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LAGUERRE AND MEIXNER SYMMETRIC FUNCTIONS, AND INFINITE-DIMENSIONAL DIFFUSION PROCESSES

ABSTRACT. The Laguerre symmetric functions introduced in the note are indexed by arbitrary partitions and depend on two continuous parameters. The top degree homogeneous component of every Laguerre symmetric function coincides with the Schur function with the same index. Thus, the Laguerre symmetric functions form a two-parameter family of inhomogeneous bases in the algebra of symmetric functions. These new symmetric functions are obtained from the N-variate symmetric polynomials of the same name by a procedure of analytic continuation. The Laguerre symmetric functions are eigenvectors of a second order differential operator, which depends on the same two parameters and serves as the infinitesimal generator of an infinite-dimensional diffusion process $X(t)$. The process $X(t)$ admits approximation by some jump processes related to one more new family of symmetric functions, the Meixner symmetric functions.

In equilibrium, the process $X(t)$ can be interpreted as a time-dependent point process on the punctured real line $\mathbb{R} \setminus \{0\}$, and the point configurations may be interpreted as doubly infinite collections of particles of two opposite charges and with log-gas-type interaction. The dynamical correlation functions of the equilibrium process have determinantal form: they are given by minors of the so-called extended Whittaker kernel, introduced earlier in a paper by Borodin and the author.

1. INTRODUCTION

1.1. Preface. The present note is a research announcement; the detailed exposition will appear elsewhere. The goal of the work is twofold: (1) to introduce new bases $\{\mathfrak{L}_\nu\}$ and $\{\mathfrak{M}_\nu\}$ in the algebra Λ of symmetric functions, and (2) to construct a diffusion process $X(t)$ in an infinite-dimensional cone $\widetilde{\Omega}$. The two subjects are interrelated: The algebra Λ

serves as the algebra of "polynomial observables" on the cone $\widetilde{\Omega}$, and the basis elements $\mathfrak{L}_\nu \in \Lambda$ are the eigenfunctions of the infinitesimal generator of the process $X(t)$. As for the basis $\{\mathfrak{M}_\nu\}$, its elements are the eigenfunctions of the infinitesimal generator of an auxiliary jump process.

The basis elements $\mathfrak{L}_\nu \in \Lambda$ are called the *Laguerre symmetric functions*. They are indexed by arbitrary partitions ν and depend on two continuous parameters. The diffusion process $X(t)$ depends on the same parameters. It possesses an invariant symmetrizing measure W, which also serves as the orthogonality measure for the Laguerre symmetric functions.

1.2. Finite-dimensional counterparts. All the basic objects – the Laguerre symmetric functions, the cone $\widetilde{\Omega}$, the probability distribution W on $\widetilde{\Omega}$, and the diffusion $X(t)$ on $\widetilde{\Omega}$ – have finite-dimensional counterparts; I will describe them briefly.

• In dimension 1, these are the classical Laguerre orthogonal polynomials on the half-line $x > 0$ with the weight measure $x^{b-1}e^{-x}dx$ (here $b > 0$ is a parameter), and the diffusion is generated by the associated ordinary differential operator

$$x\frac{d^2}{dx^2} + (b - x)\frac{d}{dx};$$

the Laguerre polynomials are its eigenfunctions.

• In dimension $N = 2, 3, \ldots$, we deal with the algebra Λ_N of symmetric polynomials in N variables $x_1, \ldots, x_N$. Such polynomials are viewed as functions on the N-dimensional cone

$$\widetilde{\Omega}_N = \{\underline{x} := (x_1, \ldots, x_N) : x_1 \geqslant \cdots \geqslant x_N \geqslant 0\} \subset \mathbb{R}^N.$$

A relevant basis in Λ_N is formed by the N-variate symmetric Laguerre polynomials [22], [21], which are orthogonal with respect to the measure on $\widetilde{\Omega}_N$ with the density

$$(x_1 \ldots x_N)^{b-1} e^{-x_1 - \cdots - x_N} \cdot \prod_{1 \leqslant i < j \leqslant N} (x_i - x_j)^2.$$

Assuming $\underline{x}$ to be in the interior of the cone, one can interpret $\underline{x}$ as a collection of N indistinguishable particles on the half-line $\mathbb{R}_{>0}$; then the above measure determines an ensemble of random particle configurations, called the N-particle *Laguerre ensemble*. Again, there exists an associated diffusion process $X_N(t)$ with state space $\widetilde{\Omega}_N$. In the interior of the cone,

$X_N(t)$ may be interpreted as a random evolution of N interacting particles with pairwise repulsion. One may call $X_N(t)$ the N-particle *dynamical Laguerre ensemble*.

• Note also that there exists a lattice version of the Laguerre ensemble, related to natural discrete analogs of the Laguerre polynomials – the Meixner polynomials. There exists also an associated Markov dynamics, which is a Markov jump process. In the simplest case $N = 1$ this is a birth-death process with linear jump rates.

1.3. Analytic continuation. In the literature, there exist many models of such a kind, continuous and discrete, associated with various systems of orthogonal polynomials. A general recipe for building infinite-dimensional analogs of such models, often applied in Random Matrix Theory, is to use a large-N limit transition (see, e.g., the survey paper [14]). However, in the present work, a different approach is applied. In short, its main idea can be formulated as *extrapolation into complex domain via analytic continuation with respect to two parameters, the number of particles $N = 1, 2, 3, \ldots$ and the additional parameter $b > 0$.* Surprisingly enough, although parameters N and b are of a very different origin, they can be treated on equal grounds.

In comparison with the existing approaches to random-matrix-type dynamical models in infinite dimension (see [28, 26, 11–13]), our approach is to great extent more algebraic.

1.4. Point processes. In the equilibrium state corresponding to the stationary distribution W, the process $X(t)$ can be interpreted as a time-dependent point process $X^{\mathrm{stat}}(t)$. This interpretation relies on the fact that the invariant measure W is supported by a dense Borel subset $\widetilde{\Omega}' \subset \widetilde{\Omega}$ admitting a natural realization as a space of infinite particle configuration on the punctured real line $\mathbb{R}^* := \mathbb{R} \setminus \{0\}$. It turns out that the time-dependent point process $X^{\mathrm{stat}}(t)$ is *determinantal*: its dynamical correlation functions are given by minors of a kernel $K(s, u; t, v)$ on $(\mathbb{R} \times \mathbb{R}^*) \times (\mathbb{R} \times \mathbb{R}^*)$ — the so-called *extended Whittaker kernel*, which initially appeared in [4].

Note that the N-particle dynamical ensembles $X_N(t)$ live on the half-line $\mathbb{R}_{>0}$ while the particle configurations of the process $X^{\mathrm{stat}}(t)$ live on the punctured real line $\mathbb{R}^*$, which is the union of *two* half-lines. Because of this duplication effect, the claim that $X(t)$ cannot be related to the processes $X_N(t)$ through a large-N limit transition becomes intuitively evident.

1.5. Lattice approximation. Although $X(t)$ does not arise from a

large-N limit, it admits a *lattice approximation*. Namely, $X(t)$ can be obtained as a scaling limit of some jump processes depending on an additional parameter $\xi \in (0, 1)$, as $\xi \uparrow 1$. These jump processes were studied in detail in [4]. Their state space is the set of all partitions. This countable set also can be realized as a set of particle configurations on the lattice $\mathbb{Z} + \frac{1}{2}$ of half-integers; here the number of particles is finite but not restricted. The second basis $\{\mathfrak{M}_\nu\} \subset \Lambda$ mentioned above just arises in connection with these jump processes. The elements $\mathfrak{M}_\nu$ are called the *Meixner symmetric functions*; they depend on three parameters: to the two parameters of the Laguerre symmetric functions one adds the third parameter $\xi \in (0, 1)$. In the limit as $\xi \uparrow 1$, one has $\mathfrak{M}_\nu \to \mathfrak{L}_\nu$, similarly to the approximation of the classical Laguerre polynomials by the Meixner polynomials.

1.6. Concluding remarks. The results announced in the present note continue those of [1, 4–6], and all these works have a connection with the asymptotic representation theory of the symmetric groups. It is interesting to compare the results of these papers and the present note with those of the papers [2, 3, 7], which are related to representations of the unitary groups. Although the construction of a Markov dynamics in [7] relies on a different approach, [7, Appendix] also exploits the idea of analytic continuation. Note that in the context of the unitary groups, the Laguerre polynomials are (to some extent) replaced by the Jacobi polynomials.

Finally, I would like to note that the representation theory of reductive groups and Lie algebras also affords examples in which finite-dimensional objects arise as a degeneration of infinite-dimensional ones, and, conversely, infinite-dimensional objects may be reconstructed from finite-dimensional ones through analytic continuation in parameters. For instance, the principal series representations or highest weight modules may be viewed as analytic continuation of the irreducible finite-dimensional representations.

2. THE ONE-PARTICLE CASE

There is a well-known relationship between systems $\{\phi_n; n = 0, 1, 2, \dots\}$ of orthogonal polynomials of hypergeometric type on $\mathbb{R}$ and some one-dimensional Markov processes $X(t)$ (see, e.g., [27]). Namely, the state space of $X(t)$ is a closed subset $I \subset \mathbb{R}$ – the support of the orthogonality measure of $\{\phi_n\}$, and the infinitesimal generator of $X(t)$ is a second order

differential (or difference) operator D, such that

$$D\phi_n = -\mu_n \phi_n, \qquad \mu_n \geqslant 0.$$

The orthogonality measure w for the polynomials ϕ_n serves as an invariant and symmetrizing measure of $X(t)$. The transition function of $X(t)$ has the form

$$P(t; x, dy) = \left(\sum_{n=0}^{\infty} e^{-\mu_n t} \frac{\phi_n(x)\phi_n(y)}{\int\limits_I \phi_n^2(u) w(du)} \right) w(dy). \qquad (2.1)$$

The simplest examples are provided by the classical Jacobi, Laguerre, and Hermite polynomials:

• *Jacobi polynomials*: I is the closed interval $[-1, 1]$, w has density $(1 - x)^{a-1}(1 + x)^{b-1}$ with parameters $a, b > 0$, and

$$D = (1 - x^2)\frac{d^2}{dx^2} + [b - a - (a + b)x]\frac{d}{dx}.$$

• *Laguerre polynomials*: I is the closed half-line $[0, +\infty)$, w has density $x^{b-1}e^{-x}$ with parameter $b > 0$, and

$$D = x\frac{d^2}{dx^2} + (b - x)\frac{d}{dx}.$$

• *Hermite polynomials*: I is the whole real line, w has density $e^{-x^2/2}$, and

$$D = \frac{d^2}{dx^2} - x\frac{d}{dx}.$$

In the Hermite case, $X(t)$ is the Ornstein–Uhlenbeck process. In the Laguerre case, $X(t)$ is closely related to a squared Bessel process (see, e.g., [9]).

3. The N-particle case

Here we recall a well-known construction providing a multidimensional generalization of the above picture.

Fix $N = 1, 2, 3, \ldots$. Instead of univariate polynomials we will deal with *symmetric* polynomials in N variables $x_1, \ldots, x_N$. Denote by Λ_N the

algebra formed by such polynomials (the base field is $\mathbb{R}$ or $\mathbb{C}$ depending on convenience). The interval I is replaced by the subset

$$I_{\mathrm{ord}}^N := \{(x_1, \ldots, x_N) \subset I^N : x_1 \geqslant \cdots \geqslant x_N\},$$

and we regard Λ_N as an algebra of functions on I_{ord}^N. Let $\nu = (\nu_1, \ldots, \nu_N)$ range over the set of partitions of length $\ell(\nu) \leqslant N$. We set

$$\phi_{\nu|N}(x_1, \ldots, x_N) = \frac{\det\left[\phi_{\nu_i + N - i}(x_j)\right]}{V_N(x_1, \ldots, x_N)}, \qquad (3.1)$$

where the determinant is of order N and V_N is the Vandermonde,

$$V_N = V_N(x_1, \ldots, x_N) = \prod_{1 \leqslant i < j \leqslant N} (x_i - x_j).$$

The $\phi_{\nu|N}$'s are symmetric polynomials forming a basis in Λ_N. Moreover, it is readily verified that they are pairwise orthogonal with respect to the measure

$$w_N(dx_1 \ldots dx_N) = (V_N)^2 \prod_{i=1}^N w(dx_i) \qquad (3.2)$$

on I_{ord}^N.

This construction seems to be well known; see, e.g., Lassalle's papers [19–21]. Formula (3.1) is similar to the classical expression for the Schur symmetric polynomials (the Schur polynomials appear if one substitutes $\phi_n(x) = x^n$; they are not orthogonal polynomials though).

Further, the analog of D is the operator

$$D_N := V_N^{-1} \circ (D^{x_1} + \cdots + D^{x_N}) \circ V_N + d_N 1, \qquad (3.3)$$

where D^{x_i} stands for a copy of D acting on variable x_i and

$$d_N = \mu_0 + \cdots + \mu_{N-1}.$$

If D is a differential operator,

$$D = A(x)\frac{d^2}{dx^2} + B(x)\frac{d}{dx},$$

then D_N is a partial differential operator,

$$D_N := \sum_{i=1}^{N} \left(A(x_i) \frac{\partial^2}{\partial x_i^2} + \left[B(x_i) + \sum_{j:\, j \neq i} \frac{2A(x_i)}{x_i - x_j} \right] \frac{\partial}{\partial x_i} \right).$$

Due to the special choice of d_N, the constant term in D_N vanishes.

Although the coefficients in front of the first derivatives have singularities along the hyperplanes $x_i = x_j$, the operator D_N is applicable to symmetric polynomials and preserves the space Λ_N. The polynomials $\phi_{\nu|N}$ are eigenfunctions of D_N:

$$D_N \phi_{\nu|N} = - \left(\sum_{i=1}^{N} (\mu_{\nu_i + N - i} - \mu_{N-i}) \right) \phi_{\nu|N}.$$

Finally, D_N serves as the (pre-)generator of a Markov process $X_N(t)$ on I_{ord}^N with invariant symmetrizing measure w_N.

The first example of such a process $X_N(t)$ has been investigated in [8]; it corresponds to the system of Hermite polynomials. As shown in that paper, $X_N(t)$ is obtained from a matrix-valued Ornstein-Uhlenbeck process through the projection onto the matrix eigenvalues. One may say that $X_N(t)$ is the radial part of this matrix-valued Markov process.

In the present work we focus on the Laguerre case.

4. The Laguerre symmetric functions

Let $e_1, e_2, \ldots$ denote the elementary symmetric polynomials,

$$e_1 = \sum_i x_i, \quad e_2 = \sum_{i<j} x_i x_j, \quad e_3 = \sum_{i<j<k} x_i x_j x_k,$$

and so on. Here it is tacitly assumed that the indices range over $\{1, \ldots, N\}$, where N is the number of variables. As well known, the algebra Λ_N of N-variate symmetric polynomials is isomorphic to the algebra of ordinary polynomials in $e_1, \ldots, e_N$.

Our first step is to make a change of variables: take $\{e_1, \ldots, e_N\}$ as new (formal) variables instead of natural coordinates $x_1, \ldots, x_N$.

Theorem 4.1. *The N-variate Laguerre operator $D_N : \Lambda_N \to \Lambda_N$ can be rewritten as the following second order partial differential operator in variables $e_1, \ldots, e_N$:*

$$D_N = \sum_{m,n=1}^{N} A_{mn} \frac{\partial^2}{\partial e_m \partial e_n} + \sum_{n=1}^{N} B_n \frac{\partial}{\partial e_n},$$

where

$$A_{mn} = \sum_{k=0}^{\min(m,n)-1} (m+n-1-2k) e_{m+n-1-k} e_k$$

and

$$B_n = -n e_n + (N-n+1)(N+b-n) e_{n-1}$$

with the convention that $e_0 = 1$ and $e_k = 0$ for $k > N$.

The next step is to replace Λ_N by the *algebra Λ of symmetric functions*. For our purpose, it is convenient to define Λ as the algebra of polynomials in countably many formal commuting variables $e_1, e_2, \ldots$, which are assumed to be algebraically independent.

For $N = 1, 2, \ldots$, let $J_N \subset \Lambda$ denote the ideal generated by elements e_k with $k > N$. The quotient algebra Λ / J_N is naturally isomorphic to Λ_N, so we get a canonical algebra homomorphism $\pi_N : \Lambda \to \Lambda_N$, which we call the Nth *truncation map*.

Definition 4.2. *Let z and z' be complex parameters. Consider the formal second order differential operator in countably many variables $e_1, e_2, \ldots$, obtained from the N-variate Laguerre operator D_N by removing the relations $e_{N+1} = e_{N+2} = \cdots = 0$, dropping the restriction $m, n \leqslant N$, and replacing the factor $(N-n+1)(N+b-n)$ in the definition of coefficient B_n by $(z-n+1)(z'-n+1)$:*

$$\mathfrak{D} = \sum_{m,n=1}^{\infty} A_{mn} \frac{\partial^2}{\partial e_m \partial e_n} + \sum_{n=1}^{\infty} B_n \frac{\partial}{\partial e_n},$$

where A_{mn} is given by exactly the same formula as above,

$$A_{mn} = \sum_{k=0}^{\min(m,n)-1} (m+n-1-2k) e_{m+n-1-k} e_k,$$

and

$$B_n = -ne_n + (z - n + 1)(z' - n + 1)e_{n-1}.$$

Observe that $\mathfrak{D}$ is correctly defined as an operator $\Lambda \to \Lambda$; we call it the Laguerre operator in Λ.

If $z = N$ and $z' = N + b - 1$, then $\mathfrak{D}$ preserves the ideal $J_N \subset \Lambda$ and hence factorizes to an operator in the quotient $\Lambda/J_N = \Lambda_N$; the resulting operator coincides with D_N. Note that this property, combined with the polynomial dependence on parameters z, z', determines the operator uniquely. In this sense, $\mathfrak{D}$ may be viewed as the result of *analytic continuation* (or extrapolation) of the N-variate Laguerre operators D_N with respect to parameters N and b.

Let $\{L_n\}$ denote the system of *monic* Laguerre polynomials with parameter $b > 0$ (see, e.g., [18]). Recall that, in our notation, the weight function is $x^{b-1}e^{-x}$. Next, let $\{L_{\nu|N,b}\}$ denote the system of N-variate symmetric Laguerre polynomials defined in accordance with the determinantal formula (3.1). We are going to define elements of Λ that may be viewed as are analogs of the polynomials $L_{\nu|N,b}$. To do this we apply the same principle of analytic continuation in N and b as we have employed in the definition of $\mathfrak{D}$.

Theorem 4.3. *For an arbitrary partition $\nu = (\nu_1, \nu_2, \dots)$, there exists a unique element $\mathfrak{L}_\nu \in \Lambda[z, z'] := \Lambda \otimes \mathbb{C}[z, z']$, such that for any natural number $N \geqslant \ell(\nu)$ and any $b > 0$,*

$$\pi_N \left(\mathfrak{L}_\nu \big|_{z=N,\, z'=N+b-1} \right) = L_{\nu|N,b}.$$

Here $\pi_N : \Lambda \to \Lambda_N$ is the Nth truncation map introduced above.

Definition 4.4. *We call the elements $\mathfrak{L}_\nu$ the Laguerre symmetric functions.*

Recall the definition of the *Schur symmetric functions*: these are elements of Λ indexed by arbitrary partitions ν and expressed through the generators e_n by the following formula (the *Nägelsbach–Kostka formula*, see [23])

$$S_\nu = \det[e_{\nu'-i+j}];$$

here ν' stands for the partition given by transposing the Young diagram corresponding to ν, and the order of the determinant is an arbitrary integer $\geqslant \ell(\nu')$. As well known, the Schur functions form a distinguished

homogeneous basis in Λ, and

$$\deg S_\nu = |\nu| := \sum \nu_i.$$

Let us explain the notation used in the next theorem. We identify partitions and the corresponding Young diagrams. Given a couple $\mu \subseteq \nu$ of Young diagrams, we denote by $\dim \nu/\mu$ the number of standard Young tableaux of the skew shape ν/μ. The symbol $\square \in \nu/\mu$ denotes a box in ν/μ, and $c(\square)$ denotes its *content*, equal to the difference $j - i$ of the column number j and the row number i of the box.

Theorem 4.5. *The expansion of the Laguerre symmetric function $\mathfrak{L}_\nu$ in the basis of the Schur symmetric functions has the form*

$$\mathfrak{L}_\nu = \sum_{\mu \subseteq \nu} C(\nu, \mu; z, z')S_\mu,$$

where

$$C(\nu, \mu; z, z') = (-1)^{|\nu|-|\mu|} \frac{\dim \nu/\mu}{(|\nu| - |\mu|)!} \prod_{\square \in \nu/\mu} (z + c(\square))(z' + c(\square))).$$

Since $C(\nu, \nu; z, z') = 1$, the top homogeneous component of $\mathfrak{L}_\nu$ is equal to S_ν:

$$\mathfrak{L}_\nu = S_\nu + \text{lower degree terms}.$$

It follows that the Laguerre symmetric functions with any fixed values of parameters z and z' form a basis in Λ.

For the empty diagram corresponding to the zero partition, $\nu = \varnothing$, we have $\mathfrak{L}_\varnothing = S_\varnothing = 1$. This is the only case when $\mathfrak{L}_\nu$ and S_ν coincide: for $\nu \neq \varnothing$, $\mathfrak{L}_\nu$ is an inhomogeneous element, so that the basis $\{\mathfrak{L}_\nu\}$ of Laguerre symmetric functions is an example of inhomogeneous basis. In this respect, it differs from other bases in Λ that are commonly used in algebraic combinatorics.

A box $\square$ in a Young diagram ν is said to be a *corner box* if the shape $\nu \setminus \square$ obtained by removing $\square$ from ν is again a Young diagram. Let ν^- denote the set of all corner boxes in ν. For instance, if $\nu = (3, 2, 2)$ then ν^- comprises two corner boxes, $\square = (1, 3)$ and $\square = (3, 2)$.

Theorem 4.6. *The action of $\mathfrak{D}$ on the Schur functions is given by*

$$\mathfrak{D}S_\nu = -|\nu|S_\nu + \sum_{\square \in \nu^-} (z + c(\square))(z' + c(\square))S_{\nu \setminus \square}.$$

Theorem 4.7. *The Laguerre symmetric functions are eigenvectors of the Laguerre operator $\mathfrak{D}$ with the same values of parameters (z, z'). More precisely,*

$$\mathfrak{D}\mathfrak{L}_\nu = -|\nu|\mathfrak{L}_\nu.$$

5. Formal orthogonality

Definition 5.1. *For any fixed $(z, z') \in \mathbb{C}^2$, introduce the formal moment functional $\psi : \Lambda \to \mathbb{C}$ by setting*

$$\psi(1) = 1, \quad \psi(\mathfrak{D}f) = 0 \quad \text{for any } f \in \Lambda,$$

where the Laguerre operator $\mathfrak{D}$ is taken with the same values of the parameters as ψ.

The definition is correct by virtue of the last theorem. Indeed, it implies that that the range of $\mathfrak{D}$ is the span of the Laguerre functions $\mathfrak{L}_\nu$ with $\nu \neq \varnothing$, while the vector $1 = \mathfrak{L}_\varnothing \in \Lambda$ is transversal to this span. Note that ψ depends polynomially on the parameters z, z'.

Theorem 5.2. *For any Young diagram ν,*

$$\psi(S_\nu) = \prod_{\square \in \nu} (z + c(\square))(z' + c(\square)) \cdot \left(\frac{\dim \nu}{|\nu|!} \right)^2.$$

This formula provides an alternative (equivalent) way of introducing the moment functional.

Theorem 5.3. *For any two Young diagrams ν and μ,*

$$\psi\left(\mathfrak{L}_\nu \mathfrak{L}_\mu\right) = \delta_{\nu\mu} \prod_{\square \in \nu} (z + c(\square))(z' + c(\square)), \tag{5.1}$$

where $\delta_{\nu\mu}$ is Kronecker's delta.

This result shows that the Laguerre symmetric functions are pairwise orthogonal with respect to the inner product in the space Λ defined by

$$(f, g) = (f, g)_{z, z'} := \psi(fg), \quad f, g \in \Lambda. \tag{5.2}$$

Obviously, the inner product is nondegenerate if (and only if) z and z' are not integers, for then the product in the right-hand side of (5.1) never vanishes.

Remark 5.4. Assume z and z' are not integers. Then the Laguerre symmetric function $\mathfrak{L}_\nu$ is characterized by the following two properties:

(1) $\mathfrak{L}_\nu$ differs from the Schur symmetric function S_ν by lower degree terms;

(2) $\mathfrak{L}_\nu$ is orthogonal, with respect to inner product (5.2), to all elements of Λ of lower degree (that is, of degree strictly less than $\deg \mathfrak{L}_\nu = |\nu|$).

Alternatively, without any assumption on the parameters, $\mathfrak{L}_\nu$ is characterized by (1) together with the following property replacing (2):

(2 ') $\mathfrak{L}_\nu$ is an eigenfunction of the Laguerre operator $\mathfrak{D}$.

6. The orthogonality measure for the Laguerre symmetric functions

Definition 6.1. *By the Thoma cone we mean the subset $\widetilde{\Omega} \subset \mathbb{R}_+^\infty \times \mathbb{R}_+^\infty \times \mathbb{R}_+$ consisting of triples $\omega = (\alpha, \beta, \delta)$, where*

$$\alpha = (\alpha_1 \geqslant \alpha_2 \geqslant \cdots \geqslant 0), \quad \beta = (\beta_1 \geqslant \beta_2 \geqslant \cdots \geqslant 0), \quad \delta \geqslant 0,$$

and

$$\sum_{i=1}^{\infty} (\alpha_i + \beta_i) \leqslant \delta.$$

The Thoma simplex Ω is the subset of $\widetilde{\Omega}$ determined by the additional condition $\delta = 1$.

Both $\widetilde{\Omega}$ and Ω are closed subsets of the product space $\mathbb{R}^{2\infty+1} := \mathbb{R}^\infty \times \mathbb{R}^\infty \times \mathbb{R}$ equipped with the product topology; Ω is compact and $\widetilde{\Omega}$ is locally compact; $\widetilde{\Omega}$ is precisely the cone with base Ω.

Let $\mathrm{Fun}(\widetilde{\Omega})$ denote the space of continuous functions on the Thoma cone; this is an algebra under pointwise multiplication. We embed Λ into $\mathrm{Fun}(\widetilde{\Omega})$ by setting

$$1 + \sum_{n=1}^{\infty} e_n t^n \to e^{\gamma t} \prod_{i=1}^{\infty} \frac{1 + \alpha_i t}{1 - \beta_i t}, \qquad (6.1)$$

where t is an auxiliary formal variable and

$$\gamma := \delta - \sum_{i=1}^{\infty} (\alpha_i + \beta_i).$$

More precisely, the correspondence, which is written in terms of the generating series for $\{e_n\}$, turns each e_n into a function $e_n(\omega)$ on the Thoma cone. This function is continuous. The correspondence $e_n \mapsto e_n(\,\cdot\,)$ is extended to the whole algebra Λ by multiplicativity. In this way we get an algebra morphism $\Lambda \to \mathrm{Fun}(\widetilde{\Omega})$, which is an embedding. For any $f \in \Lambda$ we denote the corresponding continuous function on $\widetilde{\Omega}$ by $f(\omega)$.

Equivalently, in terms of another system of generators of the algebra Λ, the Newton power sums p_n, the embedding $\Lambda \to \mathrm{Fun}(\widetilde{\Omega})$ can be defined by setting

$$p_1(\omega) = \delta, \qquad p_n(\omega) = \sum_{i=1}^{\infty} \alpha_i^n - \sum_{i=1}^{\infty} (-\beta_i)^n \quad \text{for} \quad n \geqslant 2.$$

In the algebra Λ, there is a distinguished involutive automorphism, which is defined on the generators p_n as $p_n \mapsto (-1)^{n-1} p_n$. Under this automorphism, $S_\nu \mapsto S_{\nu'}$. The above formula shows that in the realization $\Lambda \subset \mathrm{Fun}(\widetilde{\Omega})$, this automorphism amounts to transposition $\alpha \leftrightarrow \beta$.

Definition 6.2. *Let us say that $(z, z') \in \mathbb{C}^2$ is admissible if both z and z' are nonzero and*

$$\prod_{\square \in \nu} (z + c(\square))(z' + c(\square)) \geqslant 0$$

for any Young diagram ν.

The family of admissible couples (z, z') splits into the union of the three subfamilies:

• The *principal series*: both z and z' are nonreal, $z' = \bar{z}$.

• The *complementary series*: both z and z' are real and are contained inside an open interval $(m, m+1)$ with $m \in \mathbb{Z}$.

• The *degenerate series*: $(z, z') = \pm(N, N+b-1)$ or $(z', z) = \pm(N, N+b-1)$, where $N = 1, 2, \ldots$ and $b > 0$.

Theorem 6.3. *Assume (z, z') is admissible. There exists a probability measure $W = W_{z,z'}$ on the Thoma cone $\widetilde{\Omega}$ such that all functions from Λ are W-integrable and the formal moment functional ψ with parameters (z, z') coincides with expectation with respect to W:*

$$\psi(f) = \int_{\widetilde{\Omega}} f(\omega) W(d\omega) \quad \text{for all} \quad f \in \Lambda.$$

These properties determine W uniquely. Moreover, the functions from Λ are dense in the Hilbert space $L^2(\widetilde{\Omega}, W)$.

By virtue of Theorem 5.3 this implies that the measure W with admissible parameters (z, z') is the orthogonality measure for the Laguerre symmetric functions.

In the case when $z = N = 1, 2, \ldots$ and $z' = N + b - 1$ with $b > 0$ we recover the Laguerre measures w_N on the N-dimensional cone

$$\widetilde{\Omega}_N := \{(x_1 \geqslant \ldots x_N \geqslant 0)\}.$$

Here we embed $\widetilde{\Omega}_N$ into the Thoma cone by setting $\alpha_i = x_i$ for $i = 1, \ldots, N$ and $\delta = \sum x_i$ (so that all remaining α- and β-coordinates are equal to zero).

An immediate consequence of the theorem is

Corollary 6.4. *Assume that the couple (z, z') is admissible and nondegenerate, i.e., belongs to the principal or complementary series. Then the Laguerre symmetric functions, viewed as functions on the Thoma cone, form an orthogonal basis in the Hilbert space $L^2(\widetilde{\Omega}, W_{z,z'})$.*

7. PROPERTIES OF THE ORTHOGONALITY MEASURE

Here are some properties of the measures $W_{z,z'}$ with admissible parameters (z, z'):

• $W_{z,z'}$ are obtained from the so-called z-measures on the Thoma simplex by a simple integral transform along the rays of the Thoma cone. See [1, Sec. 5].

• $W_{z,z'}$ does not change under transposition $z \leftrightarrow z'$. Within this symmetry relation, the measures corresponding to different couples of parameters are pairwise disjoint: this follows from [17].

• The involutive map $\alpha \leftrightarrow \beta$ of the Thoma cone transforms $W_{z,z'}$ to $W_{-z,-z'}$.

• $W_{z,z'}$ is supported by the subset of the Thoma cone formed by those triples $\omega = (\alpha, \beta, \delta)$ for which δ exactly equals $\sum(\alpha_i + \beta_i)$, i.e. $\gamma = 0$. This subset is Borel and everywhere dense in $\widetilde{\Omega}$. (Note that γ is not a continuous function in ω, it is only lower semicontinuous.)

Theorem 7.1. *If (z, z') is in the principal or complementary series, then the topological support of $W_{z,z'}$ is the whole space $\widetilde{\Omega}$.*

(The topological support of a measure is the smallest closed subset supporting the measure.)

The theorem implies that a nonzero continuous function on $\widetilde{\Omega}$, which is square integrable with respect to $W_{z,z'}$, is a nonzero vector in the Hilbert space $L^2(\widetilde{\Omega}, W_{z,z'})$. In particular, the natural map assigning to a bounded continuous function a vector in this Hilbert space is an embedding. This assertion fails in the case when (z, z') belongs to the degenerate series.

8. The Laguerre diffusion process on the Thoma cone

In this section, (z, z') is a fixed couple of parameters from the principal or complementary series.

Recall that we may regard Λ as a dense subspace in the Hilbert space $H := L^2(\widetilde{\Omega}, W_{z,z'})$ and then $\{\mathfrak{L}_\nu\}$ becomes an orthogonal basis in H. The Laguerre operator $\mathfrak{D} : \Lambda \to \Lambda$ is diagonalized in this basis,

$$\mathfrak{D}\mathfrak{L}_\nu = -|\nu|\mathfrak{L}_\nu,$$

so that the eigenvalues of $\mathfrak{D}$ are $0, -1, -2, \ldots$, where 0 has multiplicity 1 and corresponds to the basis vector $\mathfrak{L}_\varnothing = 1$. It follows that $\mathfrak{D}$ generates a strongly continuous semigroup $P(t)$ of contractive selfadjoint operators in H,

$$P(t)\mathfrak{L}_\nu = e^{-|\nu|t}\mathfrak{L}_\nu, \quad t \geqslant 0.$$

Theorem 8.1. *The semigroup $P(t)$ is a conservative Markov L^2-semigroup*

By definition, this means that $P(t)1 = 1$ and $P(t)$ preserves the cone of nonnegative functions in H. The first claim is obvious, the second one is nontrivial; its proof relies on the approximation by some jump Markov processes, see Sec. 14.

The next claim says that $P(t)$ is actually a Feller semigroup (in one of the versions of this property). Let $C(\widetilde{\Omega}) \subset \text{Fun}(\widetilde{\Omega})$ be the space of bounded continuous functions on $\widetilde{\Omega}$ with the supremum norm and $C_0(\widetilde{\Omega}) \subset C(\widetilde{\Omega})$ be its closed subspace formed by the functions vanishing at infinity. Both $C(\widetilde{\Omega})$ and $C_0(\widetilde{\Omega})$ are Banach spaces contained in H, but $C_0(\widetilde{\Omega})$ is separable while $C(\widetilde{\Omega})$ is not.

Theorem 8.2. *The semigroup* $P(t)$ *preserves* $C_0(\widetilde{\Omega})$ *and induces a strongly continuous contractive semigroup in this Banach space.*

One of the ingredients of the proof is the separation of variables construction described in the next section.

The theorem implies that $P(t)$ gives rise to a Markov process $X(t)$ on $\widetilde{\Omega}$ with càdlàg sample trajectories. Actually, more can be said:

Theorem 8.3. *With probability one, the trajectories of* $X(t)$ *are continuous.*

By the very construction of the Markov process $X(t)$, the probability measure $W_{z,z'}$ is its invariant and symmetrizing measure.

9. Separation of variables

Extend the algebra Λ by allowing division by e_1 (that is, localize over the multiplicative semigroup generated by e_1) and denote the resulting algebra by Λ^{ext}:

$$\Lambda^{\text{ext}} := \mathbb{C}[e_1, e_1^{-1}; e_2, e_3, \dots] \supset \Lambda.$$

Since the coefficients of the Laguerre differential operator $\mathfrak{D} : \Lambda \to \Lambda$ (see Definition 4.2) are polynomials in variables e_n, $\mathfrak{D}$ can be extended to an operator $\mathfrak{D}^{\text{ext}} : \Lambda^{\text{ext}} \to \Lambda^{\text{ext}}$.

Set

$$r := e_1, \quad e_n^\circ := e_n e_1^{-n}, \qquad n \geqslant 2,$$

and

$$e_0^\circ = e_1^\circ = 1.$$

The algebra Λ^{ext} can be identified with $\mathbb{C}[r, r^{-1}; e_2^\circ, e_3^\circ, \dots]$.

Theorem 9.1. *Under this identification, the operator* $\mathfrak{D}^{\text{ext}} : \Lambda^{\text{ext}} \to \Lambda^{\text{ext}}$ *takes the form*

$$\mathfrak{D}^{\text{ext}} = \left(r\frac{\partial^2}{\partial r^2} + (c - r)\frac{\partial}{\partial r} \right) + \frac{1}{r}\mathfrak{D}^\circ,$$

where

$$c := zz'$$

and

$$\mathfrak{D}^\circ = \sum_{m,n \geqslant 2} A_{mn}^\circ \frac{\partial^2}{\partial e_m^\circ \partial e_n^\circ} + \sum_{n \geqslant 1} B_n^\circ \frac{\partial}{\partial e_n^\circ}$$

with

$$A^\circ_{mn} = -mn e^\circ_m e^\circ_n + \sum_{k=0}^{\min(m,n)-1} (m+n-1-2k) e^\circ_{m+n-1-k} e^\circ_k$$

and

$$B^\circ_n = -n(n-1+c)e^\circ_n + (z-n+1)(z'-n+1)e^\circ_{n-1}.$$

This result shows (at least on algebraic level) that the process $X(t)$ is the skew product of the one-dimensional Laguerre diffusion with parameter $c = zz' > 0$ and a Markov process on the Thoma simplex Ω, which is generated by the operator $\mathfrak{D}^\circ$. This can be compared to the splitting of the multidimensional Brownian motion into the skew product of a one-dimensional diffusion (a Bessel process) and the spherical Brownian motion, see, e.g., [10].

In more detail: The algebra Λ^{ext} is realized, in a natural way, as an algebra of functions on $\widetilde{\Omega} \setminus \{0\}$. In this realization, elements of the subalgebra Λ° turn into homogeneous functions of degree 0 with respect to homotheties of the cone $\widetilde{\Omega}$. Since the Thoma simplex Ω is a base of the Thoma cone, we may regard Λ° as an algebra of functions on Ω. Note that Λ° is dense in the Banach space $C(\Omega)$ of continuous functions on Ω. Thus, $\mathfrak{D}^\circ$ becomes a densely defined operator in $C(\Omega)$. As shown in [6], the closure of $\mathfrak{D}^\circ$ generates a diffusion process $X^\circ(t)$ in the Thoma simplex.[1] Continuing the analogy with the multidimensional Brownian motion one can say that in this picture, $X^\circ(t)$ is a counterpart of the spherical Brownian motion and the variables e°_n play the role of spherical coordinates.

10. CORRELATION FUNCTIONS

Set $\mathbb{R}^* = \mathbb{R} \setminus \{0\}$ and let $\mathrm{Conf}(\mathbb{R}^*)$ be the space of locally finite point configurations on $\mathbb{R}^*$. Following [1], we define a projection $\widetilde{\Omega} \to \mathrm{Conf}(\mathbb{R}^*)$ by setting

$$\omega = (\alpha, \beta, \delta) \mapsto \{\alpha_i : \alpha_i \neq 0\} \sqcup \{-\beta_i : \beta_i \neq 0\}.$$

Two points of $\widetilde{\Omega}$ are mapped to one and the same configuration if and only if they differ only by the value of the coordinate δ. Consequently, the restriction of the projection on the subset

$$\widetilde{\Omega}' := \{\omega : \delta = \sum(\alpha_i + \beta_i)\}$$

---

[1] In [6], the operator $\mathfrak{D}^\circ$ is written down in a different coordinate system.

is injective.

The *equilibrium version* $X^{\text{stat}}(t)$ of the process $X(t)$ is obtained by taking the stationary distribution W as the initial one. The process $X^{\text{stat}}(t)$ is stationary in time; moreover, since W is a symmetrizing measure, one may extend the time parameter t from the half-line $[0, +\infty)$ to the whole real line $\mathbb{R}$. We know that the stationary distribution W is concentrated on the subset $\widetilde{\Omega}'$. It follows that the finite-dimensional distributions of the equilibrium process may be viewed as probability measures on the spaces $(\widetilde{\Omega}')^k$, $k = 1, 2, \ldots$. Next, applying the above projection, we may interpret every k-dimensional distribution ($k = 1, 2, \ldots$) as a probability measure on

$$(\text{Conf}(\mathbb{R}^*))^k = \text{Conf}(\underbrace{\mathbb{R}^* \sqcup \cdots \sqcup \mathbb{R}^*}_{k \text{ times}}),$$

which is again a space of configurations. Such measures can be described in terms of the correlation functions. In other words, these are the *dynamical* or *space-time* correlation functions of a time-dependent point process.

Theorem 10.1. *The space-time correlation functions of the equilibrium process $X^{\text{stat}}(t)$ are determinantal. That is, they are given by minors of a kernel $K(s, x; t, y)$, where s and t are time variables and $x, y \in \mathbb{R}^*$.*

The correlation kernel $K(s, x; t, y)$ is called the *extended Whittaker kernel*. It was derived in [4, Theorem B] as a scaling limit of the correlation kernels of some equilibrium Markov jump processes on partitions. Explicit expressions for $K(s, x; t, y)$ are contained in that paper.

Note, however, that the paper [4] left open the question whether the kernel $K(s, x; t, y)$ determines a *Markov* process (a subtlety here is that, in principle, it may happen that the Markov property is destroyed in a limit transition). The results of the present section settle this question in the positive.

11. The Meixner symmetric functions

Let $\mathbb{Z}_+ \subset \mathbb{Z}$ denote the set of nonnegative integers. Fix two parameters b and ξ, where $b > 0$ as before and $0 < \xi < 1$.

The *classical Meixner polynomials* $M_n(x)$ are defined as the orthogonal polynomials corresponding to the following discrete probability measure supported by $\mathbb{Z}_+$:

$$w^{\mathsf{M}} = (1 - \xi)^b \sum_{x \in \mathbb{Z}_+} \frac{(b)_x}{x!} \xi^x \delta_x,$$

where δ_x denotes the Dirac measure at x (see, e.g., [18]). The measure w^{M} is is known under the name of the *negative binomial distribution*. We use the standardization in which the M_n's are monic polynomials: $M_n(x) = x^n + \ldots$.

Consider the following second order difference operator on $\mathbb{Z}_+$:

$$D^{\mathsf{M}}f(x) = \frac{\xi(b+x)}{1-\xi}f(x+1) + \frac{x}{1-\xi}f(x-1)$$
$$- \frac{\xi(b+x)+x}{1-\xi}f(x)$$

(the factor $1-\xi$ in the denominator is introduced to simplify some formulas below). D^{M} is formally symmetric with respect to the weight function and annihilates the constants. The Meixner polynomials are eigenfunctions of this operator,

$$D^{\mathsf{M}}M_n = -nM_n.$$

Moreover, they can be characterized as the only polynomial eigenfunctions of D^{M}.

The *N-variate symmetric Meixner polynomials*

$$M_\nu(x_1, \ldots, x_N) = M_{\nu|N,b,\xi}(x_1, \ldots, x_N)$$

are introduced following the recipe (3.1).

Set

$$\mathbb{Z}^N_{+,\mathrm{ord}} = \{(x_1, \ldots, x_N) \in \mathbb{Z}^N_+ : x_1 > \cdots > x_N\}$$

and regard polynomials from Λ_N as functions on $\mathbb{Z}^N_{+\,\mathrm{ord}}$. Then the Meixner polynomials M_ν become orthogonal polynomials with respect to the atomic measure w^{M}_N on $\mathbb{Z}^N_{+,\mathrm{ord}}$ defined according to (3.2).

The polynomials M_ν are eigenfunctions of an operator $D^{\mathsf{M}}_N : \Lambda_N \to \Lambda_N$, which is defined according to (3.3):

$$D^{\mathsf{M}}_N M_\nu = -|\nu|M_\nu.$$

This operator can be realized as a difference operator on $\mathbb{Z}^N_{+\,\mathrm{ord}}$ acting on a function f by

$$D^{\mathsf{M}}_N f(x) = \sum_{i=1}^N A_i(x)f(x+\varepsilon_i) + \sum_{i=1}^N B_i(x)f(x-\varepsilon_i) - C(x)f(x)$$
$$= \sum_{i=1}^N A_i(x)(f(x+\varepsilon_i) - f(x)) + \sum_{i=1}^N B_i(x)(f(x-\varepsilon_i) - f(x)).$$

Here $x = (x_1, \ldots, x_N)$ ranges over $\mathbb{Z}_{+,\mathrm{ord}}^N$, $\{\varepsilon_1, \ldots, \varepsilon_N\}$ is the canonical basis in $\mathbb{R}^N$, and the coefficients are given by

$$A_i(x) = \frac{\xi(b + x_i)}{1 - \xi} \prod_{j:\, j \neq i} \frac{x_i - x_j + 1}{x_i - x_j},$$

$$B_i(x) = \frac{x_i}{1 - \xi} \prod_{j:\, j \neq i} \frac{x_i - x_j - 1}{x_i - x_j}$$

$$C(x) = \frac{\xi bN + (1 + \xi) \sum\limits_{i=1}^{N} x_i}{1 - \xi} - \frac{N(N-1)}{2}.$$

We will need a modified version of the truncation map $\pi_N : \Lambda \to \Lambda_N$; this is an algebra morphism $\pi_N' : \Lambda \to \Lambda_N$, which is defined on the generators p_k (Newton power sums) by

$$\pi_N'(p_k)(x_1, \ldots, x_N) = \sum_{i=1}^{N} [(x_i - N + \tfrac{1}{2})^k - (-i + \tfrac{1}{2})^k].$$

Since the right-hand side is a symmetric polynomial, the definition makes sense. It can be better understood in terms of the realization $\Lambda \subset \mathrm{Fun}(\mathbb{Y})$, see Remark 12.2 below.

Theorem 11.1 (cf. Theorem 4.3). *Let z, z', and ξ be complex parameters. For an arbitrary partition $\nu = (\nu_1, \nu_2, \ldots)$, there exists a unique element $\mathfrak{M}_\nu \in \Lambda$, which depends polynomially on z, z' and rationally on ξ, and such that for any natural number $N \geqslant \ell(\nu)$ and any $b > 0$ and $\xi \in (0, 1)$, one has*

$$\pi_N'\left(\mathfrak{M}_\nu\big|_{z=N,\, z'=N+b-1}\right) = M_{\nu|N,b,\xi}.$$

Definition 11.2. *We call the elements $\mathfrak{M}_\nu$ the Meixner symmetric functions with parameters z, z', and ξ.*

Next, we need the *Frobenius-Schur symmetric functions*. These are some inhomogeneous elements $FS_\nu \in \Lambda$ indexed by arbitrary partitions ν and such that

$$FS_\nu = S_\nu + \text{lower degree terms}.$$

For their definition, properties, and explicit expressions, see [25]. In particular, one disposes of a simple explicit expression of the Frobenius–Schur functions through the Schur functions.

Theorem 11.3 (cf. Theorem 4.5). *The Meixner symmetric function $\mathfrak{M}_\nu$ with parameters (z, z', ξ) are given by the following expansion in the Frobenius–Schur symmetric functions:*

$$\mathfrak{M}_\nu = \sum_{\mu \subseteq \nu} C'(\nu, \mu; z, z', \xi) FS_\mu,$$

where

$$C'(\nu, \mu; z, z', \xi) = (-1)^{|\nu| - |\mu|} \left(\frac{\xi}{1 - \xi} \right)^{|\nu| - |\mu|}$$
$$\times \frac{\dim \nu / \mu}{(|\nu| - |\mu|)!} \prod_{\square \in \nu/\mu} (z + c(\square))(z' + c(\square))).$$

Definition 11.4 (cf. Theorem 4.6). *The Meixner operator $\mathfrak{D}^{\mathsf{M}} : \Lambda \to \Lambda$ with complex parameters (z, z', ξ) is defined in the basis $\{FS_\nu\}$ of the Frobenius-Schur functions by*

$$\mathfrak{D}^{\mathsf{M}} FS_\nu = -|\nu| FS_\nu + \frac{\xi}{1 - \xi} \sum_{\square \in \nu^-} (z + c(\square))(z' + c(\square)) FS_{\nu \setminus \square}.$$

From this definition one sees that $\mathfrak{D}^{\mathsf{M}}$ preserves the filtration of Λ and depends polynomially on (z, z') and rationally on ξ, with the only possible pole at $\xi = 1$. The operator $\mathfrak{D}^{\mathsf{M}}$ is uniquely determined by these properties together with the following one: If $z = N = 1, 2, \ldots$, $z' = N + b - 1$ with $b > 0$, and $\xi \in (0, 1)$, then $\mathfrak{D}^{\mathsf{M}}$ preserves the kernel of the map $\pi'_N : \Lambda \to \Lambda_N$ and the induced operator in Λ_N coincides with the N-variate Meixner operator D_N^{M} with parameter b.

Theorem 11.5 (cf. Theorem 4.7). *The Meixner symmetric functions are eigenvectors of the Meixner operator $\mathfrak{D}$ with the same values of parameters (z, z', ξ). More precisely,*

$$\mathfrak{D}^{\mathsf{M}} \mathfrak{M}_\nu = -|\nu| \mathfrak{M}_\nu.$$

Definition 11.6 (cf. Definition 5.1). *For any fixed $(z, z', \xi) \in \mathbb{C}^3$ with $\xi \neq 1$, introduce the formal moment functional $\psi^{\mathsf{M}} : \Lambda \to \mathbb{C}$ by setting*

$$\psi^{\mathsf{M}}(1) = 1, \quad \psi(\mathfrak{D}^{\mathsf{M}} f) = 0 \quad \text{for any } f \in \Lambda.$$

where the Meixner operator $\mathfrak{D}^{\mathsf{M}}$ is taken with the same values of the parameters as ψ^{M}.

Theorem 11.7 (cf. Theorem 5.2). *For any Young diagram ν,*

$$\psi^{\mathsf{M}}(FS_\nu) = \left(\frac{\xi}{1-\xi}\right)^{|\nu|} \prod_{\square \in \nu} (z + c(\square))(z' + c(\square)) \cdot \left(\frac{\dim \nu}{|\nu|!}\right)^2.$$

This formula provides an alternative (equivalent) way of introducing the moment functional.

Theorem 11.8 (cf. Theorem 5.3). *For any two Young diagrams ν and μ,*

$$\psi^{\mathsf{M}}(\mathfrak{M}_\nu \mathfrak{M}_\mu) = \delta_{\nu\mu} \frac{\xi^{|\nu|}}{(1-\xi)^{2|\nu|}} \prod_{\square \in \nu} (z + c(\square))(z' + c(\square)),$$

where $\delta_{\nu\mu}$ is Kronecker's delta.

This result shows that the Meixner symmetric functions are pairwise orthogonal with respect to the inner product in the space Λ defined by

$$(f, g) = (f, g)_{z,z',\xi} := \psi^{\mathsf{M}}(fg), \quad f, g \in \Lambda.$$

The inner product is nondegenerate provided that z and z' are not integers.

Remark 11.9. The two characterizations of the Laguerre symmetric functions from Remark 5.4 extend, with obvious modifications, to the Meixner symmetric functions.

12. THE ORTHOGONALITY MEASURE FOR THE MEIXNER SYMMETRIC FUNCTIONS

To speak about the orthogonality measure we have first to find an appropriate realization of Λ as an algebra of functions on a space. In the context of the Laguerre symmetric functions that space was the Thoma cone $\widetilde{\Omega}$. Now the relevant space is different: it is the countable set $\mathbb{Y}$ of Young diagrams.

We will need the notion of *modified Frobenius coordinates* of a diagram $\lambda \in \mathbb{Y}$. This is a double collection $(a; b) = (a_1, \ldots, a_d; b_1, \ldots, b_d)$ of half-integers, where d stands for the number of diagonal boxes in λ, $a_i = \lambda_i - i + \frac{1}{2}$ equals the number of boxes in the ith row of λ plus one-half, and b_i is the similar quantity for transposed diagram λ'. For instance, if $\lambda = (3, 2, 2)$ then $(a; b) = (2\frac{1}{2}, \frac{1}{2}; 2\frac{1}{2}, 1\frac{1}{2})$.

Definition 12.1. *Let $\mathbb{A}$ be the unital algebra of functions on $\mathbb{Y}$ generated by the functions of the form*

$$p_k(\lambda) = p_k(a; -b) := \sum_{i=1}^{d} [a_i^k - (-b_i)^k], \quad k = 1, 2, \ldots,$$

where $(a; b)$ is the collection of the modified Frobenius coordinates of a diagram $\lambda \in \mathbb{Y}$. Elements of $\mathbb{A}$ are called polynomial functions on $\mathbb{Y}$ (see [15]).

Consider the generators $p_1, p_2, \ldots$ of Λ (the Newton power sums). The assignment $p_k \mapsto p_k(\cdot)$ extends by multiplicativity to an isomorphism $\Lambda \to \mathbb{A}$ and hence defines an embedding of Λ into the algebra $\mathrm{Fun}(\mathbb{Y})$ of functions on the set $\mathbb{Y}$. This is the desired realization.

Remark 12.2. Now we can explain the origin of the map $\pi'_N : \Lambda \to \Lambda_N$ introduced in Sec. 11. Similarly to the realization $\Lambda \subset \mathrm{Fun}(\mathbb{Y})$, realize Λ_N as an algebra of functions on $\mathbb{Y}_N \subset \mathbb{Y}$, the subset of Young diagrams with at most N nonzero rows, by letting the arguments x_i of N-variate symmetric polynomials to be equal to $\lambda_i + N - i$, where λ ranges over $\mathbb{Y}_N$, $i = 1 \ldots, N$. Then π'_N is implemented by the natural map $\mathrm{Fun}(\mathbb{Y}) \to \mathrm{Fun}(\mathbb{Y}_N)$ assigning to a function on $\mathbb{Y}$ its restriction to $\mathbb{Y}_N$.

For a diagram $\lambda \in \mathbb{Y}$, denote by λ^+ the set of the boxes that can be appended to λ. As before, λ^- is the set of the boxes that can be removed from λ. By $\dim \lambda$ we denote the number of standard tableaux of the shape λ.

Theorem 12.3. *Under the realization $\Lambda = \mathbb{A} \subset \mathrm{Fun}(\mathbb{Y})$, the Meixner operator $\mathfrak{D}^{\mathsf{M}} : \Lambda \to \Lambda$ with parameters (z, z', ξ) is implemented by the following operator in $\mathrm{Fun}(\mathbb{Y})$, which will be denoted by the same symbol,*

$$\mathfrak{D}^{\mathsf{M}} f(\lambda) = \sum_{\square \in \lambda^+} A(\lambda, \square) f(\lambda \cup \square) + \sum_{\square \in \lambda^-} B(\lambda, \square) f(\lambda \setminus \square) - C(\lambda) f(\lambda)$$

$$= \sum_{\square \in \lambda^+} A(\lambda, \square)[f(\lambda \cup \square) - f(\lambda)] + \sum_{\square \in \lambda^-} B(\lambda, \square)[f(\lambda \setminus \square) - f(\lambda)],$$

where

$$A(\lambda, \square) = \frac{\xi}{1-\xi}(z+c(\square))(z'+c(\square))\frac{\dim(\lambda \cup \square)}{(|\lambda|+1)\dim\lambda}, \quad \square \in \lambda^+,$$

$$B(\lambda, \square) = \frac{1}{1-\xi}\sum_{\square \in \lambda^-}\frac{|\lambda|\dim(\lambda \setminus \square)}{\dim\lambda}, \quad \square \in \lambda^-,$$

$$C(\lambda) = \frac{1}{1-\xi}((1+\xi)|\lambda| + \xi z z').$$

Definition 12.4. *Let z, z', and ξ be complex parameters, $|\xi| < 1$. The associated complex measure on $\mathbb{Y}$, called the (mixed) z-measure, is defined by*

$$M_{z,z',\xi}(\lambda) = (1-\xi)^{zz'}\prod_{\square \in \Lambda}(z+c(\square))(z'+c(\square))\cdot \xi^{|\lambda|}\left(\frac{\dim\lambda}{|\lambda|!}\right)^2, \quad \lambda \in \mathbb{Y}.$$

One can prove that
$$\sum_{\lambda \in \mathbb{Y}}M_{z,z',\xi}(\lambda) = 1.$$

These measures were introduced in [1], and some closely related measures on the finite sets of Young diagrams with a fixed number of boxes appeared earlier in [16]. The measures $M_{z,z',\xi}$ are a special case of Okounkov's Schur measures [24].

The next theorem relates the measures $M_{z,z',\xi}$ to the formal moment functional ψ^{M} with the same parameters (see Definition 11.6).

Theorem 12.5 (cf. Theorem 6.3). *Let (z, z') be admissible and $0 < \xi < 1$. Then the mixed z-measure $M_{z,z',\xi}$ is a probability measure, all functions from $\Lambda \subset \mathrm{Fun}(\mathbb{Y})$ are integrable with respect to $M_{z,z',\xi}$, and*

$$\psi^{\mathsf{M}}(f) = \sum_{\lambda \in \mathbb{Y}}f(\lambda)M_{z,z',\xi}(\lambda), \quad \forall f \in \Lambda \subset \mathrm{Fun}(\mathbb{Y}).$$

Moreover, Λ is dense in the weight Hilbert space $\ell^2(\mathbb{Y}, M_{z,z',\xi})$.

This implies that (under the above assumptions on the parameters) $M_{z,z',\xi}$ serves as the orthogonality measure for the Meixner symmetric functions.

Remark 12.6. The classical univariate Meixner polynomials are *autodual* in the sense that, in an appropriate standardization, they are symmetric with respect to transposition of the index and the argument, which both range over $\mathbb{Z}_+$. The similar autoduality property holds for the Meixner symmetric functions viewed as functions on $\mathbb{Y}$ under the realization $\Lambda = \mathbb{A} \subset \mathrm{Fun}(\mathbb{Y})$.

Indeed, under this realization, there is a simple expression for the functions $FS_\mu(\,\cdot\,)$ on $\mathbb{Y}$:

$$FS_\mu(\lambda) = \begin{cases} \dfrac{|\lambda|!}{(|\lambda| - |\mu|)!} \dfrac{\dim \lambda/\mu}{\dim \lambda}, & \text{if } \mu \subseteq \lambda \\ 0, & \text{otherwise,} \end{cases}$$

where λ ranges over $\mathbb{Y}$; see [25]. Change the standardization of $\mathfrak{M}_\nu$ by setting

$$\mathfrak{M}_\nu = C''(\nu; z, z', \xi)\, \mathfrak{M}'_\nu,$$

where

$$C''(\nu; z, z', \xi) := \left(\frac{\xi}{1 - \xi}\right)^{|\nu|} \frac{\dim \nu}{|\nu|!} \prod_{\square \in \nu} (z + c(\square))(z' + c(\square))$$

is a normalizing factor. Then the above formula for $FS_\mu(\lambda)$ combined with Theorem 11.3 yields the following explicit expression for the function $\mathfrak{M}'_\nu(\lambda)$:

$$\begin{aligned} \mathfrak{M}'_\nu(\lambda) = \sum_{\mu \subseteq (\nu \cap \lambda)} (-1)^{|\mu|} \left(\frac{1 - \xi}{\xi}\right)^{|\mu|} \frac{|\nu|!\,|\lambda|!}{(|\nu| - |\mu|)!(|\lambda| - |\mu|)!} \\ \times \frac{\dim \nu/\mu \, \dim \lambda/\mu}{\dim \nu \, \dim \lambda} \prod_{\square \in \mu} \frac{1}{(z + c(\square))(z' + c(\square))}. \end{aligned}$$

Clearly, this expression is symmetric under $\nu \leftrightarrow \lambda$:

$$\mathfrak{M}'_\nu(\lambda) = \mathfrak{M}'_\lambda(\nu), \quad \nu, \lambda \in \mathbb{Y}.$$

13. The Meixner jump process on the set of Young diagrams

Return for a moment to the classical Meixner polynomials $M_n(x)$ and the associated difference operator D^{M} on $\mathbb{Z}_+$. Let $X_1^{\mathsf{M}}(t)$ denote the birth-death process $X_1^{\mathsf{M}}(t)$ on $\mathbb{Z}_+$ whose jump rates are the coefficients of D^{M}. That is, the rates of the jumps $x \to x+1$ and $x \to x-1$ are equal to $(1-\xi)^{-1}\xi(b+x)$ and $(1-\xi)^{-1}x$, respectively. This is a well-known instance of a birth-death process with linear jump rates. The negative binomial distribution w^{M} is the stationary distribution of $X_1^{\mathsf{M}}(t)$. The transition function of $X_1^{\mathsf{M}}(t)$ can be expressed through the Meixner polynomials according to formula (2.1), where one has to substitute $\phi_n = M_n$ and $w = w^{\mathsf{M}}$.

More generally, the coefficients A_i and B_i of the operator D_N^{M} (see Sec. 11) serve as the jump rates of a jump Markov process $X_N^{\mathsf{M}}(t)$ on the set $\mathbb{Z}_{+,\mathrm{ord}}^N$.

Even more generally, the following result holds (see [4]). Assume (z, z') is admissible and $0 < \xi < 1$. We know that then $M_{z,z',\xi}$ is a probability measure. Its support $\operatorname{supp} M_{z,z',\xi}$ is the whole set $\mathbb{Y}$ if (z, z') belongs to the principal or complementary series, or a proper subset of the form $\mathbb{Y}_N$ or $\{\lambda : \lambda' \in \mathbb{Y}_N\}$ if (z, z') belongs to the degenerate series.

Theorem 13.1. *Let (z, z') be admissible and $0 < \xi < 1$. Then there exists a jump Markov process $X_{z,z',\xi}^{\mathsf{M}}(t)$ whose state space is $\operatorname{supp} M_{z,z',\xi}$ and whose jump rates are the coefficients $A(\lambda, \square)$ and $B(\lambda, \square)$ of the Meixner operator $\mathfrak{D}^{\mathsf{M}}$. The measure $M_{z,z',\xi}$ is an invariant and symmetrizing measure for $X_{z,z',\xi}^{\mathsf{M}}(t)$.*

The fact that the Meixner operator $\mathfrak{D}^{\mathsf{M}}$ is diagonalized in the basis $\{\mathfrak{M}_\nu\}$ of Meixner symmetric functions leads to an expression for the transition function, which we state for the case of nondegenerate parameters.

Theorem 13.2. *Let (z, z') belongs to the principal or complementary series and $0 < \xi < 1$. The transition function $P(t; \lambda, \varkappa)$ of the Markov process $X_{z,z',\xi}^{\mathsf{M}}(t)$ on $\mathbb{Y}$ can be written in the form form*

$$P(t; \lambda, \varkappa) = \sum_{\nu \in \mathbb{Y}} e^{-t|\nu|} \frac{\mathfrak{M}_\nu(\lambda)\mathfrak{M}_\nu(\varkappa)}{\psi^{\mathsf{M}}(\mathfrak{M}_\nu \mathfrak{M}_\nu)} \cdot M_{z,z',\xi}(\varkappa).$$

Here $\lambda, \varkappa$ range over $\mathbb{Y}$, the $\mathfrak{M}_\nu$'s are viewed as functions on $\mathbb{Y}$ in accordance with the realization $\Lambda \subset \operatorname{Fun}(\mathbb{Y})$, and the explicit expression for $\psi^{\mathsf{M}}(\mathfrak{M}_\nu \mathfrak{M}_\nu)$ is given in Theorem 11.8.

14. APPROXIMATION MEIXNER → LAGUERRE

As well known, the Meixner polynomials $M_n(x)$ are discrete analogs of the classical Laguerre polynomials $L_n(x)$. Namely, fix parameter $b > 0$ and let $\xi \uparrow 1$. Then one has the limit relation

$$\lim_{\xi \uparrow 1} (1 - \xi)^n M_n((1 - \xi)^{-1} x) = L_n(x), \qquad (14.1)$$

where the scalar factor $(1 - \xi)^n$ is used to keep the coefficient of x^n to be equal to 1.

The limit relation (14.1) can be easily derived from the explicit expressions for the Laguerre and Meixner polynomials, see [18]. On the other hand, (14.1) can be explained by convergence of the weight functions: under the embedding

$$\mathbb{Z}_+ \to \mathbb{R}, \quad x \mapsto (1 - \xi)x, \qquad (14.2)$$

the push-forward of the negative binomial distribution w^{M} on $\mathbb{Z}_+$ with parameters (b, ξ) converges, as $\xi \uparrow 1$, to the Gamma distribution $(\Gamma(b))^{-1} x^{b-1} e^{-x} dx$ on $\mathbb{R}_+$.

One more explanation can be given in terms of the univariate Meixner and Laguerre operators. In the same scaling limit regime (14.2), as the mesh of the lattice goes to 0, the Meixner difference operator turns into the Laguerre differential operator.

We are going to formulate similar statements in the infinite-dimensional context.

Let $G : \Lambda \to \Lambda$ be the operator multiplying every homogeneous element by its degree. In accordance with this, the operator $(1 - \xi)^{-G} : \Lambda \to \Lambda$, which appears in the next theorem, acts in the mth homogeneous component of Λ as multiplication by $(1 - \xi)^{-m}$, for each $m \in \mathbb{Z}_+$.

The analog of (14.1) is

Theorem 14.1. *Let ν be an arbitrary partition, $\mathfrak{M}_\nu \in \Lambda$ be the corresponding Meixner symmetric function with parameters $(z, z', \xi) \in \mathbb{C}^3$, $\xi \neq 1$, and $\mathfrak{L}_\nu \in \Lambda$ be the Laguerre symmetric function with parameters (z, z'). One has*

$$\lim_{\xi \to 1} (1 - \xi)^{|\nu|} (1 - \xi)^{-G} \mathfrak{M}_\nu = \mathfrak{L}_\nu,$$

where convergence holds in the finite-dimensional subspace of Λ consisting of elements of degree $\leqslant |\nu|$.

This is a direct corollary of Theorems 4.5 and 11.3.

As a corollary of Theorem 14.1 combined with Theorems 4.7 and 11.5 one gets the following analog of the approximation of the univariate Laguerre differential operators by the Meixner difference operator:

Theorem 14.2. *Let* $\mathfrak{D}^{\mathsf{M}} : \Lambda \to \Lambda$ *be the Meixner operator with parameters* $(z, z', \xi) \in \mathbb{C}^3$, $\xi \neq 1$, *and* $\mathfrak{D} : \Lambda \to \Lambda$ *be the Laguerre operator with parameters* (z, z'). *One has*

$$\lim_{\xi \to 1} (1 - \xi)^{-G} \circ \mathfrak{D}^{\mathsf{M}} \circ (1 - \xi)^G = \mathfrak{D}.$$

Here we mean simple convergence on arbitrary elements $f \in \Lambda$. Note that both the pre-limit and limit operators preserve the filtration of Λ, so that application of the both operators to a given f is contained in a fixed finite-dimensional subspace of Λ.

For $\varepsilon > 0$ define the embedding

$$\iota_\varepsilon : \mathbb{Y} \to \widetilde{\Omega}, \quad \lambda = (a; b) \mapsto (\alpha, \beta, \delta),$$

where $(a; b) = (a_1, \ldots, a_d; b_1, \ldots, b_d)$ are the modified Frobenius coordinates of $\lambda \in \mathbb{Y}$, by setting

$$\alpha_i = \begin{cases} \varepsilon a_i, & i \leqslant d \\ 0, & i > d \end{cases}; \quad \beta_i = \begin{cases} \varepsilon b_i, & i \leqslant d \\ 0, & i > d \end{cases}; \quad \delta = \varepsilon |\lambda| = \sum (\alpha_i + \beta_i).$$

The image $\iota_\varepsilon(\mathbb{Y})$ is a discrete subset of $\widetilde{\Omega}$. As $\varepsilon \downarrow 0$, it becomes more and dense in $\widetilde{\Omega}$. This is the analog of the embedding (14.2) ($\varepsilon = 1 - \xi$).

The analog of the approximation of the Gamma distribution by the negative binomial distributions is

Theorem 14.3 (cf. Sec. 5 in [1])**.** *Let* (z, z') *be admissible and* $\xi \in (0, 1)$. *As* $\xi \uparrow 1$, *the pushforward of the measure* $M_{z,z',\xi}$ *under the embedding* $\iota_{1-\xi} : \mathbb{Y} \to \widetilde{\Omega}$ *weakly converges to the measure* $W_{z,z'}$.

Recall that the weak topology on measures means convergence on bounded continuous functions. Actually, more can be proved: convergence holds on any test function on $\widetilde{\Omega}$, which is continuous and grows, as $\omega = (\alpha, \beta, \delta) \to \infty$ not faster than a power of δ. In particular, as test functions one can take elements of $\Lambda \subset \mathrm{Fun}(\widetilde{\Omega})$; then, by virtue of Theorems 6.3 and 12.5, the claim of the theorem means convergence of the formal moment functionals,

$$\lim_{\xi \to 1} \psi^{\mathsf{M}} \circ (1 - \xi)^G = \psi,$$

which agrees with Theorems 5.3 and 11.8.

Finally, the approximation Meixner $\to$ Laguerre holds on the level of Markov dynamics, which is used in the proof of the very existence of the diffusion process $X(t)$.

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