Записки научных
семинаров ПОМИ
Том 385, 2010 г.

M. Bildhauer, M. Fuchs

A GEOMETRIC MAXIMUM PRINCIPLE FOR VARIATIONAL PROBLEMS IN SPACES OF VECTOR VALUED FUNCTIONS OF BOUNDED VARIATION

Abstract. We discuss variational integrals with density having linear growth on spaces of vector valued BV-functions and prove $\mathrm{Im}(u) \subset K$ for minimizers u provided that the boundary data take their values in the closed convex set K assuming in addition that the integrand satisfies natural structure conditions.

Given a closed convex set $K \subset \mathbb{R}^N$, we say that minimizers of some variational problem have the convex hull property if they are contained in K in a sense to be made precise provided this is true for their boundary data. A prominent example is given by mass minimizing integer multiplicity m-currents T with compact support, where $m \le N$ and where the comparison currents S are such that $\partial S = T_0$ for a $(m-1)$-current T_0 with compact support and $\partial T_0 = 0$. Then the support of T is contained in the convex hull of $\operatorname{spt} T_0$, which is a consequence of the monotonicity formula for stationary varifolds. We refer the reader to [14, 19.2 Theorem and 34.2 remarks]. Let us now pass to the setting of variational integrals

$$I[u,\Omega] = \int\limits_{\Omega} f(\nabla u)\,\mathrm{d}\,x$$

defined for functions $u\colon \mathbb{R}^n \supset \Omega \to \mathbb{R}^N$, Ω denoting a bounded Lipschitz domain. Suppose that we are given a function u_0 such that

$$u_0 \in W^1_1(\Omega;\mathbb{R}^N), \quad u_0(x) \in K \ \text{ a.e.}, \tag{1}$$

where $W^1_1(\Omega;\mathbb{R}^N)$ is the Sobolev space of vector-valued mappings (see, e.g., [1]). Let us further assume that $f(Z) = h(|Z|)$ with

$$h : [0,\infty) \to [0,\infty) \quad \text{strictly increasing and convex.} \tag{2}$$

Key words and phrases: Functions of bounded variation, linear growth problems, minimizers, convex hull property, maximum principle.

Then, if $u \in W_1^1(\Omega;\mathbb{R}^N)$ minimizes $I[\cdot,\Omega]$ w.r.t. the boundary data u_0, i.e.,

$$\begin{cases} I[u,\Omega] < \infty, & u - u_0 \in W_1^1(\Omega;\mathbb{R}^N) \quad \text{and} \\ I[u,\Omega] \leq I[v,\Omega] & \text{for all} \quad v \in u_0 + \overset{\circ}{W}{}_1^1(\Omega;\mathbb{R}^N), \end{cases}$$

it follows that $u(x) \in K$ for almost any $x \in \Omega$. A simple proof is given by the following observation: let $\Phi\colon \mathbb{R}^N \to K$ denote the nearest-point-projection being Lipschitz with $\mathrm{Lip}(\Phi) = 1$. From [4], comments given at the beginning of the proof of Theorem 3.96, we see that $v = \Phi(u)$ is admissible and satisfies $|\nabla v| \leq \mathrm{Lip}(\Phi)|\nabla u| = |\nabla u|$. Using the properties of h stated in (2) combined with $|\nabla v| \leq |\nabla u|$, we get from the minimality of u that $I[u,\Omega] = I[v,\Omega]$, and as it is outlined below, this will lead to $\nabla u = \nabla v$; hence $u = v$ and in conclusion $u \in K$, a.e. We remark first that a related maximum principle is due to D'Ottavio, Leonetti and Musciano [9], and second that a similar argument together with a proof of the chain rule in the Lipschitz setting has been presented in [6]. However, the reader should note at this stage that a much more general chain rule formula implying $|\nabla(\Phi \circ u)| \leq \mathrm{Lip}(\Phi)|\nabla u|$ is due to Ambrosio and Dal Maso [2]. As a matter of fact the existence of a minimizer u in a suitable Sobolev class requires that h is of superlinear growth, and therefore in general can not be guaranteed if in addition to (2) the function h satisfies

$$\overline{c} := \lim_{t\to\infty} \frac{h(t)}{t} \quad \text{exists in} \quad (0,\infty), \tag{3}$$

which means that now h is just of linear growth.

W.l.o.g. we will also assume that $h(0) = 0$. Based on ideas of De Giorgi (see the recent book [10] for an overview on his work), of Giusti [11], of Giaquinta, Modica, Souček [12], of Goffman and Serrin [13], of Ambrosio and Dal Maso [3] and of Buttazzo [8] it is possible to introduce suitable concepts of generalized solutions to the problem

$$I[u,\Omega] = \int_\Omega h(|\nabla u|)\,\mathrm{d}x \to \min \quad \text{in} \quad u_0 + \overset{\circ}{W}{}_1^1(\Omega;\mathbb{R}^N). \tag{$\mathcal{P}$}$$

Let

$$\mathcal{M} := \big\{u \in BV(\Omega;\mathbb{R}^N) : \quad u \text{ is a } L^1\text{-cluster point of a minimizing sequence of problem } \mathcal{P}\big\}$$

and define $K[\cdot,\Omega]: BV(\Omega;\mathbb{R}^N) \to \mathbb{R}$,

$$K[u,\Omega] := \int_\Omega h\big(|\nabla^a u|\big)\mathrm{d}\,x + \overline{c}\,|\nabla^s u|(\Omega) + \int_{\partial\Omega} \overline{c}\,|(u_0 - u)\otimes\mathcal{N}|\mathrm{d}\,\mathcal{H}^{n-1},$$

where $BV(\Omega;\mathbb{R}^N)$ is the space of functions of bounded variation (see [4] or [11]), $\mathcal{N}$ is the exterior normal of $\partial\Omega$ and where we have used the decomposition of the vector measure ∇u in its absolutely continuous part $\nabla^a u \llcorner \mathcal{L}^n$ and its singular part $\nabla^s u$. According to a theorem of Besicovitch ([4, Theorem 2.22]) we have $\nabla^a u \in L^1(\Omega;\mathbb{R}^{nN})$ and

$$\nabla^a u(x) = \lim_{\rho\downarrow 0} \frac{\nabla u(B_\rho(x))}{\mathcal{L}^n(B_\rho(x))} \tag{4}$$

holds for $\mathcal{L}^n$-a.a. $x \in \Omega$. Note that on account of (3) the recession function

$$f_\infty(Z) := \lim_{t\to 0} \frac{f(tZ)}{t}, \quad Z \in \mathbb{R}^{nN},$$

equals $\overline{c}|Z|$. Hence, we have the more familiar formula

$$\begin{aligned} K[u,\Omega] = &\int_\Omega f(\nabla^a u)\mathrm{d}\,x + \int_\Omega f_\infty\left(\frac{\nabla^s u}{|\nabla^s u|}\right)\mathrm{d}\,|\nabla^s u| \\ &+ \int_{\partial\Omega} f_\infty((u_0 - u)\otimes\mathcal{N})\mathrm{d}\,\mathcal{H}^{n-1} \end{aligned}$$

for the extension of I to the space $BV(\Omega;\mathbb{R}^N)$. We recall the following facts established in [7] (compare also [5, Appendix A1]):

(i) $I[\cdot,\Omega] = K[\cdot,\Omega]$ on $u_0 + \overset{\circ}{W}{}^1_1(\Omega;\mathbb{R}^N)$;

(ii) $K[\cdot,\Omega] \to \min$ admits at least one solution in $BV(\Omega;\mathbb{R}^N)$;

(iii) these minimizers are exactly the elements of $\mathcal{M}$;

(iv) $\inf\limits_{u_0+\overset{\circ}{W}{}^1_1(\Omega;\mathbb{R}^N)} I[\cdot,\Omega] = \inf\limits_{BV(\Omega;\mathbb{R}^N)} K[\cdot,\Omega]$.

Based on these facts it is reasonable to address the elements of the set $\mathcal{M}$ as generalized solutions of problem $(\mathcal{P})$.

Now we can state our main result:

Theorem 1. *Suppose that u_0 satisfies* (1) *for a closed and convex set $K \subset \mathbb{R}^N$. Assume further that we have* (2) *and* (3) *for the density h. Then it holds $u(x) \in K$, a.e., for any generalized solution of problem $(\mathcal{P})$.*

Corollary 1 (Maximum-principle)**.** *Suppose that h satisfies* (2) *and* (3)*. Assume further that $u_0 \in W_1^1(\Omega;\mathbb{R}^N) \cap L^\infty(\Omega;\mathbb{R}^N)$. Then any generalized minimizer $u \in BV(\Omega;\mathbb{R}^N)$ of problem $(\mathcal{P})$ satisfies $\|u\|_{L^\infty(\Omega)} \leq \|u_0\|_{L^\infty(\Omega)}$.*

Remark 1. The proof of Theorem 1 given below immediately extends to integrands of the form

$$f(Z) = \sum_{i=1}^{n} h_i(|Z_i|), \quad Z = (Z_1, \dots, Z_n) \in \mathbb{R}^{nN}, \quad Z_i \in \mathbb{R}^N,$$

with functions $h_1, \dots, h_n$ satisfying (2) and having the property that

$$\overline{c}_i := \lim_{t\to\infty} \frac{h_i(t)}{t}$$

exists in $(0,\infty)$. In this case, it holds

$$f_\infty(Z) = \sum_{i=1}^{n} \overline{c}_i |Z_i|.$$

Of course any other additive decomposition of f depending on the moduli of the Z_i can be considered, e.g.,

$$f(Z) = h_1\big(\sqrt{|Z_1|^2 + |Z_2|^2}\big) + h_2(|Z_3|)$$

or

$$f(Z) = h_1(|Z_1|) + h_2\big(\sqrt{|Z_2|^2 + |Z_3|^2}\big)$$

are admissible in the case $n = 3$. In fact, a careful inspection of the proof of the chain rule shows the validity of

$$|\partial_i(\Phi \circ u)| \leq \mathrm{Lip}(\Phi)|\partial_i u|, \quad i = 1, \dots, n,$$

so that $|\partial_i(\Phi \circ u)| \leq |\partial_i u|$.

Proof. We fix a Lipschitz domain $\widehat{\Omega} \Supset \Omega$, extend u_0 to an element of $W^1_1(\widehat{\Omega};\mathbb{R}^N)$ with values in K and let

$$BV_{u_0}(\Omega;\mathbb{R}^N) := \big\{w \in BV(\widehat{\Omega};\mathbb{R}^N) : w = u_0 \quad \text{on} \quad \widehat{\Omega} - \Omega\big\}.$$

Following [12] we define

$$\begin{aligned}\widehat{I}[w,\widehat{\Omega}] &:= \int\limits_{\widehat{\Omega}} f(\nabla^a w)\,\mathrm{d}\,x + \int\limits_{\widehat{\Omega}} f_\infty\Big(\frac{\nabla^s w}{|\nabla^s w|}\Big)\,\mathrm{d}\,|\nabla^s w| \\ &= \int\limits_{\widehat{\Omega}} h(|\nabla^a w|)\mathrm{d}\,x + \overline{c}|\nabla^s w|(\widehat{\Omega})\end{aligned}$$

for $w \in BV_{u_0}(\Omega;\mathbb{R}^N)$, and as outlined in [7] we have

$$\widehat{I}[w,\widehat{\Omega}] = K[w_{|\Omega},\Omega] + \text{const}\,.$$

Conversely, if $v \in BV(\Omega;\mathbb{R}^N)$ and if we put

$$\widehat{v} := \left\{\begin{matrix} v & \text{on} & \Omega \\ u_0 & \text{on} & \widehat{\Omega} - \Omega\end{matrix}\right\} \in BV_{u_0}(\Omega;\mathbb{R}^N),$$

then

$$\widehat{I}[\widehat{v},\widehat{\Omega}] = K[v,\Omega] + \text{const},$$

where $const = \int\limits_{\widehat{\Omega}-\Omega} h(|\nabla u_0|)\mathrm{d}\,x$. Due to this observation it is sufficient to consider a solution $u \in BV_{u_0}(\Omega;\mathbb{R}^N)$ of

$$\widehat{I}[\cdot,\widehat{\Omega}] \to \min \quad \text{in} \quad BV_{u_0}(\Omega;\mathbb{R}^N)$$

and to prove that $u(x) \in K$ almost everywhere.

For this purpose, we consider the retraction Φ: $\mathbb{R}^N \to K$ and let as before $v := \Phi \circ u$. According to the comments given at the beginning of the proof of Theorem 3.96 in [4] v is in $BV(\widehat{\Omega};\mathbb{R}^N)$ and (recall $\mathrm{Lip}(\Phi) = 1$)

$$|\nabla v| \le \mathrm{Lip}(\Phi)|\nabla u| = |\nabla u|, \tag{5}$$

where $|\nabla v|$ and $|\nabla u|$ denote the total variations of the vector measures ∇v and ∇u. Here we like to emphasize again that a general chain rule formula as stated for example in Theorem 3.101 of [4] is due to Ambrosio and Dal Maso [2], and that (5) is a simple consequence of this important formula. Clearly $v \in BV_{u_0}(\Omega;\mathbb{R}^N)$ so that

$$\widehat{I}[u,\widehat{\Omega}] \le \widehat{I}[v,\widehat{\Omega}]. \tag{6}$$

Now we use (4) for u and v which implies in combination with (5) for $\mathcal{L}^n$-a.a. $x \in \widehat{\Omega}$

$$|\nabla^a v(x)| = \lim_{\rho\downarrow 0} \frac{|\nabla v|(B_\rho(x))}{\mathcal{L}^n(B_\rho(x))} \le \lim_{\rho\downarrow 0} \frac{|\nabla u|(B_\rho(x))}{\mathcal{L}^n(B_\rho(x))} = |\nabla^a u(x)|,$$

and the monotonicity of h gives

$$\int_{\widehat{\Omega}} h(|\nabla^a v|)\,\mathrm{d}\,x \le \int_{\widehat{\Omega}} h(|\nabla^a u|)\,\mathrm{d}\,x. \tag{7}$$

Quoting [4, Proposition 3.92(a)], for a function $w \in BV(\widehat{\Omega};\mathbb{R}^N)$ we may write

$$\nabla^s w = \nabla w \llcorner S_w, \quad S_w := \left\{ x \in \widehat{\Omega} : \lim_{\rho\downarrow 0} \frac{|\nabla w|(B_\rho(x))}{\mathcal{L}^n(B_\rho(x))} = \infty \right\}, \tag{8}$$

and deduce from (5) that

$$S_v \subset S_u, \tag{9}$$

since

$$|\nabla v|(B_\rho(x)) \le |\nabla u|(B_\rho(x)).$$

Next, we use (5), (8), and (9) and obtain

$$|\nabla^s v|(\widehat{\Omega}) = |\nabla v|(S_v) \le |\nabla u|(S_u) = |\nabla^s u|(\widehat{\Omega}), \tag{10}$$

which in combination with (7) leads to

$$\widehat{I}[v,\widehat{\Omega}] \le \widehat{I}[u,\widehat{\Omega}].$$

By (6), we must have

$$\widehat{I}[v,\widehat{\Omega}] = \widehat{I}[u,\widehat{\Omega}],$$

and, by (7) and (10), this is only possible if

$$\int_{\widehat{\Omega}} h(|\nabla^a u|)\,\mathrm{d}\,x = \int_{\widehat{\Omega}} h(|\nabla^a v|)\,\mathrm{d}\,x, \tag{11}$$

$$|\nabla^s u|(\widehat{\Omega}) = |\nabla^s v|(\widehat{\Omega}). \tag{12}$$

From (11) and $|\nabla^a v| \leq |\nabla^a u|$ and requirement (2) it is immediate that

$$|\nabla^a u| = |\nabla^a v| \quad \mathcal{L}^n\text{-a.e. on } \widehat{\Omega}. \tag{13}$$

If $E \subset \widehat{\Omega}$ is a Borel set, then analogous to (10) we obtain from (5) and (9)

$$|\nabla^s v|(E) = |\nabla v|(S_v \cap E) \leq |\nabla u|(S_u \cap E) = |\nabla^s u|(E). \tag{14}$$

At the same time, using (14) with E replaced by $\widehat{\Omega} - E$ and (12), we find that

$$\begin{aligned} |\nabla^s v|(E) &= |\nabla^s v|(\widehat{\Omega}) - |\nabla^s v|(\widehat{\Omega} - E) \geq |\nabla^s v|(\widehat{\Omega}) - |\nabla^s u|(\widehat{\Omega} - E) \\ &= |\nabla^s u|(\widehat{\Omega}) - |\nabla^s u|(\widehat{\Omega} - E) = |\nabla^s u|(E). \end{aligned}$$

In view of (14), it is shown that

$$|\nabla^s u| = |\nabla^s v|. \tag{15}$$

Suppose that

$$\mathcal{L}^n\Big(\big\{x \in \widehat{\Omega} : \nabla^a u(x) \neq \nabla^a v(x)\big\}\Big) > 0. \tag{16}$$

We have

$$\int_{[\nabla^a u \neq \nabla^a v]} \big(|\nabla^a u| + |\nabla^a v| - |\nabla^a u + \nabla^a v|\big)\,\mathrm{d}\,x > 0, \tag{17}$$

since otherwise

$$|\nabla^a u + \nabla^a v| = |\nabla^a u| + |\nabla^a v|$$

a.e. on $[\nabla^a u \neq \nabla^a v]$ and, therefore,

$$\nabla^a u = \lambda \nabla^a v$$

on this set with a nonnegative function λ. However, (13) then leads to the contradiction $\lambda = 1$. From (17) we get recalling (2)

$$\int_{\widehat{\Omega}} h\left(\left|\nabla^a\left(\frac{u+v}{2}\right)\right|\right)\mathrm{d}\,x < \int_{\widehat{\Omega}} h\left(\frac{1}{2}|\nabla^a u| + \frac{1}{2}|\nabla^a v|\right)\mathrm{d}\,x$$
$$\leq \frac{1}{2}\int_{\widehat{\Omega}} h(|\nabla^a u|)\mathrm{d}\,x + \frac{1}{2}\int_{\widehat{\Omega}} h(|\nabla^a v|)\mathrm{d}\,x,$$

and since $|\nabla^s(u+v)| \leq |\nabla^s u| + |\nabla^s v|$ it follows from (13) and (15) that

$$\widehat{I}\left[\frac{u+v}{2}, \widehat{\Omega}\right] < \widehat{I}[u, \widehat{\Omega}]. \tag{18}$$

But $(u+v)/2$ belongs to $BV_{u_0}(\Omega;\mathbb{R}^N)$, thus the strict inequality (18) contradicts the minimizing property of u, and assumption (16) is wrong which means

$$\nabla^a u = \nabla^a v \quad \mathcal{L}^n\text{-a.e. on } \widehat{\Omega}. \tag{19}$$

Consider the measure $\mu := |\nabla^s u|$. Using (15) we find μ-measurable functions Θ_u, Θ_v: $\widehat{\Omega} \to \mathbb{R}^{nN}$ s.t. $|\Theta_u| = 1 = |\Theta_v|$ μ-a.e. and

$$\nabla^s u = \Theta_u \llcorner \mu, \quad \nabla^s v = \Theta_v \llcorner \mu. \tag{20}$$

Let us assume that

$$\left|\nabla^s\left(\frac{u+v}{2}\right)\right|(\widehat{\Omega}) < |\nabla^s u|(\widehat{\Omega}). \tag{21}$$

This implies on account of (19)

$$\widehat{I}\left[\frac{u+v}{2}, \widehat{\Omega}\right] = \int_{\widehat{\Omega}} h(|\nabla^a u|)\mathrm{d}\,x + \overline{c}\left|\nabla^s\left(\frac{u+v}{2}\right)\right|(\widehat{\Omega}) < \widehat{I}[u, \widehat{\Omega}],$$

which is in contradiction to the minimality of u. Therefore we have in place of (21)

$$\left|\int_{\widehat{\Omega}} \frac{1}{2}(\Theta_u + \Theta_v)\,\mathrm{d}\,\mu\right| = \mu(\widehat{\Omega}).$$

Hence,

$$\mu(\widehat{\Omega}) \leq \frac{1}{2} \int_{\widehat{\Omega}} |\Theta_u + \Theta_v| \,\mathrm{d}\,\mu \leq \frac{1}{2} \int_{\widehat{\Omega}} \big(|\Theta_u| + |\Theta_v|\big) \,\mathrm{d}\,\mu = \mu(\widehat{\Omega})$$

and in conclusion

$$|\Theta_u + \Theta_v| = |\Theta_u| + |\Theta_v| \quad \mu\text{-a.e.}$$

For this reason, we can write

$$\Theta_u = \overline{\lambda}\Theta_v$$

with $\overline{\lambda}$ nonnegative and μ-measurable, but $|\Theta_u| = 1 = |\Theta_v|$ gives $\overline{\lambda} \equiv 1$, i.e., $\Theta_u = \Theta_v$ μ-a.e. From (20) it follows $\nabla^s u = \nabla^s v$, which, together with (19) shows that $\nabla u = \nabla v$. Quoting Proposition 3.2 of [4], we see $u - v \equiv \mathit{const}$ and $u = u_0 = v$ on $\widehat{\Omega} - \Omega$ yields $u = v$ and in conclusion $u(x) \in K$, a.e. The proof of Theorem 1 is complete. □

For the sake of completeness, we have a look at the scalar case for which it is possible to give up the special structure of the integrand and to obtain a maximum principle close to the classical one. To be precise, let us assume that $F\colon \mathbb{R}^n \to [0,\infty)$ is strictly convex together with $F(0) = 0$. For $u_0 \in W_1^1(\Omega)$ we consider again the variational problem $\mathcal{P}$

$$I[u,\Omega] = \int_{\Omega} F(\nabla u)\mathrm{d}\,x \to \min \quad \text{in} \quad u_0 + \overset{\circ}{W}{}_1^1(\Omega), \tag{$\mathcal{P}$}$$

and observe

$$\inf_{\partial\Omega} u_0 \leq u \leq \sup_{\partial\Omega} u_0 \tag{22}$$

provided that we can find a soluton $u \in W_1^1(\Omega)$ of $(\mathcal{P})$. In fact, if we assume $M := \sup_{\partial\Omega} u_0 < \infty$, then from

$$I[u,\Omega] \leq I\big[\min(u,M),\Omega\big]$$

we deduce that

$$\int_{[u>M]} F(\nabla u)\mathrm{d}\,x = 0,$$

and $0 \le F(\nabla u/2) < F(\nabla u)/2$ on $[\nabla u \neq 0]$ implies $\nabla u = 0$ on $[u > M]$. Hence, $\nabla \max(u, M) = 0$, which shows $u \le M$.

Let us now assume that F is of linear growth, i.e., with constants a, $A > 0$, b, and $B \in \mathbb{R}$ it holds

$$a|\xi| + b \le F(\xi) \le A|\xi| + B \tag{23}$$

for all $\xi \in \mathbb{R}^n$. Moreover, we require

$$F(-\eta) = F(\eta) \quad \text{for all} \quad \eta \in \mathbb{R}^n. \tag{24}$$

Then we have

Theorem 2. *Let the strictly convex function F satisfy* (23) *and* (24) *together with $F(0) = 0$. If $u \in \mathcal{M}$ denotes a generalized minimizer of problem $(\mathcal{P})$, then (the slightly weaker variant of* (22)*)*

$$\inf_\Omega u_0 \le u(x) \le \sup_\Omega u_0 \tag{25}$$

is satisfied for a.a. $x \in \Omega$.

Proof. It is sufficient to consider the case $M := \sup_\Omega u_0 < \infty$ and to prove the second inequality stated in (25). We extend u_0 to a function of class $W^1_1(\widehat{\Omega})$ on a bounded Lipschitz domain $\widehat{\Omega} \Supset \Omega$ assuming that this extension – again denoted by u_0 – still satisfies $u_0 \le M$, a.e. (now on $\widehat{\Omega}$), since otherwise we may compose it with the function $\psi(t) := \min(M, t)$, $t \in \mathbb{R}$. As outlined in the proof of Theorem 1 the claim of Theorem 2 will follow if we can show that any solution $u \in BV_{u_0}(\Omega)$ of

$$\widehat{I}[w, \widehat{\Omega}] := \int_{\widehat{\Omega}} F(\nabla^a w) \mathrm{d}\, x + \int_{\widehat{\Omega}} F_\infty\left(\frac{\nabla^s w}{|\nabla^s w|}\right) \mathrm{d}\, |\nabla^s w| \to \min \quad \text{in} \quad BV_{u_0}(\Omega)$$

satisfies $u \le M$ a.e. Quoting the chain rule for real valued functions as stated in Theorem 3.99 of [4], we have $v := \psi \circ u \in BV_{u_0}(\Omega)$ together with

$$\nabla v = \psi'(u) \nabla^a u \llcorner \mathcal{L}^n + \big(\psi(u^+) - \psi(u^-)\big) \nu_u \mathcal{H}^{n-1} \llcorner J_u + \psi'(\widetilde{u}) \nabla^c u,$$

where our notation follows the terminology of [4]. Let us look at the part $\psi'(u) \nabla^a u \llcorner \mathcal{L}^n$ of the vector measure ∇v being absolutely continuous w.r.t. $\mathcal{L}^n$. It holds $\psi'(u) = 0$ a.e. on the set $[u > M]$, wheras $\psi'(u) = 1$, a.e., on $[u < M]$. Since the density $\nabla^a u$ equals the approximative differential of u (see [4, Theorem 3.83]), and since the approximative differential of u vanishes, a.e., on $[u = M]$ (see [4, Proposition 3.73(c)]), we get

$$\int_{\widehat{\Omega}} F(\nabla^a v)\,\mathrm{d}\,x = \int_{[u<M]} F(\nabla^a u)\,\mathrm{d}\,x. \tag{26}$$

Notice that the measures $\nabla^j v$ and $\nabla^c v$ are mutually orthogonal; hence, we can write

$$\begin{aligned}\int_{\widehat{\Omega}} F_\infty\left(\frac{\nabla^s v}{|\nabla^s v|}\right)\mathrm{d}\,|\nabla^s v| &= \int_{J_u} F_\infty\big(\psi(u^+) - \psi(u^-)\big)\nu_u \,\mathrm{d}\,\mathcal{H}^{n-1}\\ &\quad + \int_{\widehat{\Omega}} F_\infty\left(\psi'(\widetilde{u})\frac{\nabla^c u}{|\nabla^c u|}\right)\mathrm{d}\,|\nabla^c u|.\end{aligned} \tag{27}$$

The function $\psi'(\widetilde{u})$ has values in $\{0,1\}$, which means that

$$F_\infty\left(\psi'(\widetilde{u})\frac{\nabla^c u}{|\nabla^c u|}\right) \le F_\infty\left(\frac{\nabla^c u}{|\nabla^c u|}\right) \quad |\nabla^c u|\text{-a.e.}$$

At the same time, we have $\mathcal{H}^{n-1}$-a.e. on J_u

$$\begin{aligned}&F_\infty\big((\psi(u^+) - \psi(u^-))\nu_u\big)\\ &\qquad = |\psi(u^+) - \psi(u^-)|F_\infty\big(\mathrm{sign}\big[\psi(u^+) - \psi(u^-)\big]\nu_u\big)\\ &\qquad = |\psi(u^+) - \psi(u^-)|F_\infty(\nu_u)\\ &\qquad \le |u^+ - u^-|F_\infty(\nu_u)\\ &\qquad = F_\infty\big((u^+ - u^-)\nu_u\big).\end{aligned}$$

Here, the first equality sign follows from the fact that the recession function is positively homogeneous of degree one, the second is a consequence of (24) and the last equation is established in the same way. Combing the inequalities from above with (26) and (27) and using the minimality of u we obtain

$$\int_{[u\ge M]} F(\nabla^a u)\,\mathrm{d}\,x = 0, \tag{28}$$

together with

$$\int_{J_u} F_\infty\big((\psi(u^+)-\psi(u^-))\nu_u\big)\,\mathrm{d}\,\mathcal{H}^{n-1} = \int_{J_u} F_\infty\big((u^+-u^-)\nu_u\big)\,\mathrm{d}\,\mathcal{H}^{n-1} \quad (29)$$

and

$$\int_{\widehat{\Omega}} F_\infty\Big(\psi'(\widetilde{u})\frac{\nabla^c u}{|\nabla^c u|}\Big)\,\mathrm{d}\,|\nabla^c u| = \int_{\widehat{\Omega}} F_\infty\Big(\frac{\nabla^c u}{|\nabla^c u|}\Big)\,\mathrm{d}\,|\nabla^c u|. \quad (30)$$

From (28), we deduce using the strict convexity of F, together with $F(0) = 0$, that

$$\nabla^a u = 0 \quad \mathcal{L}^n\text{-a.e. on} \quad [u \geq M]. \quad (31)$$

From (29) and

$$F_\infty\big((\psi(u^+)-\psi(u^-))\nu_u\big) \leq F_\infty\big((u^+-u^-)\nu_u\big),$$

$\mathcal{H}^{n-1}$-a.e. on J_u it follows that

$$F_\infty\big((\psi(u^+)-\psi(u^-))\nu_u\big) = F_\infty\big((u^+-u^-)\nu_u\big) \quad (32)$$

$\mathcal{H}^{n-1}$-a.e. on J_u, since otherwise we would have a contradiction to the minimality of u. (32) gives

$$|\psi(u^+)-\psi(u^-)| = |u^+-u^-| \quad (33)$$

$\mathcal{H}^{n-1}$-a.e. on J_u (recall $F_\infty(t\xi) = |t|F_\infty(\xi)$) but by definition of ψ this means

$$\psi(u^+)-\psi(u^-) = u^+-u^- \quad (34)$$

$\mathcal{H}^{n-1}$-a.e. on J_u. In the same way, we obtain from (30), from

$$F_\infty\left(\psi'(\widetilde{u})\frac{\nabla^c u}{|\nabla^c u|}\right) \leq F_\infty\left(\frac{\nabla^c u}{|\nabla^c u|}\right)$$

and from the minimality of u that

$$\psi'(\widetilde{u}) = 1 \quad |\nabla^c u| - \text{a.e.} \quad (35)$$

Recalling the formula for ∇v and using (31), (34), and (35) we arrive at $\nabla v = \nabla u$; hence, $v = u$ and in conclusion $u \leq M$ a.e. on $\widehat{\Omega}$. □

References

1. R. A. Adams, *Sobolev spaces.* Academic Press, New York–San Francisco–London 1975.
2. L. Ambrosio, G. Dal Maso, *A general chain rule for distributional derivatives.* — Proc. Amer. Math. Soc. **108** (1990), 691–702.
3. L. Ambrosio, G. Dal Maso, *On the relaxation in* $BV(\Omega;\mathbb{R}^m)$ *of quasi-convex integrals.* — J. Funct. Anal. **109** (1992), 76–97.
4. L. Ambrosio, N. Fusco, D. Pallara, *Functions of Bounded Variation and Free Discontinuity Problems.* Oxford Science Publications, Clarendon Press, Oxford, 2000.
5. M. Bildhauer, *Convex variational problems: linear, nearly linear, and anisotropic growth conditions.* — Lect. Notes Math. **1818**, Springer, Berlin–Heidelberg–New York, 2003.
6. M. Bildhauer, M. Fuchs, *Partial regularity for a class of anisotropic variational integrals with convex hull property.* — Asymp. Anal. **32** (2002), 293–315.
7. M. Bildhauer, M. Fuchs, *Relaxation of convex variational problems with linear growth defined on classes of vector-valued functions.* — Algebra Analiz **14** (2002), 26–45.
8. G. Buttazzo, *Semicontinuity, relaxation, and integral representation in the calculus of variations.* — Pitman Res. Notes Math., Longman, Harlow, 1989.
9. A. D'Ottavio, F. Leonetti, C. Musciano, *Maximum principle for vector valued mappings minimizing variational integrals.* — Atti Sem. Mat. Fis. Uni. Modena **XLVI** (1998), 677–683.
10. E. De Giorgi, *Selected papers.* Edited by L. Ambrosio, G. Dal Maso, M. Forti, M. Miranda, and S. Spagnolo, Springer, Berlin, 2006.
11. E. Giusti, *Minimal surfaces and functions of bounded variation.* Monographs in Mathematics **80**, Birkhäuser, Boston–Basel–Stuttgart, 1984.
12. M. Giaquinta, G, Modica, J. Souček, *Functionals with linear growth in the calculus of variations.* — Comm. Math. Univ. Carolinae **20** (1979), 143–171.
13. C. Goffman, J. Serrin, *Sublinear functions of measures and variational integrals.* — Duke Math. J. **31** (1964), 159–178.
14. L. Simon, *Lectures on geometric measure theory.* — Proc. Centre Math. Anal., Australian Nat. Univ. **3** (1983).

Universität des Saarlandes
Fachbereich 6.1 Mathematik
Postfach 15 11 50
D-66041 Saarbrücken, Germany

E-mail: fuchs@math.uni-sb.de

Поступило 30 мая 2010 г.