end{aligned} \tag{5.6}$$

$$\begin{aligned} \partial_t h^m - \Delta h^m &= \operatorname{rot} \big(u^m \times (\varepsilon_m h^m + a^m) \big) \\ \operatorname{div} h^m &= 0, \end{aligned} \tag{5.7}$$

$$u^m \big|_{x_3=0} = 0, \quad h_3^m \big|_{x_3=0} = 0, \quad \frac{\partial h_1^m}{\partial x_3} \Big|_{x_3=0} = \frac{\partial h_2^m}{\partial x_3} \Big|_{x_3=0} = 0. \tag{5.8}$$

3. Conditions (5.5) imply in particular the boundedness

$$\sup_m \Big(\|u^m\|_{L_3(Q^+)} + \|h^m\|_{L_3(Q^+)} + \|q^m\|_{L_{\frac{3}{2}}(Q^+)} \Big) < +\infty. \tag{5.9}$$

From the local energy inequality near the boundary (3.5) we obtain

$$\begin{aligned} &\|u^m\|_{L_{2,\infty}(Q^+(\frac{9}{10}))} + \|h^m\|_{L_{2,\infty}(Q^+(\frac{9}{10}))} \\ &\qquad + \|u^m\|_{W_2^{1,0}(Q^+(\frac{9}{10}))} + \|h^m\|_{W_2^{1,0}(Q^+(\frac{9}{10}))} \leqslant C. \end{aligned} \tag{5.10}$$

From Eqs. (5.6), (5.7), and (5.8) we also obtain the estimate

$$\|\partial_t u^m\|_{L_{\frac{5}{3}}(-1,0;W_{\frac{5}{3}}^{-1}(B^+))} + \|\partial_t h^m\|_{L_{\frac{5}{3}}(-1,0;W_{\frac{5}{3}}^{-1}(B^+))} \leqslant C.$$

4. Hence we can extract subsequences

$$\begin{aligned} u^m &\rightharpoonup u \quad \text{in} \quad L_3(Q^+), \\ h^m &\rightharpoonup h \quad \text{in} \quad L_3(Q^+), \\ q^m &\rightharpoonup q \quad \text{in} \quad L_{\frac{3}{2}}(Q^+), \end{aligned} \tag{5.11}$$

$$\begin{aligned} u^m &\rightharpoonup u \quad \text{in} \quad W_2^{1,0}\Big(Q^+\Big(\frac{9}{10}\Big)\Big), \\ h^m &\rightharpoonup h \quad \text{in} \quad W_2^{1,0}\Big(Q^+\Big(\frac{9}{10}\Big)\Big), \end{aligned} \tag{5.12}$$

$$\begin{aligned} u^m &\to u \quad \text{in} \quad L_3\Big(Q^+\Big(\frac{9}{10}\Big)\Big), \\ h^m &\to h \quad \text{in} \quad L_3\Big(Q^+\Big(\frac{9}{10}\Big)\Big), \\ a^m &\to a \quad \text{in} \quad \mathbb{R}^3. \end{aligned} \tag{5.13}$$

5. Passing to the limit in Eqs. (5.6)–(5.8), we obtain

$$\begin{aligned} \partial_t u - \Delta u + \nabla q = \operatorname{rot} h \times a \qquad & \text{in} \quad Q^+, \\ \operatorname{div} u = 0 \qquad & \text{in} \quad Q^+, \\ u|_{x_3=0} = 0, \qquad & \end{aligned} \tag{5.14}$$

$$\begin{aligned} \partial_t h - \Delta h = \operatorname{rot}(u \times a) \qquad & \text{in} \quad Q^+, \\ \operatorname{div} h = 0 \qquad & \text{in} \quad Q^+, \\ h_3|_{x_3=0} = 0, \quad \frac{\partial h_1}{\partial x_3}\Big|_{x_3=0} = \frac{\partial h_2}{\partial x_3}\Big|_{x_3=0} = 0. & \end{aligned} \tag{5.15}$$

6. From (5.13) we conclude

$$\lim_{m\to+\infty} Y_\tau(u^m) = Y_\tau(u), \qquad \lim_{m\to+\infty} \widetilde{Y}_\tau(h^m) = \widetilde{Y}_\tau(h)$$

and hence

$$\limsup_{m\to\infty} Y_\tau(u^m, h^m, q^m) \leqslant Y_\tau(u) + \widetilde{Y}_\tau(h) + \limsup_{m\to\infty} \widehat{Y}_\tau(q^m). \tag{5.16}$$

As (u,h,q) is a solution to the linear problem (5.14)–(5.15), from Theorem 4.1, we obtain

$$Y_\tau(u)+\widetilde{Y}_\tau(h)\leqslant C(M)\ \tau^{1/3}\ Y_1(u,h,q). \tag{5.17}$$

7. Now we are going to estimate $\limsup\limits_{m\to\infty}\widehat{Y}_\tau(q^m)$. For this purpose we decompose (u^m,q^m) and (u,q) as

$$\begin{aligned} u^m&=u_1^m+u_2^m, & q^m&=q_1^m+q_2^m,\\ u&=u_1+u_2, & q&=q_1+q_2, \end{aligned}$$

where $(u_1^m,q_1^m)\in W^{2,1}_{\frac98,\frac32}(\Pi^+)\times W^{1,0}_{\frac98,\frac32}(\Pi^+)$ are determined as a solutions of the following initial boundary-value problems in $\Pi^+=\mathbb{R}^3_+\times(-1,0)$:

$$\begin{aligned} \partial_t u_1^m-\Delta u_1^m+\nabla q_1^m&=f^m &&\text{in}\quad \Pi_+,\\ \operatorname{div} u_1^m&=0 &&\text{in}\quad \Pi_+,\\ u_1^m|_{t=-1}&=0, && u_1^m|_{x_3=0}=0, \end{aligned}$$

where f^m is defined by the expression $\operatorname{rot} h^m\times(\varepsilon^m h^m+a^m)-\varepsilon^m u^m\cdot\nabla u^m$ on the set $Q^+(\frac{9}{10})$ and extended by zero onto the whole Π^+. Similarly, (u_1,q_1) are determined by the relations

$$\begin{aligned} \partial_t u_1-\Delta u_1+\nabla q_1&=f &&\text{in}\quad \Pi_+,\\ \operatorname{div} u_1&=0 &&\text{in}\quad \Pi_+,\\ u_1|_{t=-1}&=0, && u_1|_{x_3=0}=0, \end{aligned} \tag{5.18}$$

where f determined by the expression $\operatorname{rot} h\times a$ on the set $Q^+(\frac{9}{10})$ and extended by zero onto the whole Π^+.

8. As functions $u_1^m-u_1$ and $q_1^m-q_1$ are the solution of the first initial boundary-value problem in Π^+ with the right-hand side f^m-f and zero initial and boundary conditions, we obtain the estimate (see, for example, [14, Proposition 2.1])

$$\begin{aligned} &\|u_1^m\|_{W^{2,1}_{\frac98,\frac32}(\Pi^+)}+\|\nabla q_1^m\|_{L_{\frac98,\frac32}(\Pi^+)}\\ &\leqslant C\|f^m\|_{L_{\frac98,\frac32}(Q^+(\frac{9}{10}))}\|u_1^m-u_1\|_{W^{2,1}_{\frac98,\frac32}(\Pi^+)}+\|\nabla q_1^m-\nabla q_1\|_{L_{\frac98,\frac32}(\Pi^+)}\\ &\qquad\leqslant C\|f^m-f\|_{L_{\frac98,\frac32}(Q^+(\frac{9}{10}))} \end{aligned} \tag{5.19}$$

Note that

$$\begin{aligned} &\|f^m\|_{L_{\frac{9}{8},\frac{3}{2}}(Q^+(\frac{9}{10}))} \leqslant C(M) \\ &\|f^m - f\|_{L_{\frac{9}{8},\frac{3}{2}}(Q^+(\frac{9}{10}))} \to 0, \quad \text{as} \quad m \to \infty. \end{aligned} \tag{5.20}$$

So, taking into account the imbedding $W^{1,0}_{\frac{9}{8},\frac{3}{2}}(Q^+(\frac{9}{10})) \hookrightarrow L_{\frac{3}{2}}(Q^+(\frac{9}{10}))$ we can conclude that

$$q_1^m \to q_1 \quad \text{in} \quad L_{\frac{3}{2}}(Q^+(\frac{9}{10}))$$

and, hence, for any $\tau \in (0, \frac{9}{10})$

$$\lim_{m\to\infty} Y_\tau(q_1^m) = Y_\tau(q_1).$$

On the other hand, (u_1, q_1) is a solution of the linear Stokes problem in Q^+. Hence from Corollary 4.1, we conclude

$$Y_\tau(q_1) \leqslant C(M)\ \tau^{1/3}\ Y_{\frac{9}{10}}(q_1).$$

9. We need to estimate $Y_{\frac{9}{10}}(q_1)$. From imbedding theorem $L_{\frac{3}{2}}(B^+(\frac{9}{10})) \hookrightarrow W^1_{\frac{9}{8}}(B^+(\frac{9}{10}))$ we conclude

$$Y_{\frac{9}{10}}(q_1) \leqslant C\ \|\nabla q_1\|_{L_{\frac{9}{8},\frac{3}{2}}(B^+(\frac{9}{10}))}.$$

For the solution (u_1, q_1) of the initial-boundary value problem (5.18) we have the estimate

$$\|u_1\|_{W^{2,1}_{\frac{9}{8},\frac{3}{2}}(Q^+(\frac{9}{10}))} + \|\nabla q_1\|_{L_{\frac{9}{8},\frac{3}{2}}(Q^+(\frac{9}{10}))} \leqslant C(M)\ \|\nabla h\|_{L_{\frac{9}{8},\frac{3}{2}}(Q^+(\frac{9}{10}))}.$$

Using Hölder inequality $\|\nabla h\|_{L_{\frac{9}{8},\frac{3}{2}}(Q^+(\frac{9}{10}))} \leqslant C\ \|\nabla h\|_{L_2(Q^+(\frac{9}{10}))}$ and taking into account the weak convergence (5.12) from which we conclude

$$\|\nabla h\|_{L_2(Q^+(\frac{9}{10}))} \leqslant \liminf_{m\to\infty} \|\nabla h^m\|_{L_2(Q^+(\frac{9}{10}))}$$

and using (5.10) we obtain

$$Y_{\frac{9}{10}}(q_1) \leqslant C(M).$$

10. Now we consider functions (u_2^m, q_2^m) determined by relations

$$u_2^m := u^m - u_1^m, \quad q_2^m := q^m - q_1^m. \tag{5.21}$$

These functions satisfy the homogeneous Stokes problems in $Q^+(\frac{9}{10})$:

$$\begin{aligned} \partial_t u_2^m - \Delta u_2^m + \nabla q_2^m &= 0 \qquad \text{in} \quad Q^+\left(\frac{9}{10}\right), \\ \operatorname{div} u_2^m &= 0 \qquad \text{in} \quad Q^+\left(\frac{9}{10}\right), \\ u_2^m|_{x_3=0} &= 0, \end{aligned}$$

$$\begin{aligned} \partial_t u_2 - \Delta u_2 + \nabla q_2 &= 0 \qquad \text{in} \quad Q^+\left(\frac{9}{10}\right), \\ \operatorname{div} u_2 &= 0 \qquad \text{in} \quad Q^+\left(\frac{9}{10}\right), \\ u_2|_{x_3=0} &= 0. \end{aligned}$$

Then

$$\|u_2^m\|_{W^{2,1}_{9,\frac32}(Q^+(\frac45))} + \|\nabla q_2^m\|_{L_{9,\frac32}(Q^+(\frac45))} \leqslant C\Big(\|u_2^m\|_{L_3(Q^+(\frac{9}{10}))} + \|q_2^m\|_{L_{\frac32}(Q^+(\frac{9}{10}))}\Big).$$

Note that due to (5.21), (5.9) and the first inequalities in (5.19), (5.20) we have the estimate

$$\begin{aligned} &\|u_2^m\|_{L_3(Q^+(\frac{9}{10}))} + \|q_2^m\|_{L_{\frac32}(Q^+(\frac{9}{10}))} \\ &\leqslant \|u^m\|_{L_3(Q^+(\frac{9}{10}))} + \|q^m\|_{L_{\frac32}(Q^+(\frac{9}{10}))} + \|u_1^m\|_{L_3(Q^+(\frac{9}{10}))} + \|q_1^m\|_{L_{\frac32}(Q^+(\frac{9}{10}))} \\ &\leqslant C(M). \end{aligned}$$

On the other hand, from the Hölder inequality, we obtain for any $\tau \in (0, \frac45)$

$$\begin{aligned} \widehat{Y}_\tau(q_2^m) &= \tau\Bigg(\fint\limits_{Q^+(\tau)} |q_2^m - [q_2^m]_{B^+(\tau)}|^{\frac32}\, dxdt\Bigg)^{\frac23} \\ &\leqslant C\tau^2 \Bigg(\fint\limits_{Q^+(\tau)} |\nabla q_2^m|^{\frac32}\, dxdt\Bigg)^{\frac23} \\ &\leqslant C\, \tau^{\frac76}\, \|\nabla q_2^m\|_{L_{9,\frac32}(Q^+(\frac45))} \leqslant C(M)\, \tau^{\frac76}. \end{aligned}$$

11. Summarizing all previous estimates we arrive at

$$\limsup_{m\to\infty} \widehat{Y}_\tau(q^m) \leqslant \lim_{m\to\infty} \widehat{Y}_\tau(q_1^m) + \limsup_{m\to\infty} \widehat{Y}_\tau(q_2^m) \leqslant C(M)\ \tau^{\frac{1}{3}}.$$

Taking into account (5.16) and (5.17), finally, we obtain

$$\limsup_{m\to\infty} Y_\tau(u^m, h^m, q^m) \leqslant C(M)\ \tau^{\frac{1}{3}}.$$

This estimate contradicts (5.5) whenever $C_* > C(M)$. Theorem 5.1 is proved.

Theorem 2.2 follows from Theorem 5.1 in the standard way by iterations of the estimate (5.4), scaling arguments, and combination of boundary estimates with the internal estimates obtained in [15]. See details in [8, 9, 11, 12].

6. Estimates of Solutions to the Heat Equation

In this section we study solutions of the heat equations with the lower order terms:

$$\begin{gathered} \partial_t H \ - \ \Delta H \ = \ \operatorname{div}(v\otimes H - H\otimes v) \qquad \text{in} \quad Q^+. \\ v|_{x_3=0} = 0, \\ H_3|_{x_3=0} = 0, \qquad H_{\alpha,3}|_{x_3=0} = 0, \qquad \alpha = 1,2. \end{gathered}$$

Namely, we assume the functions (v, H) possess the following properties:

$$\begin{gathered} v,\ H \in W_2^{1,0}(Q^+), \\ v|_{x_3=0} = 0, \quad H_3|_{x_3=0} = 0 \quad \text{in the sense of traces,} \end{gathered} \tag{6.1}$$

for any $\eta \in C_0^\infty(Q;\mathbb{R}^3)$ such that $\eta_3|_{x_3=0} = 0$ the following integral identity holds

$$\int\limits_{Q^+} \Big(- H\cdot\partial_t\eta + \nabla H : \nabla\eta \Big)\ dxdt \ = \ -\int\limits_{Q^+} G : \nabla\eta\ dxdt, \tag{6.2}$$

where $G = v\otimes H - H\otimes v$, and

$$\operatorname{div} v = 0, \qquad \operatorname{div} H = 0 \qquad \text{a.e. in} \quad Q^+. \tag{6.3}$$

We start with the auxiliary results:

Teorem 6.1. *Assume conditions* (6.1), (6.2) *hold. Denote by* v^* *and* H^* *the extensions of the functions* v *and* H *from* Q^+ *onto* Q *obtained in the following way: the components* v^*_α, H^*_α, $\alpha = 1,2$ *are the even extensions of the components* v_α, H_α, $\alpha = 1,2$, *and the components* v^*_3, H^*_3 *are the odd extensions of* v_3, H_3. *Define also the function*

$$G^* = v^* \otimes H^* - H^* \otimes v^*.$$

Then the following relation holds:

$$\partial_t H^* - \Delta H^* = \operatorname{div} G^* \qquad \textit{in} \qquad \mathcal{D}'(Q). \tag{6.4}$$

Moreover, if (6.3) is satisfied, then

$$\operatorname{div} v^* = 0, \qquad \operatorname{div} H^* = 0, \qquad \textit{in} \quad \mathcal{D}'(Q).$$

Proof of Theorem 6.1. The result is a direct consequence of the boundary conditions $v|_{x_3=0} = 0$ and $H_3|_{x_3=0} = 0$. Theorem 6.1 is proved.

Now we introduce the following functionals

$$\begin{aligned} E(r) &= \Big(\frac{1}{r} \int\limits_{Q^+(r)} |\nabla v|^2 \, dxdt \Big)^{1/2}, \\ E_*(r) &= \Big(\frac{1}{r} \int\limits_{Q^+(r)} |\nabla H|^2 \, dxdt\Big)^{1/2}, \\ F_q(r) &= \Big(\frac{1}{r^{5-q}} \int\limits_{Q^+(r)} |H|^q \, dxdt\Big)^{1/q}. \end{aligned} \tag{6.5}$$

Theorem 6.2. *Assume conditions* (6.1)–(6.3) *hold. Then for any* $r \in (0, \frac{1}{2})$ *the following estimate holds*

$$\|H\|_{L_{1,\infty}(Q^+(r))} \leqslant cr^2 \Big(1 + E(2r)\Big)\Big(F_2(2r) + E_*(2r) + r\Big) \tag{6.6}$$

Proof of Theorem 6.2. Cosider the function

$$\eta = \frac{\zeta H^*}{(1 + |H^*|^2)^{1/2}},$$

where $\zeta \in C_0^\infty(Q(2r))$ is a standard cut-off function. The following relations are true:

$$\int\limits_{Q(2r)} \nabla H^* : \nabla \eta \, dxdt = \int\limits_{Q(2r)} \frac{\nabla H^* : H^* \otimes \nabla \zeta}{(1+|H^*|^2)^{\frac{1}{2}}} \, dxdt + \int\limits_{Q(2r)} \zeta \left(\frac{|\nabla H^*|^2}{(1+|H^*|^2)^{\frac{1}{2}}} - \frac{|\nabla |H^*|^2|^2}{4(1+|H^*|^2)^{\frac{3}{2}}} \right) \, dxdt,$$

and

$$(1+|H^*|^2)|\nabla H^*|^2 - \frac{1}{4}|\nabla |H^*|^2|^2 \ \geqslant \ |\nabla H^*|^2.$$

Hence we obtain the estimate

$$\int\limits_{Q(2r)} \nabla H^* : \nabla \eta \, dxdt \ \geqslant \ \int\limits_{Q(2r)} \zeta \frac{|\nabla H^*|^2}{(1+|H^*|^2)^{\frac{3}{2}}} \, dxdt - \int\limits_{Q(2r)} |\nabla H^*||\nabla \zeta| \, dxdt.$$

Testing the equation (6.4) by the function η we obtain

$$\sup_{t\in(-r^2,0)} \int\limits_{B(2r)} \zeta(1+|H^*|^2)^{\frac{1}{2}} \, dx + \int\limits_{Q(2r)} \zeta \frac{|\nabla H^*|^2}{(1+|H^*|^2)^{\frac{3}{2}}} \, dxdt \leqslant c \int\limits_{Q(2r)} |\partial_t \zeta|(1+|H^*|^2)^{\frac{1}{2}} \, dxdt + \int\limits_{Q(2r)} |\nabla H^*||\nabla \zeta| \, dxdt + \int\limits_{Q(2r)} \operatorname{div} G^* \cdot \eta \, dxdt.$$

Note that

$$\operatorname{div} G^* = (H^* \cdot \nabla) v^* - (v^* \cdot \nabla) H^* \qquad \text{a.e. in} \quad Q.$$

Integrating by parts we obtain

$$\int\limits_{Q(2r)} (v^* \cdot \nabla) H^* \cdot \eta \, dxdt = \int\limits_{Q(2r)} \zeta v^* \cdot \nabla (1+|H^*|^2)^{1/2} \, dxdt = -\int\limits_{Q(2r)} v^* \cdot \nabla \zeta (1+|H^*|^2)^{1/2} \, dxdt$$

Hence

$$\int\limits_{Q(2r)} \operatorname{div} G^* \cdot \eta \; dxdt \leqslant c\; \|\nabla v\|_{L_2(Q^+(2r))} \|H\|_{L_2(Q^+(2r))}$$
$$+ \frac{c}{r}\; \|v\|_{L_2(Q^+(2r))} \|H\|_{L_2(Q^+(2r))} + \frac{c}{r}\; \|v\|_{L_1(Q^+(2r))}$$

Taking into account the Poincare inequality for v and the Hölder inequality, we obtain the estimate

$$\|H\|_{L_{1,\infty}(Q^+(r))} \leqslant c\, \Big(r^{\frac12} + \|\nabla v\|_{L_2(Q^+(2r))}\Big) \|H\|_{L_2(Q^+(2r))}$$
$$+ cr^{\frac32} \|\nabla H\|_{L_2(Q^+(2r))} + c\Big(r^3 + r^{\frac52} \|\nabla v\|_{L_2(Q^+(2r))}\Big)$$

which implies (6.6). Theorem 6.2 is proved.

The main result of this section is following.

Theorem 6.3. *Assume conditions* (6.1)–(6.3) *hold. Then there exist absolute positive constants* ε_1, α *and* c *such that for any* $\varepsilon \in (0, \varepsilon_1)$ *and any* $K > 0$ *if*

$$\sup_{r\in(0,1)} E(r) < \varepsilon \quad \textit{and} \quad \sup_{r\in(0,1)} E_*(r) < K \tag{6.7}$$

then for any $0 < r < \rho \leqslant 1$

$$F_2(r) \leqslant c \left(\frac{r}{\rho}\right)^{\alpha} F_2(\rho) + c\varepsilon(K+1). \tag{6.8}$$

Proof of Theorem 6.3.

1. Denote by v^* and H^* the extensions of functions v and H from Q^+ onto Q described in Theorem 6.1. Fix arbitrary $r \in (0,1)$ and let $\zeta \in C^\infty(\bar Q)$ be a cut off function such that $\zeta \equiv 1$ on $Q(r)$ and $\operatorname{supp} \zeta \subset B \times (-1, 0]$. Denote $\Pi = \mathbb{R}^3 \times (-1, 0)$ and denote by $\widehat G$ the function which coincides with G^* on $Q(\frac{r}{2})$ and additionally possesses the following properties: $\widehat G \in W^{1,0}_1(\Pi) \cap L_{\frac{18}{11},\frac65}(\Pi)$, $\widehat G$ is compactly supported in Π, and

$$\|\widehat G\|_{L_{\frac{18}{11},\frac65}(\Pi)} \leqslant c\|G^*\|_{L_{\frac{18}{11},\frac65}(Q(\frac r2))} \leqslant c\|G\|_{L_{\frac{18}{11},\frac65}(Q^+(\frac r2))} \tag{6.9}$$

2. We decompose H^* as

$$H^* = \widehat{H} + \widetilde{H},$$

where $\widehat{H}$ is a solution of the Cauchy problem for the heat equation

$$\begin{cases} \partial_t \widehat{H} - \Delta \widehat{H} = \operatorname{div} \widehat{G} & \text{in} \quad \Pi, \\ \widehat{H}|_{t=-1} = 0, \end{cases} \tag{6.10}$$

defined by the formula $\widehat{H} = \Gamma * \operatorname{div} \widehat{G} = -\nabla\Gamma * \widehat{G}$, where Γ is the fundamental solution of the heat operator. The function $\widetilde{H}$ satisfies the homogeneous heat equation

$$\partial_t \widetilde{H} - \Delta \widetilde{H} = 0 \quad \text{in} \quad Q\big(\frac{r}{2}\big). \tag{6.11}$$

3. Take arbitrary $\theta \in (0, \frac{1}{2})$. We estimate $\|H\|_{L_2(Q^+(\theta r))}$ in the following way

$$\begin{aligned} \|H\|_{L_2(Q^+(\theta r))} \leqslant \ & \|H^*\|_{L_2(Q(\theta r))} \\ & \leqslant \|\widehat{H}\|_{L_2(Q(\theta r))} + \|\widetilde{H}\|_{L_2(Q(\theta r))}, \end{aligned} \tag{6.12}$$

For $\|\widehat{H}\|_{L_2(Q(\theta r))}$ we have

$$\|\widehat{H}\|_{L_2(Q(\theta r))} \leqslant c\ \|\widehat{H}\|_{L_2(Q(\frac{r}{2}))}. \tag{6.13}$$

As $\widetilde{H}$ satisfies (6.11) by local estimate of the maximum of $\widetilde{H}$ via its L_2-norm we obtain

$$\begin{aligned} \|\widetilde{H}\|_{L_2(Q(\theta r))} & \leqslant c\, \theta^{\frac{5}{2}}\ \|\widetilde{H}\|_{L_2(Q(\frac{r}{2}))} \\ & \leqslant\ c\, \theta^{\frac{5}{2}}\ \big(\|H^*\|_{L_2(Q(r))} + \|\widehat{H}\|_{L_2(Q(\frac{r}{2}))}\big) \end{aligned} \tag{6.14}$$

4. So, we need to estimate $\|\widehat{H}\|_{L_2(Q(\frac{r}{2}))}$. As singular integrals are bounded on the anisotropic Lesbegue space $L_{s,l}$ (see, for example, [14]) for the convolution $\widehat{h} = \Gamma * \widehat{G}$ we obtain the estimate

$$\|\widehat{h}\|_{W^{2,1}_{\frac{18}{11},\frac{6}{5}}(Q(r))} \leqslant c\|\widehat{G}\|_{L_{\frac{18}{11},\frac{6}{5}}(\Pi)}.$$

On the other hand, from the 3D-parabolic imbedding theorem (see [1])

$$W^{2,1}_{s,l}(Q) \hookrightarrow W^{1,0}_{p,q}(Q), \quad \text{as} \quad 1 - \left(\frac{3}{s} + \frac{2}{l} - \frac{3}{p} - \frac{2}{q}\right) \geqslant 0,$$

for $p = q = 2$ and $s = \frac{18}{11}$, $l = \frac{6}{5}$ and for $\widehat{H} = -\nabla \widehat{h}$ we obtain

$$\|\widehat{H}\|_{L_2(Q(r))} \leqslant c\ \|\widehat{G}\|_{L_{\frac{18}{11},\frac{6}{5}}(\Pi)}.$$

(Note that the constant c in this inequality does not depend on r). Taking into account (6.9) we arrive at

$$\|\widehat{H}\|_{L_2(Q(r))} \leqslant c\ \|G\|_{L_{\frac{18}{11},\frac{6}{5}}(Q^+(\frac{r}{2}))}. \tag{6.15}$$

5. From the definition of G we obtain

$$\|G\|_{L_{\frac{18}{11},\frac{6}{5}}(Q^+(\frac{r}{2}))} \leqslant c \left(\int\limits_{-r^2/4}^{0} \|v \otimes H\|^{\frac{6}{5}}_{L_{\frac{18}{11}}(B^+(r/2))}\ dt \right)^{\frac{5}{6}}$$

Applying the Hölder inequality and Sobolev imbedding $W^1_2(B^+(r)) \hookrightarrow L_6(B^+(r))$ for v we obtain

$$\|G\|_{L_{\frac{18}{11},\frac{6}{5}}(Q^+(\frac{r}{2}))} \leqslant c \left(\int\limits_{-r^2/4}^{0} \|v\|^{\frac{6}{5}}_{L_6(B^+(r))}\|H\|^{\frac{6}{5}}_{L_{\frac{9}{4}}(B^+(r/2))}\ dt \right)^{\frac{5}{6}}$$

$$\leqslant c \left(\int\limits_{-r^2/4}^{0} \|\nabla v\|^{\frac{6}{5}}_{L_2(B^+(r))}\|H\|^{\frac{6}{5}}_{L_{\frac{9}{4}}(B^+(r/2))}\ dt \right)^{\frac{5}{6}}.$$

Interpolating $L_{\frac{9}{4}}$-norm between L_1 and L_6 and using the imbedding $W^1_2(B^+(r)) \hookrightarrow L_6(B^+(r))$ again we obtain

$$\|H\|_{L_{\frac{9}{4}}(B^+(r/2))} \leqslant \|H\|^{\frac{1}{3}}_{L_1(B^+(r/2))}\|H\|^{\frac{2}{3}}_{L_6(B^+(r/2))}$$

$$\leqslant \|H\|^{\frac{1}{3}}_{L_1(B^+(r/2))}\Big(\|\nabla H\|^{\frac{2}{3}}_{L_2(B^+(r))} + r^{-\frac{2}{3}}\|H\|^{\frac{2}{3}}_{L_2(B^+(r))}\Big)$$

Hence we arrive at

$$\|G\|_{L_{\frac{18}{11},\frac{6}{5}}(Q^+(\frac{r}{2}))} \leqslant\ c\ \|H\|^{\frac{1}{3}}_{L_{1,\infty}(Q^+(r/2))}$$

$$\times \left(\int\limits_{-r^2/4}^{0} \|\nabla v\|^{\frac{6}{5}}_{L_2(B^+(r))}\Big(\|\nabla H\|^{\frac{4}{5}}_{L_2(B^+(r))} + r^{-\frac{4}{5}}\|H\|^{\frac{4}{5}}_{L_2(B^+(r))}\Big)\ dt \right)^{\frac{5}{6}}.$$

Applying the Hölder inequality we obtain

$$\|G\|_{L_{\frac{18}{11},\frac{6}{5}}(Q^+(\frac{r}{2}))} \leqslant c\ \|H\|^{\frac{1}{3}}_{L_{1,\infty}(Q^+(r/2))}\|\nabla v\|_{L_2(Q^+(r))} \times \Big(\|\nabla H\|^{\frac{2}{3}}_{L_2(Q^+(r))} + r^{-\frac{2}{3}}\|H\|^{\frac{2}{3}}_{L_2(Q^+(r))}\Big)$$

6. Estimating $\|H\|_{L_{1,\infty}(Q^+(r/2))}$ using Theorem 6.2 we obtain

$$\|G\|_{L_{\frac{18}{11},\frac{6}{5}}(Q^+(\frac{r}{2}))} \leqslant cr^{\frac{2}{3}}\ \Big(1+E(r)\Big)^{\frac{1}{3}}\Big(F_2(r)+E_*(r)+r\Big)^{\frac{1}{3}} \times \|\nabla v\|_{L_2(Q^+(r))}\Big(\|\nabla H\|^{\frac{2}{3}}_{L_2(Q^+(r))} + r^{-\frac{2}{3}}\|H\|^{\frac{2}{3}}_{L_2(Q^+(r))}\Big) \tag{6.16}$$

7. Gathering estimates (6.12)–(6.16) together we arrive at

$$\|H\|_{L_2(Q^+(\theta r))} \leqslant c\ \theta^{\frac{5}{2}}\|H\|_{L_2(Q^+(r))} + c\ r^{\frac{2}{3}}\|\nabla v\|_{L_2(Q^+(r))} \times\Big(1+E(r)\Big)^{\frac{1}{3}}\Big(F_2(r)+E_*(r)+r\Big)^{\frac{1}{3}}\Big(\|\nabla H\|^{\frac{2}{3}}_{L_2(Q^+(r))} + r^{-\frac{2}{3}}\|H\|^{\frac{2}{3}}_{L_2(Q^+(r))}\Big)$$

Dividing this inequality by $(\theta r)^{\frac{3}{2}}$ we arrive at

$$F_2(\theta r) \leqslant\ c\ \theta\ F_2(r) +\ c(\theta)\ E(r)\Big(1+E(r)\Big)^{\frac{1}{3}}\Big(F_2(r)+E_*(r)+r\Big).$$

Hence we obtain

$$F_2(\theta r) \leqslant \Big(\ c\theta + c(\theta)E(r)\big(1+E(r)\big)^{\frac{1}{3}}\ \Big)F_2(r) + c(\theta)\ E(r)\Big(1+E(r)\Big)^{\frac{1}{3}}\left(E_*(r)+1\right).$$

Taking into account assumptions (6.7) for $\varepsilon < 1$ we obtain

$$F_2(\theta r) \leqslant \Big(\ c\theta + c(\theta)\varepsilon\ \Big)F_2(r) + c(\theta)\varepsilon(K+1),$$

valid for any $r \in (0,1)$ and any $\theta \in (0,\frac{1}{2}]$.

8. Choosing $\theta \in (0, \frac{1}{2}]$ so that

$$c\theta = \frac{1}{4}$$

and then choosing $\varepsilon_1 \in (0,1)$ so that

$$\frac{1}{4} + c(\theta)\varepsilon_1 \leqslant \frac{1}{2}$$

we obtain the estimate

$$F_2(\theta r) \leqslant \frac{1}{2}\, F_2(r) + c\varepsilon(K+1).$$

Iterating this estimate we derive (6.8). Theorem 6.3 is proved.

7. Estimates of Energy Functionals

In the previous section we defined functionals $F_q(r)$, $E(r)$, and $E_*(r)$, see (6.5). Now we define few more functionals. Note that all these functionals are invariant with respect to the natural scaling of the MHD system. For $r \leqslant 1$, $q \in [1, \frac{10}{3}]$ and $s \in [1, \frac{9}{8}]$ we introduce the following quantities:

$$A(r) \equiv \Big(\frac{1}{r} \sup_{t\in(-r^2,0)} \int\limits_{B^+(r)} |v|^2\, dy\Big)^{1/2},$$

$$A_*(r) \equiv \Big(\frac{1}{r} \sup_{t\in(-r^2,0)} \int\limits_{B^+(r)} |H|^2\, dy\Big)^{1/2},$$

$$C_q(r) \equiv \Big(\frac{1}{r^{5-q}} \int\limits_{Q^+(r)} |v|^q\, dydt\Big)^{1/q},$$

$$D(r) \equiv \Big(\frac{1}{r^2} \int\limits_{Q^+(r)} |p - [p]_{B^+(r)}|^{3/2}\, dydt\Big)^{2/3},$$

$$D_s(r) = R^{\frac{5}{3}-\frac{3}{s}} \Big(\int\limits_{-r^2}^{0} \Big(\int\limits_{B^+(r)} |\nabla p|^s\, dy\Big)^{\frac{1}{s}\cdot\frac{3}{2}}\, dt\Big)^{2/3},$$

$$C(r) = C_3(r), \quad F(r) = F_3(r), \quad D_*(r) = D_{\frac{36}{35}}(r).$$

First we formulate the set of results following from the general theory of functions:

Theorem 7.1. *Assume v, $H \in W^{1,0}_2(Q^+)$ and $p \in W^{1,0}_{\frac98,\frac32}(Q^+)$ are arbitrary functions. Assume $v|_{x_3=0} = 0$. Then the following inequalities hold:*

$$C(r) \leqslant A^{\frac12}(r)E^{\frac12}(r), \quad F(r) \leqslant A_*^{\frac12}(r)[E_*^{\frac12}(r) + F_2^{\frac12}(r)] \tag{7.1}$$

$$D(r) \leqslant cD_1(r), \quad D_1(r) \leqslant cD_s(r), \quad \forall s > 1. \tag{7.2}$$

Proof of Theorem 7.1. The proof follows from interpolation inequalities and imbedding theorems. Proof of the similar inequalities for the Navier-Stokes system can be found in [5].

Now we formulate a theorem concerning boundary suitable weak solutions to the MHD system.

Theorem 7.2. *Assume (v, H, p) is a boundary suitable weak solution to the MHD equations in Q^+. Then for any $r \in (0,1)$ and $\theta \in (0, \frac12)$ the following inequalities hold*

$$\begin{aligned} A(r/2) &+ A_*(r/2) + E(r/2) + E_*(r/2) \\ &\leqslant c\left(C_2(r) + F_2(r) + C^{\frac12}(r)D^{\frac12}(r) + C^{\frac32}(r)\right) \\ &\quad + c\left(C^{\frac12}(r)A_*^{\frac12}(r)E_*^{\frac12}(r) + F^{\frac12}(r)A_*^{\frac12}(r)E^{\frac12}(r)\right) \end{aligned} \tag{7.3}$$

$$\begin{aligned} D_*(\theta r) &\leqslant c\,\theta^{\frac43}\left(D_*(r) + E(r)\right) \\ &\quad + c(\theta)\left(A^{\frac23}(r)E^{\frac43}(r) + A_*^{\frac56}(r)F^{\frac16}(r)E_*(r)\right) \end{aligned} \tag{7.4}$$

Proof of Theorem 7.2, estimate (7.3).
1. Estimate (7.3) follows from (1.5) in a standard way. We just explain the specific estimates of the terms

$$I_1 := \int\limits_{Q^+(r)} |H|^2(v\cdot\nabla\zeta)\,dxdt \quad \text{and} \quad I_2 := \int\limits_{Q^+(r)} (v\cdot H)(H\cdot\nabla\zeta)\,dxdt.$$

2. I_1 we transform in the following way

$$I_1 = \int\limits_{Q^+(r)} \left(|H|^2 - [|H|^2]_{B^+(r)}\right)(v\cdot\nabla\zeta)\,dxdt$$

Applying the Hölder inequality we obtain

$$|I_1| \leqslant \frac{c}{r} \int\limits_{-r^2}^{0} \big\| |H|^2 - [|H|^2]_{B^+(r)} \big\|_{L_{\frac{3}{2}}(B^+(r))} \|v\|_{L_3(B^+(r))} \ dt$$

Applying the inequality $\|f - [f]_{B^+(r)}\|_{L_{\frac{3}{2}}(B^+(r))} \leqslant c\|\nabla f\|_{L_1(B^+(r))}$, we arrive at

$$\begin{aligned}
|I_1| &\leqslant \frac{c}{r} \int\limits_{-r^2}^{0} \|\nabla |H|^2\|_{L_1(B^+(r))} \|v\|_{L_3(B^+(r))} \ dt \\
&\leqslant \frac{c}{r} \int\limits_{-r^2}^{0} \|H\|_{L_2(B^+(r))} \|\nabla H\|_{L_2(B^+(r))} \|v\|_{L_3(B^+(r))} \ dt \\
&\leqslant \frac{c}{r^{2/3}} \ \|H\|_{L_{2,\infty}(Q^+(r))} \|\nabla H\|_{L_2(Q^+(R))} \|v\|_{L_3(Q^+(r))} \\
&\leqslant cr \ A_*(r) E_*(r) C(r)
\end{aligned}$$

3. For I_2 we obtain relations

$$I_2 = \int\limits_{Q^+(r)} \Big((v \cdot H) - [v \cdot H]_{B^+(r)} \Big) (H \cdot \nabla \zeta) \ dxdt$$

Hence

$$\begin{aligned}
&|I_2| \leqslant \frac{c}{r} \int\limits_{-r^2}^{0} \big\| (v \cdot H) - [v \cdot H]_{B^+(r)} \big\|_{L_2(B^+(r))} \|H\|_{L_2(B^+(r))} \ dt \\
&\leqslant \frac{c}{r} \ \|H\|_{L_{2,\infty}(Q^+(r))} \int\limits_{-r^2}^{0} \|\nabla (v \cdot H)\|_{L_{\frac{6}{5}}(B^+(r))} \ dt \leqslant \frac{c}{r} \ \|H\|_{L_{2,\infty}(Q^+(r))} \\
&\times \int\limits_{-r^2}^{0} \Big(\|\nabla v\|_{L_2(B^+(r))} \|H\|_{L_3(B^+(r))} + \|\nabla H\|_{L_2(B^+(r))} \|v\|_{L_3(B^+(r))} \Big) \ dt \\
&\leqslant \frac{c}{r^{2/3}} \ \|H\|_{L_{2,\infty}(Q^+(r))} \\
&\times \Big(\|\nabla v\|_{L_2(Q^+(r))} \|H\|_{L_3(Q^+(r))} + \|\nabla H\|_{L_2(Q^+(r))} \|v\|_{L_3(Q^+(r))} \Big)
\end{aligned}$$

So, we obtain

$$|I_2| \leqslant cr\ A_*(r)\ \Big(\ E(r)F(r) + E_*(r)C(r)\ \Big)$$

Proof of Theorem 7.2, estimate (7.4).

1. To obtain (7.4) we apply the method developed in [8, 10], see also [11]. Denote $\Pi_r = \mathbb{R}^3_+ \times (-r^2, 0)$. We fix $r \in (0,1]$ and $\theta \in (0, \frac{1}{2})$ and define a function $g : \Pi_r^+ \to \mathbb{R}^3$ by the formula

$$g = \begin{cases} \operatorname{rot} H \times H - (v\cdot\nabla)v, & \text{in } \ Q^+(r), \\ 0, & \text{in } \ \Pi_r^+ \setminus Q^+(r) \end{cases}$$

Then we decompose v and p as

$$v = \widehat{v} + \widetilde{v}, \qquad p = \widehat{p} + \widetilde{p},$$

where $(\widehat{v}, \widehat{p})$ is a solution of the Stokes initial boundary value problem in a half-space

$$\begin{cases} \partial_t \widehat{v} - \Delta \widehat{v} + \nabla \widehat{p} = g, \\ \qquad \operatorname{div} \widehat{v} = 0 \end{cases} \quad \text{in} \quad \Pi_r^+,$$
$$\widehat{v}|_{t=0} = 0, \qquad \widehat{v}|_{x_3=0} = 0,$$

and $(\widetilde{v}, \widetilde{p})$ is a solution of the homogeneous Stokes system in $Q^+(r)$:

$$\begin{cases} \partial_t \widetilde{v} - \Delta \widetilde{v} + \nabla \widetilde{p} = 0, \\ \qquad \operatorname{div} \widetilde{v} = 0 \end{cases} \quad \text{in} \quad Q^+(r),$$
$$\widetilde{v}|_{x_3=0} = 0.$$

2. For $\nabla\widehat{p}$ and $\nabla\widetilde{p}$ the following estimates hold (see [10], see also [13]):

$$\|\nabla \widehat{p}\|_{L_{\frac{36}{35},\frac{3}{2}}(Q^+(r))} + \frac{1}{r}\|\nabla \widehat{v}\|_{L_{\frac{36}{35},\frac{3}{2}}(Q^+(r))}$$
$$\leqslant c\ \Big(\ \|H \times \operatorname{rot} H\|_{L_{\frac{36}{35},\frac{3}{2}}(Q^+(r))} + \|(v\cdot\nabla)v\|_{L_{\frac{36}{35},\frac{3}{2}}(Q^+(r))}\ \Big),$$

$$\|\nabla \widetilde{p}\|_{L_{\frac{36}{35},\frac{3}{2}}(Q^+(\theta r))} \leqslant c\ \theta^{\frac{31}{12}}\ \Big(\ \frac{1}{r}\|\nabla \widetilde{v}\|_{L_{\frac{36}{35},\frac{3}{2}}(Q^+(r))} + \|\nabla \widetilde{p}\|_{L_{\frac{36}{35},\frac{3}{2}}(Q^+(r))}\ \Big).$$

3. From the Hölder inequality we obtain

$$\|H\times \operatorname{rot} H\|_{L_{\frac{36}{35},\frac{3}{2}}(Q^+(r))} \leqslant c\, r^{\frac{2}{9}}\ \|H\|^{\frac{5}{6}}_{L_{2,\infty}(Q^+(r))}\|\nabla H\|_{L_2(Q^+(r))}\|H\|^{\frac{1}{6}}_{L_3(Q^+(r))}$$

$$\|(v\cdot\nabla)v\|_{L_{\frac{36}{35},\frac{3}{2}}(Q^+(r))} \leqslant c\, r^{\frac{1}{4}}\ \|(v\cdot\nabla)v\|_{L_{\frac{9}{8},\frac{3}{2}}(Q^+(r))} \leqslant c\, r^{\frac{1}{4}}\ \|v\|^{\frac{2}{3}}_{L_{2,\infty}(Q^+(r))}\|\nabla v\|^{\frac{4}{3}}_{L_2(Q^+(r))}$$

Representing $\widetilde{v}=v-\widehat{v}$, $\widetilde{p}=p-\widehat{p}$ and gathering all above estimates for $\widehat{p}$ and $\widehat{v}$ we obtain

$$D_*(\theta r)\leqslant c\,\theta^{\frac{4}{3}}\ \Big(\ D_*(r)+E(r)+A^{\frac{2}{3}}(r)E^{\frac{4}{3}}(r)+A_*^{\frac{5}{6}}(r)E_*(r)F^{\frac{1}{6}}(r)\ \Big) + c(\theta)\ \Big(\ A^{\frac{2}{3}}(r)E^{\frac{4}{3}}(r)+A_*^{\frac{5}{6}}(r)E_*(r)F^{\frac{1}{6}}(r)\ \Big)$$

Theorem 7.2 is proved.

8. CKN condition and Partial Regularity of Solutions

In this section we present the proofs of Theorems 2.2 and 2.3.

Theorem 8.1. *Denote by* $\mathcal{E}(r)$ *the following functional*

$$\mathcal{E}(r)=A(r)+A_*(r)+D_*(r),$$

and let $\varepsilon_1>0$ *be the absolute constant defined in Theorem* 6.3. *For any* $K>0$ *there exists a constant* $c(K)>0$ *such that for any* $\varepsilon\in(0,\varepsilon_1]$ *and any boundary suitable weak solution* (v,H,p) *of the MHD system in* Q^+ *if*

$$\sup_{r\in(0,1)} E(r)\leqslant\varepsilon,\qquad \sup_{r\in(0,1)} E_*(r)\leqslant K, \tag{8.1}$$

and

$$F_2(1)\leqslant M, \tag{8.2}$$

then for any $0<r<\rho\leqslant 1$

$$\mathcal{E}(r)\leqslant c\left(\frac{r}{\rho}\right)^{\beta}\mathcal{E}(\rho)\ +\ c(K)(1+\rho^{\alpha}M). \tag{8.3}$$

where $\beta > 0$ is some absolute constant.

Proof of Theorem 8.1.

1. Without loss of generality we can assume $K \geqslant 1$. Then from (6.8) we obtain

$$F_2(r) \leqslant cr^\alpha M + cK.$$

From this inequality and (7.1) we obtain

$$C(r) \leqslant c\, \mathcal{E}^{\frac12}(r)\varepsilon_1^{\frac12}, \quad F(r) \leqslant c\, \mathcal{E}^{\frac12}(r)\Big(K^{\frac12} + r^{\frac{\alpha}{2}} M^{\frac12}\Big) \tag{8.4}$$

2. Assume $r \in (0,1)$ and $\theta \in (0, \frac12)$. From (7.3) with the help of (7.2) and the Young inequality we obtain

$$\mathcal{E}(\theta r) \leqslant c\, \Big(F_2(2\theta r) + D_*(2\theta r)\Big)$$
$$+c(\theta)\Big(C_2(r) + C(r) + C^{\frac32}(r) + C^{\frac12}(r)A_*^{\frac12}(r)E_*^{\frac12}(r) + F^{\frac12}(r)A_*^{\frac12}(r)E^{\frac12}(r)\Big)$$

Taking into account (8.4) and (8.1) we obtain

$$\begin{aligned}\mathcal{E}(\theta r) \leqslant c\, &\Big(F_2(2\theta r) + D_*(2\theta r)\Big)\\ &+ c(\theta)\Big(\varepsilon_1 + \mathcal{E}^{\frac12}(r)\varepsilon_1^{\frac12} + \mathcal{E}^{\frac34}(r)\varepsilon_1^{\frac34} + \varepsilon_1^{\frac14}\mathcal{E}^{\frac34}(r)K^{\frac12}\\ &\qquad + (K^{\frac14} + r^{\frac{\alpha}{4}}M^{\frac14})\mathcal{E}^{\frac34}(r)\varepsilon_1^{\frac12}\Big)\end{aligned} \tag{8.5}$$

Applying the Young inequality $ab \leqslant \varepsilon a^p + C_\varepsilon b^{p'}$ we obtain

$$\mathcal{E}(\theta r) \leqslant \frac14 \mathcal{E}(r) + c\Big(F_2(2\theta r) + D_*(2\theta r)\Big) + c(\theta)c(K) + c(\theta)r^\alpha M.$$

3. From (6.8) and (7.4) we obtain

$$\begin{aligned}F_2(2\theta r) + D_*(2\theta r) \leqslant c\theta^\alpha &\Big(F_2(r) + D_*(r)\Big) + c\varepsilon_1(1+K)\\ &+ c(\theta)\, \Big(A^{\frac23}(r)E^{\frac43}(r) + A_*^{\frac56}(r)F^{\frac16}(r)E_*(r)\Big)\end{aligned}$$

Taking into account (8.4) and the obvious inequality $F_2(r) \leqslant A_*(r)$ we arrive at

$$\begin{aligned}F_2(2\theta r) + D_*(2\theta r) \leqslant c\theta^\alpha &\mathcal{E}(r) + c(K)\\ &+ c(\theta)\Big(\mathcal{E}^{\frac23}(r)\varepsilon_1^{\frac43} + \mathcal{E}^{\frac{11}{12}}(r)(K^{\frac1{12}} + r^{\frac{\alpha}{12}}M^{\frac1{12}})K\Big)\end{aligned}$$

Applying the Young inequality we get

$$F_2(2\theta r) + D_*(2\theta r) \leqslant \Big(\frac{1}{4} + c\theta^\alpha\Big)\mathcal{E}(r) + c(\theta)c(K)(1 + r^\alpha M)$$

4. Gathering the estimates we obtain

$$\mathcal{E}(\theta r) \leqslant \Big(\frac{1}{4} + c\theta^\alpha\Big)\mathcal{E}(r) + c(\theta)c(K)(1 + r^\alpha M).$$

We fix $\theta \in (0, \frac{1}{2})$ so that

$$\frac{1}{4} + c\theta^\alpha = \frac{1}{2}.$$

Hence

$$\mathcal{E}(\theta r) \leqslant \frac{1}{2}\mathcal{E}(r) + c(\theta)c(K)(1 + r^\alpha M).$$

Iterating this inequality we obtain (8.3). Theorem 8.1 is proved.

Theorem 8.2. *Assume all conditions of Theorem* 8.1 *hold and fix* $\rho_0 \in (0,1)$ *so that*

$$\rho_0^\alpha M \leqslant 1. \tag{8.6}$$

Then for any $0 < r < \rho \leqslant \rho_0$ *the following estimates hold:*

$$A(r) + A_*(r) \leqslant c\left(\frac{r}{\rho}\right)^\gamma \Big(A(\rho) + A_*(\rho)\Big) + \varepsilon^{\frac{1}{4}}D(\rho) + G(K,\varepsilon) \tag{8.7}$$

$$D(r) \leqslant c\left(\frac{r}{\rho}\right)^\gamma D(\rho) + c(K)\Big(A^{\frac{11}{12}}(\rho) + A_*^{\frac{11}{12}}(\rho)\Big) + G(K,\varepsilon) \tag{8.8}$$

where $\gamma > 0$ *is some absolute constant and* G *is a continuous function possessing the following property:*

$$\text{for any fixed} \quad K > 0 \quad G(K,\varepsilon) \to 0 \quad \text{as} \quad \varepsilon \to 0. \tag{8.9}$$

Proof of Theorem 8.2.

1. From (7.1) taking into account (8.6) we obtain

$$C(r) \leqslant A^{\frac{1}{2}}(r)\varepsilon^{\frac{1}{2}}, \qquad F(r) \leqslant A_*^{\frac{1}{2}}(r)(K^{\frac{1}{2}} + 1) \tag{8.10}$$

2. Take arbitrary $r \in (0, \rho_0)$ and $\theta \in (0, \frac{1}{2})$. Denote by $\mathcal{E}(r)$ the following functional

$$\mathcal{E}_*(r) = A(r) + A_*(r),$$

Then from (7.3) similar to (8.5) using (8.10) we derive

$$\mathcal{E}_*(\theta r) \leqslant F_2(2\theta r) + C^{\frac{1}{2}}(2\theta r) D^{\frac{1}{2}}(2\theta r) \\ + c(\theta)\Big(\mathcal{E}_*^{\frac{1}{2}}(r)\varepsilon^{\frac{1}{2}} + \mathcal{E}_*^{\frac{3}{4}}(r)\varepsilon^{\frac{3}{4}} + \mathcal{E}_*^{\frac{3}{4}}(r) K^{\frac{1}{2}}\varepsilon^{\frac{1}{4}} + \mathcal{E}_*^{\frac{3}{4}}(r)(K^{\frac{1}{4}}+1)\varepsilon^{\frac{1}{2}}\Big)$$

Applying the Young inequality and using (7.2) we obtain

$$\mathcal{E}_*(\theta r) \leqslant \frac{1}{8}\, \mathcal{E}_*(r) + c(\theta) G(K,\varepsilon) + F_2(2\theta r) + C^{\frac{1}{2}}(2\theta r) D_*^{\frac{1}{2}}(2\theta r) \qquad (8.11)$$

3. From (6.8) we conclude

$$F(2\theta r) \leqslant c\theta^{\alpha}\mathcal{E}_*(r) + G(K,\varepsilon). \qquad (8.12)$$

4. From (7.4) for $r \leqslant \rho_0$ with the help of (8.10) and the Young inequality we obtain

$$D_*(2\theta r) \leqslant c\ \theta^{\beta} D_*(r) + c(\theta)c(K)\mathcal{E}_*^{\frac{11}{12}}(r) + c(\theta)G(K,\varepsilon) \qquad (8.13)$$

Hence from (8.10) we obtain

$$C^{\frac{1}{2}}(2\theta r) D^{\frac{1}{2}}(2\theta r) \leqslant c(\theta)\mathcal{E}_*^{\frac{1}{4}}(r)\varepsilon^{\frac{1}{4}} D_*^{\frac{1}{2}}(r) \\ + c(\theta)c(K)\varepsilon^{\frac{1}{4}}\mathcal{E}_*^{\frac{17}{24}}(r) + c(\theta)G(K,\varepsilon)$$

Applying the Young inequality we arrive at

$$C^{\frac{1}{2}}(2\theta r) D^{\frac{1}{2}}(2\theta r) \leqslant \frac{1}{8}\mathcal{E}_*(r) + \frac{1}{2}\varepsilon^{\frac{1}{4}} D_*(r) + \ c(\theta)G(K,\varepsilon) \qquad (8.14)$$

5. Gathering estimates (8.11)–(8.14) we obtain the inequality

$$\mathcal{E}_*(\theta r) \leqslant \Big(\frac{1}{4} + c\theta^{\gamma}\Big)\ \mathcal{E}_*(r) + \frac{1}{2}\varepsilon^{\frac{1}{4}} D_*(r) + c(\theta)G(K,\varepsilon)$$

Choosing $\theta \in (0, \frac{1}{2})$ so that

$$\frac{1}{4} + c\theta^{\alpha} = \frac{1}{2}$$

we obtain

$$\mathcal{E}_*(\theta r) \leqslant \frac{1}{2}\ \mathcal{E}_*(r) + \frac{1}{2}\varepsilon^{\frac{1}{4}} D_*(r) + c(\theta) G(K,\varepsilon)$$

Iterating this inequality we obtain (8.7).

6. Choosing in (8.13) $\theta \in (0, \frac{1}{2})$ so that

$$c\theta^{\beta} = \frac{1}{2}$$

and iterating the obtained inequality we derive (8.8). Theorem 8.2 is proved.

Theorem 8.3. *For any $K > 0$ there exists a constant $\varepsilon_0(K) > 0$ such that if the condition* (8.1) *holds with $\varepsilon \leqslant \varepsilon_0$, then there exists $\rho_* \in (0,1)$ such that*

$$\Big(C(\rho_*) + F(\rho_*) + D(\rho_*)\Big) < \varepsilon_*^{\frac{1}{3}},$$

where the constant $\varepsilon_* > 0$ is defined in Theorem 2.1.

Proof of Theorem 8.3.

1. From (8.3) we obtain

$$\limsup_{r\to 0} D_*(r) \leqslant c(K).$$

2. From (8.7) we derive

$$\limsup_{r\to 0} \Big(A(r) + A_*(r)\Big) \leqslant \varepsilon^{\frac{1}{4}} \limsup_{\rho\to 0} D(\rho) + G(K,\varepsilon) \\ \leqslant \varepsilon^{\frac{1}{4}} c(K) + G(K,\varepsilon).$$

3. From (8.8) we obtain

$$\limsup_{r\to 0} D_*(r) \leqslant c(K) \limsup_{\rho\to 0} \Big(A^{\frac{11}{12}}(\rho) + A_*^{\frac{11}{12}}(\rho)\Big) + G(K,\varepsilon) \\ \leqslant c(K)\Big(\varepsilon^{\frac{1}{4}} c(K) + G(K,\varepsilon)\Big)^{\frac{11}{12}} + G(K,\varepsilon).$$

4. From (7.1) we conclude

$$\limsup_{r\to 0} \Big(C(r) + F(r)\Big) \leqslant (\varepsilon^{\frac{1}{2}} + K^{\frac{1}{2}}) \limsup_{r\to 0} \Big(A(r) + A_*(r)\Big) \\ \leqslant (\varepsilon^{\frac{1}{2}} + K^{\frac{1}{2}})\Big(\varepsilon^{\frac{1}{4}} c(K) + G(K,\varepsilon)\Big)^{\frac{1}{2}}.$$

5. Taking into account (8.9) for any $K > 0$ we can find $\varepsilon_0(K) > 0$ such that for any $\varepsilon \in (0, \varepsilon_0)$

$$c(K)\Big(\varepsilon^{\frac{1}{4}} c(K) + G(K, \varepsilon)\Big)^{\frac{11}{12}} + G(K, \varepsilon) < \frac{\varepsilon_*^{\frac{1}{3}}}{2}$$

and

$$(\varepsilon^{\frac{1}{2}} + K^{\frac{1}{2}})\Big(\varepsilon^{\frac{1}{4}} c(K) + G(K, \varepsilon)\Big)^{\frac{1}{2}} < \frac{\varepsilon_*^{\frac{1}{3}}}{2}.$$

Hence for $\varepsilon \in (0, \varepsilon_0)$

$$\limsup_{r \to 0} \Big(C(r) + F(r) + D_*(r)\Big) < \varepsilon_*^{\frac{1}{3}}.$$

Theorem 8.3 is proved.

Proof of Theorem 2.2. Assume condition (2.6) holds and let $\varepsilon_0(K) > 0$ be the constant defined in Theorem 8.3. From (2.6), (2.7) we obtain there exists $R > 0$ such that

$$\sup_{r \in (0,R)} E(r) < \varepsilon_0 \qquad \text{and} \qquad \sup_{r \in (0,R)} E_*(r) < K.$$

Denote (v^R, H^R, p^R) the functions

$$\begin{aligned} v^R(x,t) &= Rv(x_0 + Rx, t_0 + R^2 t), \\ H^R(x,t) &= RH(x_0 + Rx, t_0 + R^2 t), \\ p^R(x,t) &= R^2 p(x_0 + Rx, t_0 + R^2 t). \end{aligned}$$

Then functions (v^R, H^R, p^R) satisfy all conditions of Theorem 8.3. The result follows from Theorem 8.3 and Theorem 2.1.

Proof of Theorem 2.3. The result is a direct consequence of Theorem 2.2 and measure theory, see [2, 7, 5, 8].

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