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BOUNDS FOR THE CUBIC WEYL SUM

Abstract. Subject to the *abc*-conjecture, we improve the standard Weyl estimate for cubic exponential sums in which the argument is a quadratic irrational. Specifically we show that

$$\sum_{n\leqslant N} e(\alpha n^3) \ll_{\varepsilon,\alpha} N^{\frac{5}{7}+\varepsilon}$$

for any $\varepsilon > 0$ and any quadratic irrational $\alpha \in \mathbb{R} - \mathbb{Q}$. Classically one would have had the (unconditional) exponent $\frac{3}{4} + \varepsilon$ for such α.

1. Introduction

In this paper, we shall consider bounds for the cubic Weyl sum

$$S(\alpha, N) = \sum_{n\leqslant N} e(\alpha n^3),$$

where $e(x) = \exp(2\pi i x)$ as usual. The classical bound, due essentially to Weyl [5], shows that $S(\alpha, N) \ll_\varepsilon N^{\frac{3}{4}+\varepsilon}$ for any $\varepsilon > 0$, providing that there is a rational number $\frac{a}{q}$ with denominator in the range $N \leqslant q \leqslant N^2$, for which we have $\left|\alpha - \frac{a}{q}\right| \leqslant q^{-2}$. It is clear that a condition on rational approximations to α will be necessary, and the exact condition here is unimportant. Of greater significance is the exponent $\frac{3}{4}$ in the Weyl estimate, which has never been improved on. An alternative method to bound $S(\alpha, N)$ has been given by Vaughan [4, Theorem 3], leading to exactly the same exponent $\frac{3}{4}$. If α is a real algebraic irrational then Roth's Theorem shows that the Diophantine approximation condition is met, so that

$$S(\alpha, N) \ll_{\varepsilon,\alpha} N^{\frac{3}{4}+\varepsilon}.$$

The goal of this paper is to show how an improvement can be made for special values of α. Unfortunately our result depends on an unproved

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hypothesis, namely the *abc*-conjecture. This states that if $\varepsilon > 0$ is given there is a constant $K(\varepsilon)$ such that

$$c \leqslant K(\varepsilon)\left(\prod_{p|abc} p\right)^{1+\varepsilon}$$

for any coprime positive integers a, b, c with $a + b = c$.

We shall then prove the following bound.

Theorem 1. *Let $\alpha \in \mathbb{R} - \mathbb{Q}$ be a quadratic irrational. Assume the truth of the abc-conjecture. Then*

$$S(\alpha, N) \ll_{\varepsilon,\alpha} N^{\frac{5}{7}+\varepsilon}$$

for any $\varepsilon > 0$.

Note that $\frac{5}{7} = \frac{3}{4} - \frac{1}{28}$.

The underlying idea is to apply the q-analogue of van der Corput's method, which requires a suitable approximation $\frac{a}{q}$ to α, in which q factorizes in a suitable way. Results of this type were proved in an Oxford DPhil thesis by Ringrose [3] in 1985, but not otherwise published. We therefore establish a variant of Ringrose's result here.

Theorem 2. *Suppose that a and q are coprime integers with $N \leqslant q \leqslant N^{\frac{3}{2}}$. Suppose further that $q = q_1q_2q_3$ with the factors q_1, q_2, q_3 coprime in pairs and q_3 square-free. Then if $N \leqslant \min\{q_1q_3, q_2q_3\}$ we have*

$$S(\alpha, N) \ll_\varepsilon \left(1 + N^3\left|\alpha - \frac{a}{q}\right|\right) \times \left(Nq^{-\frac{1}{3}} + N^{\frac{1}{2}}q_1^{\frac{1}{2}} + N^{\frac{1}{4}}q^{\frac{1}{4}}q_2^{\frac{1}{4}} + N^{\frac{1}{4}}q^{\frac{1}{4}}q_3^{\frac{1}{8}}\right)q^{\varepsilon},$$

i.e., for any $\varepsilon > 0$.

Theorem 1 is an easy consequence of Theorem 2, along with the following result on Diophantine approximation with smooth denominators.

Theorem 3. *Let $\alpha \in \mathbb{R}$ be a quadratic irrational, and let $\varepsilon > 0$ be given. Then there is a constant $C(\alpha, \varepsilon)$ such that, for any $N \in \mathbb{N}$, one can solve*

$$\left|\alpha - \frac{a}{q}\right| \leqslant \frac{C(\alpha, \varepsilon)}{qN}, \quad (a \in \mathbb{Z},\ \ q \in \mathbb{N},\ \ q \leqslant N)$$

with q having no prime factors $p > q^{\varepsilon}$.

This result provides approximations almost as strong as Dirichlet's Theorem yields. However the hypothesis that α is quadratic makes the proof rather simple. One can quite easily improve the statement of the theorem slightly to say that any prime power factor p^e of q has $p^e \leqslant q^{\varepsilon}$. However we do not need this for our application. Unfortunately we are unable to produce values of q which are "nearly square-free," and it is at this point that we must call on the *abc*-conjecture. Indeed the reader may notice that it suffices for Theorem 1 that the largest power-full divisor of q should be at most $q^{\frac{11}{21}}$, but we are unable to produce numbers q of this type.

2. Preliminary steps

If $\delta = \alpha - \frac{a}{q}$ we have

$$S(\alpha, N) = S\Big(\frac{a}{q}, N\Big)e(\delta N^3) - \int_0^N (2\pi i\delta)3t^2 S\Big(\frac{a}{q}, t\Big)dt$$
$$\ll (1 + N^3|\delta|) \max_{t \leqslant N} \Big|S\Big(\frac{a}{q}, t\Big)\Big|,$$

by partial summation. Moreover, if we set

$$S(a, h; q) = \sum_{n=1}^{q} e\Big(\frac{(an^3 + hn)}{q}\Big)$$

and

$$T(h, t; q) = \sum_{n \leqslant t} e\Big(\frac{-hn}{q}\Big)$$

we have

$$S\Big(\frac{a}{q}, t\Big) = q^{-1} \sum_{-\frac{q}{2} < h \leqslant \frac{q}{2}} S(a, h; q)T(h, t; q).$$

Since $S(a, 0; q) \ll q^{\frac{2}{3}}$ the term $h = 0$ contributes $\ll Nq^{-\frac{1}{3}}$, which is satisfactory.

For the remaining terms we note that

$$T(h, t; q) \ll \min\Big(N, \frac{q}{|h|}\Big) \quad \text{and} \quad \frac{\partial T(h, t; q)}{\partial h} \ll q^{-1}N \min\Big(N, \frac{q}{|h|}\Big)$$

for $|h| \leqslant \frac{q}{2}$ and $t \leqslant N$. From now on we shall assume that

$$\left|\sum_{-\frac{q}{2}<h<0} S(a,h;q)T(h,t;q)\right| \leqslant \left|\sum_{0<h\leqslant\frac{q}{2}} S(a,h;q)T(h,t;q)\right|,$$

the alternative case being treated in exactly the same way.

We proceed to define $K = \left[\frac{q}{N}\right]$ and

$$\eta(r) = \max_{0\leqslant L\leqslant K}\left|\sum_{(r-1)K<h\leqslant(r-1)K+L} S(a,h;q)\right|.$$

We then find by partial summation that

$$\sum_{(r-1)K<h\leqslant(r-1)K+K'} S(a,h;q)T(h,t;q) \ll \eta(r)\min\left(N, \frac{q}{(r-1)K}\right) \ll \frac{N}{r}\eta(r)$$

for any integer $r \geqslant 1$ and any $K' \leqslant K$. Summing for $r \leqslant q$ (which is more than adequately large) we deduce that

$$\sum_{0<h\leqslant\frac{q}{2}} S(a,h;q)T(h,t;q) \ll N\sum_{r\leqslant q}\frac{\eta(r)}{r}.$$

We may therefore conclude as follows.

Lemma 1. *With the definitions above, if $N \leqslant q \leqslant N^{\frac{3}{2}}$ we have*

$$S\Big(\frac{a}{q}, t\Big) \ll \frac{N}{q}\sum_{r\leqslant q}\frac{\eta(r)}{r}.$$

It follows as a special case of Loxton and Vaughan [2, Theorem 1] that

$$S(a,h;q) \ll_{\varepsilon} q^{\frac{1}{2}+\varepsilon}(q,h)^{\frac{1}{4}} \tag{1}$$

for any $\varepsilon > 0$, whence

$$\sum_{r\leqslant q}\frac{\eta(r)}{r} \ll \sum_{r\leqslant q}\sum_{n\leqslant K}\frac{|S(a,(r-1)K+n;q)|}{r}$$
$$\ll \sum_{m\leqslant qK}|S(a,m;q)|\min\Big(1,\frac{K}{m}\Big)$$
$$\ll_\varepsilon q^{\frac12+\varepsilon}\sum_{m\leqslant qK}(q,m)^{\frac14}\min\Big(1,\frac{K}{m}\Big).$$

We now use the following easy lemma, which we shall prove at the end of this section.

Lemma 2. *Let positive integers v and $H_1 \leqslant H_2$ be given. Then, for any fixed $\varepsilon > 0$, we have*

$$\sum_{H_2-H_1<h\leqslant H_2}(h,v)^\rho \ll_\varepsilon \big\{H_1+\min(v,H_2)\big\}v^\varepsilon$$

uniformly for $\rho \leqslant 1$, and in particular

$$\sum_{1\leqslant h\leqslant H_2}(h,v)^\rho \ll_\varepsilon H_2 v^\varepsilon.$$

From this it follows by partial summation that

$$\sum_{r\leqslant q}\frac{\eta(r)}{r} \ll_\varepsilon q^{\frac12+\varepsilon}\sum_{m\leqslant qK}(q,m)^{\frac14}\min\Big(1,\frac{K}{m}\Big) \ll_\varepsilon Kq^{\frac{1}{(2+2\varepsilon)}}, \tag{2}$$

whence

$$S\Big(\frac{a}{q},t\Big) \ll_\varepsilon \frac{N}{q}Kq^{\frac{1}{(2+2\varepsilon)}} \ll_\varepsilon q^{\frac{1}{(2+2\varepsilon)}} \ll N^{\frac34+3\varepsilon},$$

since we are assuming that $q \leqslant N^{\frac32}$. We thus recover the classical exponent $\frac34$. The argument above is equivalent to that given by Vaughan, mentioned in the introduction.

To prove Lemma 2 we merely note that

$$\begin{aligned}\sum_{H_2-H_1<h\leqslant H_2}(h,v)^{\rho} &\leqslant \sum_{H_2-H_1<h\leqslant H_2}(h,v)\\ &\leqslant \sum_{d|v,\, d\leqslant H_2} d\#\{H_2-H_1<h\leqslant H_2 : d|h\}\\ &\leqslant \sum_{d|v,\, d\leqslant H_2} d\left(\frac{H_1}{d}+1\right)\\ &\leqslant H\sum_{d|v}\big\{H_1+\min(v,H_2)\big\}\\ &\ll_{\varepsilon} \big\{H_1+\min(v,H_2)\big\}v^{\varepsilon}.\end{aligned}$$

3. The first iteration

In order to improve on the classical bound we must demonstrate some cancellation amongst the terms $S(a,h;q)$ in the sum $\eta(r)$. We begin by noting the factorization property

$$S(a,h;uv)=S(av^2,h;u)S(au^2,h;v) \tag{3}$$

for coprime u and v. We begin the van der Corput argument by writing $\eta(r)=|\Sigma|$, where

$$\Sigma=\sum_{h\in I}S(a,h;q)$$

for an appropriate interval $I\subseteq((r-1)K,rK]$. Recall that $q=q_1q_2q_3$ in Theorem 2. Thus, if we impose the condition

$$q_2q_3\geqslant N, \tag{4}$$

we will have $K\geqslant q_1$. We then set $M=\left[\frac{K}{q_1}\right]\geqslant 1$ and observe that

$$\begin{aligned}M\Sigma &= \sum_{m\leqslant M}\ \sum_{h:\, h+mq_1\in I} S(a,h+mq_1;q)\\ &= \sum_{(r-2)K<h<rK}\ \sum_{m\leqslant M:\, h+mq_1\in I} S(a,h+mq_1;q).\end{aligned}$$

We now apply (3) with $u = q_1$ and $v = q_2q_3$, so that

$$\begin{aligned} S(a, h + mq_1; q) &= S(a', h + mq_1; q_1)S(b, h + mq_1; q_2q_3) \\ &= S(a', h; q_1)S(b, h + mq_1; q_2q_3), \end{aligned}$$

where we have written $a' = aq_2^2q_3^2$ and $b = aq_1^2$. It follows that

$$M\Sigma = \sum_{(r-2)K<h<rK} S(a', h; q_1) \sum_{m\leqslant M:\, h+mq_1\in I} S(b, h + mq_1; q_2q_3).$$

We now apply Cauchy's inequality to produce

$$M^2\eta(r)^2 = M^2|\Sigma|^2 \leqslant \eta_1(r)\eta_2(r), \tag{5}$$

where

$$\eta_1(r) = \sum_{(r-2)K<h<rK} |S(a', h; q_1)|^2$$

and

$$\eta_2(r) = \sum_{(r-2)K<h<rK} \left| \sum_{m\leqslant M:\, h+mq_1\in I} S(b, h + mq_1; q_2q_3) \right|^2 .$$

We may estimate η_1 using Lemma 2. By (1) we have

$$\begin{aligned} \eta_1(r) &= \sum_{(r-2)K<h<rK} |S(a', h; q_1)|^2 \\ &\ll_\varepsilon q_1^{1+2\varepsilon} \sum_{(r-2)K<h<rK} (h, q_1)^{\frac{1}{2}} \\ &\ll_\varepsilon q_1^{1+3\varepsilon}(K + q_1) \\ &\ll_\varepsilon q_1^{1+3\varepsilon}K, \end{aligned} \tag{6}$$

since $K \geqslant q_1$, as noted above.

To handle $\eta_2(r)$ we expand the square to produce

$$\begin{aligned} \eta_2(r) &= \sum_{m_1,m_2\leqslant M} \sum_{\substack{h:\, h+m_1q_1\in I \\ h+m_2q_1\in I}} S(b, h + m_1q_1; q_2q_3)\overline{S(b, h + m_2q_1; q_2q_3)} \\ &= \sum_{\substack{n_1,n_2\in I \\ q_1|n_1-n_2}} S(b, n_1; q_2q_3)\overline{S(b, n_2; q_2q_3)}N(b_1, b_2), \end{aligned}$$

where $N(b_1,b_2)$ is the number of triples $(h,m_1,m_2)\in\mathbb{Z}\times\mathbb{N}\times\mathbb{N}$ for which $m_1,m_2\leqslant M$ and $h+m_1q_1=n_1$, $h+m_2q_1=n_2$. Then

$$N(b_1,b_2)=M-q^{-1}|n_1-n_2|,$$

whence

$$\begin{aligned}\eta_2(r)&=\sum_{\substack{n_1,n_2\in I\\ q_1|n_1-n_2}}(M-q^{-1}|n_1-n_2|)S(b,n_1;q_2q_3)\overline{S(b,n_2;q_2q_3)}\\ &=\sum_{|m|\leqslant M}(M-|m|)\sum_{n,n+mq_1\in I}S(b,n+mq_1;q_2q_3)\overline{S(b,n;q_2q_3)}\\ &=\sum_{|m|\leqslant M}(M-|m|)\sum_{n\in I(m)}S_2(b,m,n;q_2q_3),\end{aligned}$$

where $I(m)$ is a subinterval of $((r-1)K,rK]$ given by

$$I(m)=\{x\in\mathbb{R}:\,x,x+mq_1\in I\}$$

and $S_2(b,n;q_2q_3,)$ is the exponential sum

$$S_2(b,m,n;u)=S(b,n+mq_1;u)\overline{S(b,n;u)}. \tag{7}$$

We write

$$\eta_3(r,m)=\sum_{n\in I(m)}S_2(b,m,n;q_2q_3);$$

so that our bound becomes

$$\eta_2(r)\ll M\sum_{|m|\leqslant M}|\eta_3(r,m)|\ll M|\eta_3(r,0)|+\sum_{1\leqslant|m|\leqslant M}|\eta_3(r,m)|. \tag{8}$$

Notice that the only dependence of $\eta_3(r,m)$ on r is through the interval $I(m)$, which our notation has suppressed.

We now combine Lemma 1, (5), (6), and (8) to deduce that

$$\begin{aligned}S(\frac{a}{q},t)^2&\ll\left(\frac{N}{q}\right)^2(\log q)\sum_{r\leqslant q}\frac{\eta(r)^2}{r}\\ &\ll_\varepsilon N^2q^{-2+\varepsilon}M^{-2}q_1^{1+3\varepsilon}K\sum_{r\leqslant q}\frac{\eta_2(r)}{r}\\ &\ll_\varepsilon N^2q^{-2+\varepsilon}M^{-2}q_1^{1+3\varepsilon}KM\sum_{r\leqslant q}\sum_{|m|\leqslant M}\frac{\eta_3(r,m)}{r}\\ &\ll_\varepsilon N^2q^{-2+\varepsilon}q_1^{2+3\varepsilon}\sum_{r\leqslant q}\sum_{|m|\leqslant M}\frac{|\eta_3(r,m)|}{r},\end{aligned}$$

whence

$$S\left(\frac{a}{q},t\right)^2 \ll_\varepsilon T_1 + T_2,$$

where

$$T_1 = N^2 q^{4\varepsilon}(q_2q_3)^{-2}\sum_{r\leqslant q}\frac{|\eta_3(r,0)|}{r}$$

and

$$T_2 = N^2 q^{4\varepsilon}(q_2q_3)^{-2}\sum_{r\leqslant q}\sum_{1\leqslant |m|\leqslant M}\frac{|\eta_3(r,m)|}{r}. \tag{9}$$

However

$$\begin{aligned}\eta_3(r,0) &= \sum_{n\in I(0)} S_2(b,0,n;q_2q_3)\\ &= \sum_{n\in I(0)} |S(b,n;q_2q_3)|^2\\ &\ll_\varepsilon (q_2q_3)^{1+2\varepsilon}\sum_{(r-1)K<n\leqslant rK}(q_2q_3,n)^{\frac{1}{2}}\end{aligned}$$

by (1), whence

$$\begin{aligned}T_1 &\ll_\varepsilon N^2q^{4\varepsilon}(q_2q_3)^{-2}\sum_{r\leqslant q} r^{-1}(q_2q_3)^{1+2\varepsilon}\sum_{(r-1)K<n\leqslant rK}(q_2q_3,n)^{\frac{1}{2}}\\ &\ll_\varepsilon N^2q^{6\varepsilon}(q_2q_3)^{-1}\sum_{n\leqslant qK}\frac{K}{n}(q_2q_3,n)^{\frac{1}{2}}\\ &\ll_\varepsilon N^2q^{6\varepsilon}(q_2q_3)^{-1}Kq^{\varepsilon}\\ &\ll_\varepsilon Nq^{7\varepsilon}q_1\end{aligned}$$

using Lemma 2. This is satisfactory for Theorem 2.

We summarize the state of play as follows.

Lemma 3. *When $q_2q_3 \leqslant N$ and $q \leqslant N^{\frac{3}{2}}$, we have*

$$S\left(\frac{a}{q},t\right)^2 \ll_\varepsilon Nq^{\varepsilon}q_1 + T_2$$

with T_2 as in (9) for any $t \leqslant N$ and any $\varepsilon > 0$.

This completes the first application of the van der Corput "A-process." We can check that nothing of significance has been lost at this stage. Thus if we use the bound (1), ignoring the highest common factor terms for simplicity, we would get a bound

$$\ll_\varepsilon M \sum_{|m|\leqslant M} \sum_{n\in I(m)} (q_2q_3)^{1+2\varepsilon} \ll_\varepsilon K^3q_1^{-2}(q_2q_3)^{1+2\varepsilon}$$

for $\eta_2(r)$. Combining this with (5) and (6) would lead to $\eta(r) \ll_\varepsilon Kq^{\frac{1}{2}+3\varepsilon}$, allowing us to recover the bound (2). As previously observed this in turn would lead to the classical exponent $\frac{3}{4}$ for the original sum $S(\frac{a}{q}, N)$.

Although nothing has been lost, our manipulations have produced an advantage. We need to demonstrate that there is some cancellation in the sum $\eta_3(r,m)$. The range for n in this sum is of the same kind as in the previous sum Σ, and the exponential sum $S_2(b,m,n;q_2q_3)$ which occurs is more complicated than before, but crucially the modulus q_2q_3 for the exponential sum is smaller than in Σ, where it was q. Unfortunately the modulus is still too large, so that a second iteration of the A-process is necessary.

4. The second iteration

For the second iteration, we impose the condition

$$q_1q_3 \geqslant N, \tag{10}$$

whence $q_2 \leqslant \frac{q}{N}$. It follows that $q_2 \leqslant K$. We now set $U = \left[\frac{K}{q_2}\right]$, so that $U \geqslant 1$. For the sum (7) the product formula (3) leads to the relation

$$S_2(b,m,n;q_2q_3) = S_2(b',m,n;q_2)S_2(c,m,n;q_3),$$

where $b' = bq_3^2$ and $c = bq_2^2$. Then, by the arguments leading to (5) and (8), we find that

$$|\eta_3(r,m)|^2 \leqslant U^{-2}\eta_4(r,m)\eta_5(r,m), \tag{11}$$

with

$$\eta_4(r,m) = \sum_{(r-2)K<h<rK} |S_2(b',m,h;q_2)|^2$$

and

$$\eta_5(r,m) = \sum_{|u|\leqslant U} (U-|u|) \sum_{n\in I(m,u)} S_3(c,m,u,n;q_3).$$

Here, $I(m,u)$ is an appropriate subinterval of $((r-1)K, rK]$, and

$$S_3(c,m,u,n;v) = S_2(c,m,n+uq_2;v)\overline{S_2(c,m,n;v)}.$$

By (1), we have

$$\begin{aligned} S_2(b',m,h;q_2) &\ll_\varepsilon q_2^{1+2\varepsilon}(h+mq_1,q_2)^{\frac{1}{4}}(h,q_2)^{\frac{1}{4}} \\ &\ll_\varepsilon q_2^{1+2\varepsilon}\{(h+mq_1,q_2)^{\frac{1}{2}}+(h,q_2)^{\frac{1}{2}}\}. \end{aligned}$$

It follows that

$$\begin{aligned} \eta_4(r,m) &\ll_\varepsilon q_2^{2+4\varepsilon} \sum_{(r-2)K<h<rK} \{(h+mq_1,q_2)+(h,q_2)\} \\ &\ll_\varepsilon q_2^{2+4\varepsilon} \sum_{(r-3)K<h<(r+1)K} (h,q_2) \\ &\ll_\varepsilon q_2^{2+5\varepsilon}(q_2+K) \ll_\varepsilon q_2^{2+5\varepsilon}K, \end{aligned} \tag{12}$$

by Lemma 2. We now set

$$\eta_6(r,m,u) = \sum_{n\in I(m,u)} S_3(c,m,u,n;q_3),$$

whence

$$\eta_5(r,m) \ll U \sum_{|u|\leqslant U} |\eta_6(r,m,u)|. \tag{13}$$

It now follows that

$$\begin{aligned} T_2^2 &\ll_\varepsilon N^4 q^{8\varepsilon}(q_2q_3)^{-4}\Big\{M\sum_{r\leqslant q} r^{-1}\Big\}\Big\{\sum_{r\leqslant q}\ \sum_{1\leqslant|m|\leqslant M} \frac{|\eta_3(r,m)|^2}{r}\Big\} \\ &\ll_\varepsilon N^4 q^{8\varepsilon}(q_2q_3)^{-4}\left\{q_2q_3N^{-1}q^\varepsilon\right\} \\ &\times \Big\{U^{-1}Kq_2^{2+5\varepsilon}\sum_{r\leqslant q}\ \sum_{1\leqslant|m|\leqslant M}\ \sum_{|u|\leqslant U} \frac{|\eta_6(r,m,u)|}{r}\Big\} \\ &\ll_\varepsilon N^3 q^{14\varepsilon} q_3^{-3} \sum_{r\leqslant q}\ \sum_{1\leqslant|m|\leqslant M}\ \sum_{|u|\leqslant U} \frac{|\eta_6(r,m,u)|}{r}. \end{aligned} \tag{14}$$

When $u=0$ we have

$$\begin{aligned} S_3(c,m,0;q_3) &= |S_2(c,m,n;q_3)|^2 \\ &= |S(c,n+mq_1;q_3)|^2|S(c,n;q_3)|^2 \\ &\ll_\varepsilon q_3^{2+4\varepsilon}(q_3,n+mq_1)^{\frac12}(q_3,n)^{\frac12} \end{aligned}$$

by (1). The contribution to (14) from terms with $u=0$ is then

$$\begin{aligned} &\ll_\varepsilon N^3q^{18\varepsilon}q_3^{-1}\sum_{r\leqslant q}\sum_{1\leqslant|m|\leqslant M}\sum_{\substack{(r-1)K<n,\\ n+mq_1\leqslant rK}}\frac{(q_3,n+mq_1)^{\frac12}(q_3,n)^{\frac12}}{r} \\ &\ll_\varepsilon N^3q^{18\varepsilon}q_3^{-1}\sum_{r\leqslant q}\sum_{1\leqslant|m|\leqslant M}\sum_{\substack{(r-1)K<n,\\ n+mq_1\leqslant rK}}\frac{(q_3,n+mq_1)+(q_3,n)}{\frac{n}{K}} \\ &\ll_\varepsilon N^3q^{18\varepsilon}q_3^{-1}KM\sum_{r\leqslant q}\sum_{(r-1)K<n\leqslant rK}\frac{(q_3,n)}{n} \\ &\ll_\varepsilon Nq^{19\varepsilon}q_1q_2^2q_3, \end{aligned}$$

by Lemma 2.

We complete the second van der Corput A-process by combining this with Lemma 3 and (14) to deduce the following bound.

Lemma 4. *When $q_2q_3\geqslant N$, $q_1q_3\geqslant N$, and $q\leqslant N^{\frac32}$, we have*

$$\begin{aligned} &S\Big(\frac{a}{q},t\Big)^4 \\ &\ll_\varepsilon q^\varepsilon\Big\{N^2q_1^2+Nq_1q_2^2q_3+N^3q_3^{-3}\sum_{r\leqslant q}\sum_{1\leqslant|m|\leqslant M}\sum_{1\leqslant|u|\leqslant U}\frac{|\eta_6(r,m,u)|}{r}\Big\} \end{aligned}$$

for any $t\leqslant N$ and any $\varepsilon>0$.

The first two terms here are suitable for Theorem 2.

5. The van der Corput B-process

To complete the van der Corput argument, we will estimate $\eta_6(r,m,u)$. We have

$$\eta_6(r,m,n)=q_3^{-1}\sum_{-\frac{q_3}{2}<t\leqslant\frac{q_3}{2}}S_4(c,m,u,t;q_3)\sum_{n\in I(m,u)}e\Big(\frac{-nt}{q_3}\Big),$$

where

$$S_4(c,m,u,t;v)=\sum_{n=1}^{v}S_3(c,m,u,n;v)e\Big(\frac{nt}{v}\Big).$$

Since $I(m,n)$ is an interval of length at most K this leads to

$$\eta_6(r,m,u)\ll q_3^{-1}\sum_{-\frac{q_3}{2}<t\leqslant\frac{q_3}{2}}\min\left(K,\,\frac{q_3}{|t|}\right)\big|S_4(c,m,u,t;q_3)\big|. \tag{15}$$

The sum $S_4(c,m,u,n,t;v)$ has a multiplicative property

$$S_4(c,m,u,t;vw)=S_4(cw^2,m,u,\overline{w}t;v)S_4(cv^2,m,u,\overline{v}t;w), \tag{16}$$

where $w\overline{w}\equiv 1 \bmod v$ and $v\overline{v}\equiv 1 \bmod w$. It therefore suffices to bound sums to prime-power modulus. Indeed, since we are assuming q_3 to be square-free it will be enough to consider the case in which the modulus is prime. It would be good if we were able to remove the square-freeness condition, by handling S_4 for prime power moduli, but this appears to be unduly complicated.

In view of the definition of $S_3(c,m,u,n;v)$ we find that

$$\begin{aligned}S_4(c,m,u,t;v)&=\sum_{n=1}^{v}S^{(1)}\overline{S^{(2)}S^{(3)}}S^{(4)}e\Big(\frac{nt}{v}\Big)\\&=\sum_{w,x,y,z=1}^{v}\sum_{n=1}^{v}e\bigg(\frac{f(w,x,y,z,n)}{v}\bigg),\end{aligned}$$

where

$$\begin{aligned}S^{(1)}&=S(c,n+uq_2+mq_1;v), & S^{(2)}&=S(c,n+uq_2;v),\\ S^{(3)}&=S(c,n+mq_1;v), & S^{(4)}&=S(c,n;v),\end{aligned}$$

and

$$\begin{aligned}f(w,x,y,z,n)=&\,c(w^3-x^3-y^3+z^3)+w(n+uq_2+mq_1)\\&-x(n+uq_2)-y(n+mq_1)+zn+tn.\end{aligned}$$

When we perform the summation over n this produces

$$S_4=v\sum_{\substack{w,x,y,z \,(\mathrm{mod}\ v)\\ v\mid w-x-y+z+t}}e\left(\frac{c(w^3-x^3-y^3+z^3)+uq_2(w-x)+mq_1(w-y)}{v}\right).$$

We substitute $z = x + y - w - t$ so that the numerator becomes

$$\begin{aligned} &c(w^3 - x^3 - y^3 + (x + y - w - t)^3) + uq_2(w - x) + mq_1(w - y) \\ &= 3c(x + y)(w - x)(w - y) - 3ct(x + y - w)^2 + 3ct^2(x + y - w) - ct^3 \\ &\qquad + uq_2(w - x) + mq_1(w - y) \\ &= 3cWXY - \frac{3}{4}ct(W+X+Y)^2 + \frac{3}{2}ct^2(W+X+Y) - ct^3 - uq_2X - mq_1Y, \end{aligned}$$

on writing $W = x + y$ and $X = x - w$, $Y = y - w$. Thus if v is a prime p which does not divide $6t$ we find that

$$S_4 = p \sum_{W,X,Y \,(\mathrm{mod}\ p)} e\left(\frac{g(W,X,Y)}{p}\right),$$

where $g(W, X, Y)$ takes the shape

$$c'WXY + t'(W^2+W^2+Y^2+2XY+2WX+2WY) + \mu_1 W + \mu_2 X + \mu_3 Y - ct^3$$

with $p \nmid c't'$. Exponential sums of this type have been treated by Bombieri and Sperber [1, Theorem 7]. Their work shows that

$$S_4(c, m, u, t; p) \ll p^{\frac{5}{2}}$$

for such primes. When $p \mid t$ it is easy to see that $S_4(c, m, u, t; p) \ll p^2(p, m, u)$, whence in general we have $S_4(c, m, u, t; p) \ll p^{\frac{5}{2}}(p, t, m, u)^{\frac{1}{2}}$ for all primes p. While it is convenient to call on a theorem from the literature, we remark that it is possible to evaluate $S_4(c, m, u, t; p)$ in terms of exponential sums in one variable, for which it suffices to use Weil's theorem rather than Deligne's.

By the multiplicative property (16), we now deduce that

$$S_4(c, m, u, t; q_3) \ll_\varepsilon q_3^{\frac{5}{2}+\varepsilon}(q_3, t, m, u)^{\frac{1}{2}},$$

whence (15) and Lemma 2 yield

$$\begin{aligned} \eta_6(r, m, u) &\ll_\varepsilon q_3^{\frac{3}{2}+\varepsilon} \sum_{-\frac{q_3}{2} < t \leqslant \frac{q_3}{2}} \min\left(K, \frac{q_3}{t}\right)(q_3, t, m, u)^{\frac{1}{2}} \\ &\ll_\varepsilon q_3^{\frac{3}{2}+\varepsilon} \left\{ K(q_3, m, u)^{\frac{1}{2}} + \sum_{1 \leqslant t \leqslant \frac{q_3}{2}} \frac{q_3}{|t|}(q_3, t)^{\frac{1}{2}} \right\} \\ &\ll_\varepsilon q_3^{\frac{3}{2}+\varepsilon}\{K(q_3, m, u)^{\frac{1}{2}} + q_3^{1+\varepsilon}\}, \end{aligned}$$

by partial summation. It then follows using Lemma 2 that

$$\begin{aligned} N^3 q_3^{-3} \sum_{r \leqslant q} \sum_{1 \leqslant |m| \leqslant M} \sum_{1 \leqslant |u| \leqslant U} \frac{|\eta_6(r,m,u)|}{r} & \\ \ll_\varepsilon N^3 q_3^{-\frac{3}{2}+\varepsilon} \{ K M U q_3^\varepsilon + M U q_3^{1+2\varepsilon} \} & \\ \ll_\varepsilon q^{3\varepsilon} \{ q_1^2 q_2^2 q_3^{\frac{3}{2}} + N q_1 q_2 q_3^{\frac{3}{2}} \} & \\ \ll_\varepsilon N q_1 q_2 q_3^{\frac{3}{2}} q^{3\varepsilon}, & \end{aligned}$$

since

$$N = \left(\frac{N^{\frac{3}{2}}}{q_1 q_2} \right)^2 \frac{q_1^2 q_2^2}{N^2} \geqslant \left(\frac{q}{q_1 q_2} \right)^2 \frac{q_1^2 q_2^2}{N^2} = q_1 q_2 \frac{q_1 q_3}{N} \cdot \frac{q_2 q_3}{N} \geqslant q_1 q_2 .$$

Now, when we insert this estimate into Lemma 4, we see that Theorem 2 follows, with a new value for ε.

6. Proofs of Theorems 3 and 1

To prove Theorem 3 it clearly suffices to suppose that $\alpha = \sqrt{d}$ for some nonsquare $d \in \mathbb{N}$. Let $a, b \in \mathbb{N}$ be solutions of the Pell equation $a^2 - db^2 = 1$ and define p_n, q_n by

$$p_n + q_n \sqrt{d} = \eta^n ,$$

where $\eta = a + b\sqrt{d}$. It follows that

$$q_n = (2\sqrt{d})^{-1} (\eta^n - \eta^{-n}) = (2\sqrt{d})^{-1} \prod_{k|n} \Phi_k(\eta, \eta^{-1}),$$

where

$$\Phi_k(X,Y) = \prod_{1 \leqslant h \leqslant k,\, (h,k)=1} (X - e^{\frac{2\pi i h}{k}} Y)$$

is the kth cyclotomic polynomial. Thus $\Phi_k(X,Y) \in \mathbb{Z}[X,Y]$ and $\Phi_k(X, X^{-1}) = \Phi_k(X^{-1}, X)$ except for $k = 1$, in which case $\Phi_1(X,Y) = X - Y$. It follows that

$$q_n = b \prod_{k|n,\, k \geqslant 2} r_k$$

for integers $r_k = |\Phi_k(\eta, \eta^{-1})|$. Moreover,

$$r_k \leqslant (\eta + \eta^{-1})^{\phi(k)} = (2a)^{\phi(k)} \leqslant (2a)^{\phi(n)}.$$

Now fix an integer m such that

$$\frac{\phi(m)}{m} \leqslant \varepsilon \frac{\log \eta}{2\log(2a)}.$$

Then if $m|n$, we have $\frac{\phi(n)}{n} \leqslant \frac{\phi(m)}{m}$, whence

$$r_k \leqslant (2a)^{\phi(n)} \leqslant \{(2a)^n\}^{\frac{\phi(m)}{m}} \leqslant \{(2a)^n\}^{\frac{\varepsilon \log \eta}{2\log(2a)}} = (\eta^{\frac{n}{2}})^{\varepsilon}.$$

However

$$q_n = (2\sqrt{d})^{-1}(\eta^n - \eta^{-n}) \geqslant (4\sqrt{d})^{-1}\eta^n \geqslant \eta^{\frac{n}{2}}$$

for large enough n, whence $r_k \leqslant q_n^{\varepsilon}$. It follows that every prime factor of q_n is at most q_n^{ε} if $m|n$.

Finally, we observe that $q_{n+1} \ll q_n$ with an implied constant depending on the choice of a, b, and d, so that there is a value of n which is a multiple of m and for which $N \geqslant q_n \gg N$. Since

$$|p_n - q_n\sqrt{d}| = \frac{1}{p_n + q_n\sqrt{d}} \leqslant \frac{1}{q_n};$$

the theorem then follows.

To deduce Theorem 1, we write $\alpha = \frac{f+g\sqrt{d}}{c}$ and approximate $\sqrt{d}$ as above, with

$$\left|\sqrt{d} - \frac{u}{v}\right| \leqslant \frac{1}{vV}, \quad V \geqslant v \gg V,$$

where we choose $V = [\frac{N^{\frac{3}{2}}}{c}]$. Let v_0 be the product of all prime powers $p^e||q$ for which $e \geqslant 2$. Then the product of the prime divisors of v_0 can be at most $v_0^{\frac{1}{2}}$. It follows that

$$\prod_{p|u^2v^2d} p \leqslant u\frac{v}{v_0}v_0^{\frac{1}{2}}d.$$

Since $1 + v^2d = u^2$ the *abc*-conjecture would imply that $u^2 \ll_{\varepsilon,\alpha} (uvv_0^{-\frac{1}{2}})^{1+\varepsilon}$, whence $v_0 \ll_{\varepsilon,\alpha} v^{4\varepsilon}$.

We now write $a_1 = fv + gu$ and $q_1 = cv$, and set $a = \frac{a_1}{(a_1,q_1)}$ and $q = \frac{q_1}{(a_1,q_1)}$, so that a and q are coprime, with

$$q \leqslant q_1 = cv \leqslant cV \leqslant N^{\frac{3}{2}}$$

and

$$q \gg_\alpha q_1 \gg_\alpha v \gg_{\varepsilon,\alpha} V \gg_\alpha N^{\frac{3}{2}}.$$

Then

$$\left|\alpha - \frac{a}{q}\right| \ll_\alpha \left|\sqrt{d} - \frac{u}{v}\right| \leqslant \frac{1}{vV} \ll_{\varepsilon,\alpha} \frac{1}{qN^{\frac{3}{2}}}.$$

Moreover, every prime factor of q is $O_{\varepsilon,\alpha}(q^\varepsilon)$, and if q_0 is the product of all $p^e||q$ with $e \geqslant 2$, then $q_0 \ll_{\varepsilon,\alpha} q^{4\varepsilon}$.

We proceed to build up coprime square-free divisors q_2, q_3 of $\frac{q}{q_0}$, one prime factor at a time, to produce products in the ranges

$$q^{\frac{5}{21}} \leqslant q_2 \ll_{\varepsilon,\alpha} q^{\frac{5}{21}+\varepsilon}, \quad q^{\frac{10}{21}} \leqslant q_3 \ll_{\varepsilon,\alpha} q^{\frac{10}{21}+\varepsilon}.$$

We have $q = q_1 q_2 q_3$ with

$$q^{\frac{2}{7}-2\varepsilon} \ll_{\varepsilon,\alpha} q_1 \leqslant q^{\frac{2}{7}}.$$

One may then verify that the hypotheses of Theorem 2 are satisfied, and that

$$S(\alpha, N) \ll_{\varepsilon,\alpha} q^{\frac{10}{21}+2\varepsilon} \ll_{\varepsilon,\alpha} N^{\frac{5}{7}+3\varepsilon}.$$

This suffices for Theorem 1, on re-defining ε.

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