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ON THE BETA FUNCTION OF THE TUBE OF THE LIGHT CONE

Abstract. We construct the beta function of the Hermitian symmetric space $\mathrm{O}(n,2)/\,\mathrm{O}(n)\times\mathrm{O}(2)$, or, equivalently, of the tube $(\operatorname{Re} z_0)^2 > (\operatorname{Re} z_1)^2 + \cdots + (\operatorname{Re} z_n)^2$ in $\mathbb{C}^{n+1}$.

1. Formulation of the result

1.1. Preliminary references. The beta function of the symmetric cones

$$\mathrm{GL}(n,\mathbb{R})/\,\mathrm{O}(n),\quad \mathrm{GL}(n,\mathbb{C})/\,\mathrm{U}(n),\quad \mathrm{GL}(n,\mathbb{H})/\,\mathrm{Sp}(n)$$

was constructed by Gindikin in [4], see also [6]. For the remaining series of classical symmetric spaces, the beta function was obtained in [7]. The subseries $\mathrm{O}(n,2)/\,\mathrm{O}(n)\times\mathrm{O}(2)$ has two beta functions; the first one is a special case of the beta function of $\mathrm{O}(p,q)/\,\mathrm{O}(p)\times\mathrm{O}(q)$. The second beta function is discussed here, it is related to the Hermitian structure of these spaces.[1]

1.2. The tube of the light cone. Consider the space $\mathbb{C}^{n+1}$ with coordinates $z_0, z_1, \dots, z_n$. By $\mathfrak{T}_n$ we denote the tube

$$(\operatorname{Re} z_0)^2 > (\operatorname{Re} z_1)^2 + (\operatorname{Re} z_2)^2 + \cdots + (\operatorname{Re} z_n)^2, \quad \operatorname{Re} z_0 > 0.$$

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[1] This note is a kind of addendum to my paper [7] on matrix beta functions; originally, it was an omitted section of that paper. Since the Plancherel formula for Berezin representations (which was my actual purpose) for $\mathrm{O}(n,2)/\,\mathrm{O}(n)\times\mathrm{O}(2)$ was known due to Berezin himself [1] (the proof was published by Unterberger and Upmeier [11]), I omitted the 6-parameter integral (1.1). However, after a discussion with Prof. G. Roos, I understood that this integral can be interesting in itself, since this series of symmetric spaces is familiar to various non-representation-theoretic people.

The space $\mathfrak{T}_n$ is homogeneous with respect to the pseudo-orthogonal group $\mathrm{O}(n+1,2)$ (apparently, this was discovered by E. Cartan [2]; for a discussion, see, for instance, [5, 10, 6]), namely,

$$\mathfrak{T}_n \simeq \mathrm{O}(n+1,2)/\,\mathrm{O}(n+1)\times \mathrm{O}(2).$$

This group acts on $\mathfrak{T}$ by quadratic fractional transformations and is generated by the maps of the following 3 types:

(a) Shifts $z \mapsto z + ia$, where $a \in \mathbb{R}^{n+1}$.

(b) Linear transformations $z \mapsto zg$, where $g \in \mathrm{SO}_0(n,1)$, i.e., g is a real matrix preserving the quadratic form $x_0^2 - x_1^2 - \cdots - x_n^2$.

(c) Quasi-inversion $z \mapsto z/(z,z)$, where $(z,z) := z_0^2 - z_1^2 - \cdots - z_n^2$.

This space has numerous names, in particular, *Cartan domain of 4th type*, *future tube*, *Lie spheres*, *Lie balls*[2].

We prefer another realization of the same space. Namely, consider $\mathbb{C}^{n+1}$ with coordinates $u_1, u_2, z_1, \dots, z_{n-1}$,

$$\begin{aligned} u_1 &= v_1 + iw_1; \\ u_2 &= v_2 + iw_2; \\ z_j &:= x_j + y_j. \end{aligned}$$

We define our tube $\mathfrak{T}_n$ by the inequalities

$$v_1 v_2 - \sum x_j^2 > 0, \quad v_1 > 0.$$

1.3. Formulation of the result. For a nonzero complex a and complex λ, μ, denote

$$a^{\{\lambda|\mu\}} := a^\lambda \overline{a}^\mu.$$

[2]The last two terms arose as a result of noncorrect successive translations German–Chinese–English–Russian–English. Initially, a *'Lie sphere'* was an oriented subsphere in $S^n = \mathbb{R}^n \cup \infty$ or a point. *Lie spheres* was the space of Lie spheres; it is the homogeneous space $\mathrm{O}(n+2)/\,\mathrm{O}(n+2)\times\mathrm{O}(2)$. *Lie sphere geometry* (see, for instance, [3]) was the geometry of this space. Hua Loo Keng extended the term 'Lie spheres' to the dual symmetric space $\mathrm{O}(n+2,2)/\,\mathrm{O}(n+2)\times\mathrm{O}(2)$. In the Russian edition of Hua's book [5], the space $\mathrm{O}(n,2)/\,\mathrm{O}(n)\times\mathrm{O}(2)$ became a *Lie sphere* ("сфера Ли"; in fact, there are no traces of the original meaning of 'Lie spheres' in Hua's book). However, the term 'sphere' for an open domain is too peculiar, and our space turned into a *'Lie ball'* in the English edition.

We also denote

$$du_1 := dv_1\, dw_1, \qquad dz_j := dx_j\, dy_j,$$
$$du := du_1\, du_2, \qquad dz := dx_1\, dy_1 \dots dx_{n-1}\, dy_{n-1}.$$

In particular, $du\, dz$ denotes the integration with respect to the Lebesgue measure on $\mathfrak{T}_n$.

Our purpose is the following formula:

$$\begin{aligned}\int\limits_{\mathfrak{T}_n} & \frac{v_1^{\lambda_1-\lambda_2}(v_1v_2-\sum x_j^2)^{\lambda_2-n-1} du\, dz}{(1+u_1)^{\{\sigma_1-\sigma_2|\tau_1-\tau_2\}}\left((1+u_1)(1+u_2)-\sum z_j^2\right)^{\{\sigma_2|\tau_2\}}} \\ &= 2^{1-\sigma_1-\tau_1+n}\pi \frac{\Gamma\left(\lambda_1-\frac{1}{2}(n+1)\right)\Gamma\left(\sigma_1+\tau_1-\lambda_1-\frac{1}{2}(n-1)\right)}{\Gamma\left(\sigma_1-\frac{1}{2}(n-1)\right)\Gamma\left(\tau_1-\frac{1}{2}(n-1)\right)} \\ &\qquad \times 2^{2-\sigma_2-\tau_2+n}\pi^n \frac{\Gamma(\lambda_2-n)\Gamma(\sigma_2+\tau_2-\lambda_2)}{\Gamma(\sigma_2)\Gamma(\tau_2)},\end{aligned} \tag{1.1}$$

where λ_1, λ_2, σ_1, σ_2, τ_1, $\tau_2 \in \mathbb{C}$.

Remark. Let us explain the meaning of the complex powers. The base numbers v_1, $v_1v_2-\sum x_j^2$ of the numerator are positive reals. Next, the point $u_1=1$, $u_2=1$, $z=0$ is contained in $\mathfrak{T}_n$. The denominator is well defined at this point. Since the domain $\mathfrak{T}_n$ is simply connected, the corresponding branches of the power functions are well defined.

1.4. The meaning of the factors. (a) The functions $v_1^{\lambda_1-\lambda_2}(v_1v_2-\sum x_j^2)^{\lambda_2-n-1}$ are precisely the eigenfunctions of the parabolic subgroup in $\mathrm{O}(n+1,2)$. Also, $(v_1v_2-\sum x_j^2)^{-n-1}$ is the density of the $\mathrm{O}(n+1,2)$-invariant measure on $\mathfrak{T}_n$.

(b) The factor $\left((1+u_1)(1+u_2)-\sum z_j^2\right)$ is the standard term that is present in formulas for the Cauchy kernel, Bargman kernel (see [5]), and (more generally) Berezin kernels on $\mathfrak{T}_n$.

1.5. Comments. Special cases. (a) If $\lambda_1=\lambda_2=0$, $\sigma_1=\tau_1=0$, $\sigma_2=\tau_2$, then we obtain one of the Hua integrals, see [5].

(b) The Plancherel formula for the Berezin kernels for the spaces $\mathrm{O}(n+1,2)/\,\mathrm{O}(n+1)\times\mathrm{O}(2)$ reduces to our integral with $\sigma_1=\tau_1=0$, $\sigma_2=\tau_2$ (see [11]). Apparently, Berezin himself (he perished in an accident in 1980) derived this formula in some another way (however, his proof is unknown; see also [11]).

(c) For $n = 3, 4, 6$ we have the following exceptional isomorphisms of homogeneous spaces:

$$\mathrm{O}(3,2)/\,\mathrm{O}(3)\times\mathrm{O}(2) = \mathrm{Sp}(6,\mathbb{R})/\,\mathrm{U}(2),$$
$$\mathrm{O}(4,2)/\,\mathrm{O}(4)\times\mathrm{O}(2) = \mathrm{U}(2,2)/\,\mathrm{U}(2)\times\mathrm{U}(2),$$
$$\mathrm{O}(6,2)/\,\mathrm{O}(6)\times\mathrm{O}(2) = \mathrm{SO}^*(8)/\,\mathrm{U}(4).$$

In these cases, our integrals coincide with the matrix beta integrals obtained in [7].

2. Calculations

2.1. A change of variables. First, we transform our integral into

$$\int\limits_{v_1>0,\ w_1\in\mathbb{R}} \frac{v_1^{\lambda_1-n-1}}{(1+u_1)^{\{\sigma_1|\tau_1\}}} \times \Biggl\{ \int\limits_{v_2-\frac{1}{v_1}\sum x_j^2>0} \frac{\Bigl(v_2-\frac{1}{v_1}\sum x_j^2\Bigr)^{\lambda_2-n-1} du_2\,dz}{\Bigl(1+u_2-\frac{1}{1+u_1}\sum z_j^2\Bigr)^{\{\sigma_2|\tau_2\}}} \Biggr\} du_1.$$

Next, we change the variable v_2 to r:

$$r := v_2 - \frac{1}{v_1}\sum x_j^2$$

(the Jacobian of this substitution is equal to 1). The inner integral now is reduced to

$$\int\limits_{\substack{r>0,\,w_1\in\mathbb{R},\\ x\in\mathbb{R}^{n-1},\,y\in\mathbb{R}^{n-1}}} \frac{r^{\lambda_2-n-1}dr\,dw_2\,dx\,dy}{\Bigl(1+r+iw_2+\frac{1}{v_1}\sum x_j^2-\frac{1}{1+u_1}\sum z_j^2\Bigr)^{\{\sigma_2|\tau_2\}}}. \qquad (2.1)$$

2.2. Another change of variables. Now we wish to decompose the expression

$$H := 1+r+iw_1+\frac{1}{v_1}\sum x_j^2-\frac{1}{1+u_1}\sum z_j^2$$

into the sum of imaginary and real parts. For this purpose, we write

$$\operatorname{Re}\frac{1}{1+u_1}\sum z_j^2 = \operatorname{Re}\frac{\sum(x_j^2-y_j^2+2ix_jy_j)}{1+v_1+iw_1}$$
$$= \sum_j \frac{(x_j^2-y_j^2)(1+v_1)+2x_jy_jw_1}{(1+v_1)^2+w_1^2}.$$

Therefore,

$$\operatorname{Re} H = 1 + r + \sum_j \begin{pmatrix} x_j & y_j \end{pmatrix} S \begin{pmatrix} x_j \\ y_j \end{pmatrix},$$

where

$$S := \begin{pmatrix} \frac{1}{v_1} - \frac{1+v_1}{(1+v_1)^2+w_1^2} & \frac{-w_1}{(1+v_1)^2+w_1^2} \\ \frac{-w_1}{(1+v_1)^2+w_1^2} & \frac{1+v_1}{(1+v_1)^2+w_1^2} \end{pmatrix}.$$

Note that

$$\det S = \frac{1}{v_1\left((1+v_1)^2+w_1^2\right)} = v_1^{-1}(1+v_1+iw_1)^{-\{1|1\}} = v_1^{-1}(1+u_1)^{-\{1|1\}}.$$

Thus $\det S > 0$, and the diagonal elements of S are also positive. Hence S is positive definite; therefore $S^{1/2}$ is well defined.

Our next change of variables is

$$\begin{pmatrix} x_j & y_j \end{pmatrix} = \begin{pmatrix} p_j & q_j \end{pmatrix} S^{-1/2}.$$

Its Jacobian is equal to

$$v_1^{(n-1)/2}(1+u_1)^{(n-1)/2\,\cdot\,\{1|1\}}.$$

Finally, the "inner integral" (2.1) takes the form

$$v_1^{(n-1)/2}(1+u_1)^{(n-1)/2\,\cdot\,\{1|1\}}$$
$$\times \int\limits_{\substack{r>0,\, w_2\in\mathbb{R}, \\ p\in\mathbb{R}^{n-1},\, q\in\mathbb{R}^{n-1}}} \frac{r^{\lambda_2-n-1}\,dr\,dw_2\,dp\,dq}{\left(1+r+\sum(p_j^2+q_j^2)+iw_2+iQ(p,q,v_1,w_1)\right)^{\{\sigma_2|\tau_2\}}},$$

where $Q(p,q,v_1,w_1)$ is a real expression (its explicit form is unessential for us).

2.3. Separation of variables. Now we change the variable w_2 to

$$h := w_2 + Q(\cdot).$$

The Jacobian is equal to 1, and we reduce our initial integral to the product

$$I \cdot J := \int\limits_{v_1>0, w_1\in\mathbb{R}} \frac{v_1^{\lambda_1-n/2-3/2} dv_1\, dw_1}{(1+v_1+iw_1)^{\{\sigma_1-(n-1)/2|\tau_1-(n-1)/2\}}}$$

$$\times \int\limits_{r>0,\, h\in\mathbb{R},\, p\in\mathbb{R}^{n-1},\, q\in\mathbb{R}^{n-1}} \frac{r^{\lambda_2-n-1} dr\, dh\, dp\, dq}{\Bigl(1+r+\sum(p_j^2+q_j^2)+ih\Bigr)^{\{\sigma_2|\tau_2\}}}, \qquad (2.2)$$

where I denotes the first integral factor and J is the second one.

2.4. An auxiliary integral. First, we derive the identity

$$\int\limits_{x>0, y\in\mathbb{R}} \frac{x^{\alpha-1} dx\, dy}{(1+x+iy)^{\{\beta|\gamma\}}} = 2^{2-\beta-\gamma}\pi \frac{\Gamma(\alpha)\Gamma(\beta+\gamma-\alpha-1)}{\Gamma(\beta)\Gamma(\gamma)}. \qquad (2.3)$$

We represent the left-hand side as

$$\int\limits_0^\infty dv \cdot v^{\alpha-1} \int\limits_{-\infty}^\infty \frac{dw}{(1+v+iw)^{\{\beta|\gamma\}}}.$$

The inner integral is the Cauchy beta integral (see [9, 2.2.6.31]), and we obtain

$$2\pi \cdot 2^{1-\beta-\gamma} \frac{\Gamma(\beta+\gamma-1)}{\Gamma(\beta)\Gamma(\gamma)} \int\limits_0^\infty \frac{v^{\alpha-1} dv}{(1+v)^{\beta+\gamma-1}}.$$

The last integral is a rephrasing of the definition of the beta function (see [9, 2.2.4.29]). We obtain

$$2\pi \cdot 2^{1-\beta-\gamma} \frac{\Gamma(\beta+\gamma-1)}{\Gamma(\beta)\Gamma(\gamma)} \cdot \frac{\Gamma(\alpha)\Gamma(\beta+\gamma-\alpha-1)}{\Gamma(\beta+\gamma-1)}.$$

2.5. The first factor. By (2.3),

$$I := 2^{1-\sigma_1-\tau_1+n}\pi \frac{\Gamma\left(\lambda_1 - \frac{1}{2}(n+1)\right)\Gamma\left(\sigma_1+\tau_1-\lambda_1-\frac{1}{2}(n-1)\right)}{\Gamma\left(\sigma_1-\frac{1}{2}(n-1)\right)\Gamma\left(\tau_1-\frac{1}{2}(n-1)\right)}. \quad (2.4)$$

2.6. The second factor. Now we evaluate the factor J in (2.2). First, we pass to the spherical coordinates

$$R^2 = \sum (p_j^2 + q_j^2)$$

in $\mathbb{R}^{2n-2}$. We obtain (see [9, 3.3.2.1])

$$\frac{2\pi^{n-1}}{\Gamma(n-1)} \int\limits_{r>0,\, R>0,\, h\in\mathbb{R}} \frac{r^{\lambda_2-n-1}R^{2n-3}dr\, dR\, dh}{(1+r+R^2+ih)^{\{\sigma_2|\tau_2\}}}.$$

Next, we substitute $\rho := R^2$:

$$J = \frac{\pi^{n-1}}{\Gamma(n-1)} \int\limits_{r>0,\, R>0,\, h\in\mathbb{R}} \frac{r^{\lambda_2-n-1}\rho^{n-2}\, dr\, d\rho\, dh}{(1+r+\rho+ih)^{\{\sigma_2|\tau_2\}}}.$$

Next, we pass from the variables (r,ρ) to

$$(x, r) := (r+\rho, r),$$

i.e.,

$$J = \frac{\pi^{n-1}}{\Gamma(n-1)} \int\limits_{x>0,\, 0<r<x,\, h\in\mathbb{R}} \frac{r^{\lambda_2-n-1}(x-r)^{n-2}\, dr\, dx\, dh}{(1+x+ih)^{\{\sigma_2|\tau_2\}}}.$$

Now we integrate in r using the standard definition of the beta function:

$$J = \frac{\pi^{n-1}}{\Gamma(n-1)} \mathrm{B}\left(\lambda - n, n-1\right) \int\limits_{x>0,\, h\in\mathbb{R}} \frac{x^{\lambda_2-2}\, dx\, dh}{(1+x+ih)^{\{\sigma_2|\tau_2\}}}.$$

The last integral is of the form (2.3). Finally,

$$J = 2^{2-\sigma_2-\tau_2-n}\pi^n \frac{\Gamma(\lambda_2-n)\Gamma(\sigma_2+\tau_2-\lambda_2)}{\Gamma(\sigma_2)\Gamma(\tau_2)}.$$

References

1. F. A. Berezin, *The connection between covariant and contravariant symbols of operators on classical complex symmetric spaces.* — Dokl. Akad. Nauk SSSR **241** (1978), no. 1, 15–17.
2. E. Cartan, *Sur les domaines bornés homogènes de l'espace des n variables complexes.* — Abhandl. Hamburg **11** (1935), 116–162.
3. Th. E. Cecil, *Lie Sphere Geometry. With Applications to Submanifolds.* Springer-Verlag, New York, 1992.
4. S. G. Gindikin, *Analysis in homogeneous domains.*— Uspekhi Mat. Nauk **19** (1964), no. 4 (118), 3–92.
5. L. K. Hua, *Harmonic Analysis of Functions of Several Complex Variables in the Classical Domains.* Science Press, Peking, 1958.
6. J. Faraut, A. Koranyi, *Analysis on Symmetric Cones.* Clarendon Press, Oxford, 1994.
7. Yu. A. Neretin, *Matrix analogues of the beta function and Plancherel's formula for Berezin kernel representations.* — Sb. Math. **191** (2000), no. 5, 683–715.
8. Yu. A. Neretin, *Plancherel formula for Berezin deformation of* L^2 *on Riemannian symmetric space.* — J. Funct. Anal. **189** (2002), no. 2, 336–408.
9. A. P. Prudnikov, Yu. A. Brychkov, O. I. Marichev, *Integrals and Series.* Vol. 1, *Elementary Functions.* Gordon & Breach Science Publishers, New York, 1986.
10. I. I. Pjateckiĭ-Šapiro, *Geometry of Classical Domains and Theory of Automorphic Functions.* Gosudarstv. Izdat. Fiz.-Mat. Lit., Moscow, 1961.
11. A. Unterberger, H. Upmeier, *The Berezin transform and invariant differential operators.* — Comm. Math. Phys. **164** (1994), 563–597.

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