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# BASE CHANGE FOR HILBERT EIGENVARIETIES OF UNITARY GROUPS

Abstract. The construction of eigenvarieties by Chenevier is extended to the Hilbert case, that is to unitary groups over a totally real field $F$ which are anisotropic at each archimedean place. This permits us to ask about the relationship of the eigenvarieties which we construct for two totally real fields, one being a cyclic extension of the other.

## 1. Introduction

R. Coleman [7, 8], continuing works of Serre, Katz, Dwork, Hida, Gouvea–Mazur, showed that each eigenform of level $\Gamma_1(p)$ at a fixed prime $p$ belongs to a $p$-adic family of eigenforms. Thus he constructed an analytic family of overconvergent modular forms and applied to the operator $U_p$ the spectral theory of compact operators of orthonormalizable Banach modules, which he introduced.

These ideas were developed by Coleman and Mazur [9] to construct a $p$-adic analytic curve, which they named the eigencurve, parametrizing the overconvergent modular forms of finite slope and fixed tame ramification. The existence of $p$-adic families of cusp eigenforms *ordinary* at $p$ had been studied by Hida.

Buzzard [3] showed that the construction of families in the case of quaternion algebras over $\mathbb{Q}$ which are ramified at infinity can be dealt with directly by elementary means, also in the Hilbert modular case, namely over a totally real field $F$ instead of $\mathbb{Q}$.

Chenevier [5] developed the theory to apply to unitary groups over $\mathbb{Q}$ which are compact at the real place. The compactness at infinity eliminates many geometric difficulties, in fact $G(\mathbb{Q})\backslash G(\mathbb{A})/U$ is finite for any open compact subgroup $U$ of $G(\mathbb{A})$, and so all automorphic forms on $G$ contribute only to cohomology in degree zero.

Following Chenevier closely, we extend his theory to the Hilbert case, that is to unitary groups over a totally real field $F$ which are anisotropic at

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each archimedean place. This permits us to ask about the relationship of the eigenvarieties which we construct for two totally real fields, one being a cyclic extension of the other. Thus we pose the problem of constructing a base change morphism from one eigenvariety to the other, compatible with the basechange lifting for automorphic representations of these two groups. This leads to new questions in representation theory of $p$-adic groups.

We now continue with a more detailed description of the contents.

Let $F'$ be a totally imaginary quadratic extension of a totally real field $F$. Let $G_F$ be a unitary group over $F$ with $G_F(F') = \mathrm{GL}(n, F')$ and anisotropic $G_F(F_{\infty_r})$ for any archimedean completion $F_{\infty_r}$ of $F$ ($1 \leqslant r \leqslant d = [F : \mathbb{Q}]$). Let $p$ be a rational prime which splits completely in $F$ such that $G_F(F_{p_r}) = \mathrm{GL}(n, F_{p_r})$ for each $F$-prime $p_r$ over $p$. Let $U_0(p)_F \subset G_F(\mathbb{A}_{F,f})$ be a compact open subgroup which is a product of local such groups, $G_F(F_v^0)$ at almost all finite places $v$ of $F$, where $F_v^0$ is the ring of integers in $F_v$, whose component at each place $p_r$ of $F$ over $p$ is $I_{p_r}$. Here $I_{p_r}$ is the Iwahori subgroup of matrices in $\mathrm{GL}(n, F_{p_r}^0) = \mathrm{GL}(n, \mathbb{Z}_p)$ which are upper triangular mod $p_r$.

The space of *automorphic forms of weight* $\mu_F = (\mu_r; 1 \leqslant r \leqslant d)$, $\mu_r \in \mathbb{Z}^n_{\geqslant 0}$ for each real place $\infty_r$, and *of level* $U_0(p)_F$, is the $\mathbb{Q}_p$-vector space of $U_0(p)_F$-equivariant functions $\phi$ on $G_F(F)\backslash G_F(\mathbb{A}_{F,f})/U_0(p)_F^p$ with values in $\otimes_r {}_{\mu_r}V(\mathbb{Q}_p)$. Here ${}_{\mu_r}V(\mathbb{Q}_p)$ is the irreducible algebraic representation of highest weight $\mu_r$ (thus $\mu_r$ is a dominant character $\mu'$ of the diagonal torus of $\mathrm{GL}(n)$, of the form

$$t = \mathrm{diag}(t_1, \ldots, t_n) \mapsto \prod_{1 \leqslant i \leqslant n} t_i^{\mu'_i},$$

$\mu' = (\mu'_1 \geqslant \cdots \geqslant \mu'_n) \in \mathbb{Z}^n_{\geqslant 0})$ of $\mathrm{GL}(n, \mathbb{Q}_p)$, and $U_0(p)_F = U_0(p)_F^p \cdot \prod_{1 \leqslant r \leqslant d} I_{p_r}$ acts through its components at the primes $p_r$ over $p$. It is a finite dimensional $\mathbb{Q}_p$-vector space (since $G_F(F)\backslash G_F(\mathbb{A}_{F,f})/U_0(p)_F$ is finite) on which the ring of Hecke correspondences of $(G_F(\mathbb{A}_{F,f}), U_0(p)_F)$ acts.

To vary these spaces continuously $p$-adically as a function of $\mu_F = (\mu_r)$, $\mu_r \in \mathbb{Z}^n_{\geqslant 0}$, one deforms $p$-adically the restrictions to $I_{p_r}$ of the algebraic representations of $\mathrm{GL}(n, F_{p_r}) = \mathrm{GL}(n, \mathbb{Q}_p)$ as functions of their highest weights. For this purpose we recall in Sec. 2 a convenient and explicit model for the algebraic representations of $\mathrm{GL}(n)$ in characteristic zero in terms of (lower triangular) unipotent invariants.

Following Coleman, Chenevier interpolates when ($F=\mathbb{Q}$) first by including each of the representations of $I_{p_r}$ in a representation of $I_{p_r}$ on a $p$-adic Banach space of infinite dimension, then by showing that these form an orthonormalizable analytic family. A key tool is to consider the orbit under $\prod_r I_{p_r}$ of the origin in the flag variety $\overline{B}\backslash\operatorname{GL}(n)$, where $\overline{B}$ is the lower triangular minimal parabolic subgroup. The set of $\mathbb{Q}_p$-points of this orbit has a natural structure as the set of $\mathbb{Q}_p$-points of an affinoid over $\mathbb{Q}_p$, denoted by $Y_{1,F}$ below. The Borel–Weil–Bott theorem implies that every irreducible algebraic representation of $\operatorname{GL}(n,\mathbb{Q}_p)$ can be realized on the space of global sections of a fiber bundle on $\overline{B}\backslash\operatorname{GL}(n)$. Every fiber restricts to a trivial fiber on $Y_{1,F}$. We trivialize it using a vector of highest weight. Its rigid sections over $Y_{1,F}$ provide the desired representation of $I_{p_r}$. These representations have a common underlying Banach space. It is the space $A(Y_{1,F})$ of analytic functions on $Y_{1,F}$ which are twisted by a product of fundamental 1-cocycles of $\prod_r \operatorname{GL}(n,F_{p_r})$ raised to the power $\mu_F=(\mu_r)$, $\mu_r=(\mu_{ri})$. This permits realizing and defining the orthonormal family $A(Y_{1,F};\chi_F,W_F)$ of representations of $\prod_r I_{p_r}$ parametrized by the rigid space of analytic weights $W_F^+(\mathbb{C}_p)=\operatorname{Hom}\big((\prod_{1\leqslant r\leqslant d} F_{p_r}^{0,\times})^n,\mathbb{C}_p^\times\big)$. Here $\mathbb{C}_p$ is the completion of an algebraic closure $\overline{\mathbb{Q}}_p$ of $\mathbb{Q}_p$. These weights are the characters of the maximal compact subgroup of the diagonal torus in $\prod_{1\leqslant r\leqslant d}\operatorname{GL}(n,F_{p_r})$ whose restrictions to $\prod_r(1+p_rF_{p_r}^0)^n$ are analytic. One can view $A(Y_{1,F};\chi_F,W_F)$ as an analytic family of principal series representations of $\prod_r I_{p_r}$. The choice of the big open orbit under $\prod_r I_{p_r}$ in the flag manifold lets $A(Y_{1,F};\chi_F,W_F)$ extend to a representation of the semigroup $I_F\cdot p^{\{a_F\}}\cdot I_F$ in the Hecke algebra of $\operatorname{GL}(n,F\otimes\mathbb{Q}_p)$ generated by $I_F=\Gamma_0(F\otimes\mathbb{Q}_p)=\prod_r I_{p_r}$ and the diagonal operators $p^{a_F}=\prod_r p_r^{a_{(r)}}$, $a_{(r)}=(a_{r1},\dots,a_{rn})$ with $(a_{r1}\leqslant\cdots\leqslant a_{rn})$ in $\mathbb{Z}^n$, $p_r^{a_{(r)}}=\operatorname{diag}(p_r^{a_{r1}},\dots,p_r^{a_{rn}})$. If the $(a_{ri})_i$ are strictly increasing then $p_r^{a_{(r)}}$ acts completely continuously on $A(Y_{1,F};\chi_F,W_F)$.

These local constructions permit us to define the analytic family of the Banach spaces of *$p$-adic automorphic forms* of type $(G_F,U_0(p)_F)$ and weight in $W_F(\mathbb{C}_p)$, denoted

$$A(Y_{1,F};\chi_F,W_F)_\#=\{A(Y_{1,F};\chi_F,W_F,\mu_F)_\#;\ \mu_F\in W_F(\mathbb{C}_p)\},$$

on considering the space of $U_0(p)_F$-equivariant functions on the homogeneous space $G_F(F)\backslash G_F(\mathbb{A}_{F,f})$ with values in $A(Y_{1,F};\chi_F,W_F)$. This family admits an action of the Hecke algebra of $U_0(p)_F$ outside $p$, and at $p$ of the commutative subalgebra of Iwahori biinvariant functions with support in $I_F \cdot p^{\{a_F\}} \cdot I_F$, $I_F = \prod_r I_{p_r}$.

Let $\mu_F = (\mu_{(r)})$, $\mu_{(r)} \in \mathbb{Z}^n_{\geqslant 0}$, be an element of the space $W_F(\mathbb{C}_p)$. Then the fiber $A(Y_{1,F}; \chi_F, W_F, \mu_F)_\#$ contains as a sub-Hecke-module the finite dimensional $\mathbb{C}_p$-vector space $A(Y_{1,F}; \chi_F, W_F, \mu_F)^{\mathrm{cl}}_\#$ of "classical" automorphic forms on $G_F$ of weight $\mu_F$, level $U_0(p)_F$ and a certain behaviour at $p$. The $W_F(\mathbb{C}_p)$-linear endomorphism of $A(Y_{1,F}; \chi_F, W_F)_\#$ defined by $p^F = \prod_r p^F_r$, where

$$p^F_r = U_0(p)_F \cdot p^F_{r1} \cdot U_0(p)_F, \quad p^F_{r1} = \mathrm{diag}(1, p_r, p_r^2, \ldots, p_r^{n-1}), \quad 1 \leqslant r \leqslant d,$$

is completely continuous (= compact). A $p$-adic automorphic form in the space $A(Y_{1,F}; \chi_F, W_F, \mu_F)_\#$ has *finite slope* $\alpha_F = (\alpha_r)$ if it lies in the largest subspace $A(Y_{1,F}; \chi_F, W_F, \mu_F)^{\alpha_F}_\#$ on which each $p^F_r$ has eigenvalue of valuation $\alpha_r$, $1 \leqslant r \leqslant d$. Generalizing Chenevier [5] we prove:

**Theorem 1.1.** *For $\mu_F = (\mu_{(r)})$, $\mu_{(r)} = (\mu_{r1} \geqslant \cdots \geqslant \mu_{rn}) \in \mathbb{Z}^n_{\geqslant 0}$ and $\alpha_F = (\alpha_r)$, $\alpha_r \geqslant 0$ in $\mathbb{Q}$, if $\min_i(\mu_{ri} - \mu_{r,i+1})$ is bigger than $\alpha_r - 1$ for all $r$ $(1 \leqslant r \leqslant d)$, then $A(Y_{1,F}; \chi_F, W_F, \mu_F)^{\alpha_F}_\# \subset A(Y_{1,F}; \chi_F, W_F, \mu_F)^{\mathrm{cl}}_\#$.*

Let $P_F(\mu_F, t) = \det[I - tp^F | A(Y_{1,F}; \chi_F, W_F, \mu_F)_\#] \in 1 + tA(W_F)\{\{t\}\}$ be the Fredholm series of $p^F$ acting on $A(Y_{1,F}; \chi_F, W_F)_\#$. Let $Z_F \subset W_F \times \mathbb{A}^{1,\mathrm{an}}$ be the *spectral variety*, namely the Fredholm hypersurface defined by $P_F(\mu_F, t) = 0$. Let $\mathrm{pr}_{W_F} : Z_F \to W_F$ and $\mathrm{pr}_{\mathbb{A}^1} : Z_F \to \mathbb{A}^{1,\mathrm{an}}$ be the two natural projections. Let $H_F$ denote a commutative subalgebra of the Hecke algebra of $G_F(\mathbb{A}_{F,f})$ with respect to $U_0(p)_F$ whose component at $p_r$ is $I_{p_r} \cdot p_r^{a_{(r)}} \cdot I_{p_r}$, $a_{(r)} = (a_{r1} \leqslant \cdots \leqslant a_{rn})$ for each $r$ $(1 \leqslant r \leqslant d)$. Following Coleman–Mazur and Chenevier one concludes the existence of the eigenvariety associated with the group $G_F$ over $F$.

**Theorem 1.2.** *There exists a rigid analytic space $D_F = D[G_F, U_0(p)_F, H_F]$ together with a ring homomorphism $a : H_F \to A(D_F)^0$ (= space of integral rigid analytic functions on $D_F$), and a finite morphism $\pi : D_F \to Z_F$ such that $a(p^F)^{-1} = \mathrm{pr}_{\mathbb{A}^1} \circ \pi$ (thus $a(p^F)$ is invertible on $D_F$). It satisfies:*

(i) *For each $x \in D_F(\mathbb{C}_p)$ there is an open affinoid neighborhood $D_x$ of $x$ such that $|a(p^F)(z)|$ is constant on $z \in D_x(\mathbb{C}_p)$, $\kappa(D_x)$ is an open affinoid of $W_F$, where $\kappa = \mathrm{pr}_{W_F} \circ \pi : D_F \to W_F$, and the restriction of $\kappa : D_x \to \kappa(D_x)$ to any irreducible component of $D_x$ is finite and surjective.*

(ii) *The map* $D_F(\mathbb{C}_p) \to \mathrm{Hom}(H_F, \mathbb{C}_p)$, $x \mapsto (\chi_x : h \mapsto a(h)(x))$, *induces for each* $\mu_F \in W_F(\mathbb{C}_p)$ *a bijection between the set of points* $x \in D_F(\mathbb{C}_p)$ *with* $\kappa(x) = \mu_F$, *and the set of systems of eigenvalues of* $H_F$ *acting on the space of* $p$*-adic automorphic forms of weight* $\mu_F$ *and finite slope* (*counted without multiplicity*).

(iii) *The restriction of* $\pi$ *to any irreducible component of* $D_F$ *is finite and surjective on an irreducible component of* $Z_F$, *and* $\kappa$ *maps this irreducible component of* $D_F$ *onto a Zariski open subset of* $W_F$.

(iv) *The closure of* $a(H_F)$ *in* $A(D_F)^0$ *is compact.*

(v) *A point* $x \in D_F(\mathbb{C}_p)$ *is called* classical *if* $\chi_x$ *is a system of eigenvalues of a vector in* $A(Y_{1,F}; \chi_F, W_F, \kappa(x))^{\mathrm{cl}}_{\#}$. *Then the classical points are Zariski dense in* $D_F(\mathbb{C}_p)$.

In particular, the points $x \in D_F(\mathbb{C}_p)$ with $\kappa(x) = \mu_F \in W_F(\mathbb{C}_p)$ parametrize the systems of eigenvalues of the Hecke algebra $H_F$ acting on the space of $p$-adic automorphic forms of weight $\mu_F$ and finite slope. Further, the classical points in $D_F(\mathbb{C}_p)$, those which parametrize the automorphic representations in the usual sense, are Zariski dense in $D_F(\mathbb{C}_p)$. We write $D[X]$ to emphasize that $D$ depends on $X$. We write $D(X)$ for the set of $X$-points of $D$.

In the speculative final part of this work we raise the question of a possible application of the theory of base change of automorphic representations to construct a base change lifting theory of eigenvarieties from the group $G_F$ to the group $G_E = G_F \times_F E$, where $E$ is a totally real cyclic extension of $F$ of odd prime degree not dividing $n$. Thus $G_E$ is a unitary group defined using the totally imaginary quadratic extension $E' = EF'$ of $E$. Let $p$ be a finite prime which splits completely in $E'$. Note that each archimedean place of $E$ is real. As with $G_F$, since each real place of $F$ splits completely in $E$, $G_E$ is anisotropic at each infinite place, and its group of $E_{p_{ra}}$-points, for each prime divisor $p_{ra}$ in $E$ of $p_r$ in $F$, is $\mathrm{GL}(n, \mathbb{Q}_p)$. Let $\sigma$ denote a generator of the Galois group of $E$ over $F$. We freely use conjectural statements in the theory of automorphic representations on the unitary group $\mathrm{U}(n, F'/F)$, proven so far only for $n = 3$ (see [13, 4]).

The automorphic representations of $G_F(\mathbb{A}_F)$ are partitioned into packets. Each packet base change lifts to a packet of automorphic representations of $G_E(\mathbb{A}_E)$ which consists of $\sigma$-invariant representations; see [4] when $n = 3$. Each such $\sigma$-invariant packet of $G_E(\mathbb{A}_E)$ is the base change lift of a unique packet of $G_F(\mathbb{A}_F)$. Now such an automorphic representation is a system of Hecke eigenvalues of a classical $p$-adic automorphic form. These points are Zariski dense in the eigenvariety $D_F(\mathbb{C}_p)$. We discuss, using density and continuity, following [8], the existence of a rigid spaces morphism $D_F[b^*H_E] \to D_E[H_E]$, the base change map, which should be injective and its image should consist of $\sigma$-invariant points in $D_E$. Here $b^* : H_E \to H_F$ is dual to $b : {}^LG_F \to {}^LG_E$, see Sec. 3. Previously an analogous morphism of eigenvarieties had been obtained by Chenevier [8] from the eigenvariety of the multiplicative group of a quaternion algebra over $\mathbb{Q}$ to the eigenvariety of $\mathrm{GL}(2, \mathbb{Q})$.

Our discussion is definitely not complete for various reasons, but we think the question we raise, about the existence of a base change morphism, is interesting. We hope it would lead to further work in this area. In particular we are led to a question in representation theory of $p$-adic groups, of whether the following holds. *Admissible Control* Statement: For each $k > 0$ there exists $k_E > 0$, depending only on $n$, the local field $F_v$ and its cyclic field extension $E_v$ of prime degree, such that if the admissible irreducible representation $\pi_v$ of $G_F(F_v)$ $(= \mathrm{GL}(n, F_v)$ or the quasisplit $\mathrm{U}(n, F'_v/F_v))$ has a nonzero vector invariant under $\mathrm{U}(k)$ $(= (I + \boldsymbol{\pi}_v^k M(n, R'_v)) \cap G_F(F_v))$, then the base change $\pi_{E_v}$ to $G_E(E_v)$ $(= \mathrm{GL}(n, E_v)$ or $\mathrm{U}(n, E'_v/E_v))$ has a nonzero vector invariant under $\mathrm{U}_E(k_E)$ $(= (I + \boldsymbol{\pi}_{E_v}^{k_E} M(n, R'_{E_v})) \cap G_E(E_v))$. The Statement follows if the characteristic functions $f_{\mathrm{U}(k)}$ of $\mathrm{U}(k)$ and $f_{\mathrm{U}_E(k_E)}$ of $\mathrm{U}_E(k_E)$ have matching stable orbital integrals. Then $\mathrm{tr}\{\pi\}(f_{\mathrm{U}(k)}) = \mathrm{tr}\{\pi_E\}(f_{\mathrm{U}_E(k_E)} \times \sigma)$ for corresponding packets $\{\pi\}$ and $\{\pi_E\}$ (which are singletons for $\mathrm{GL}(n)$). We prove that $k_E$ is $k$ if $E_v/F_v$ is unramified and $k[E_v : F_v]$ if it is ramified, for $G = \mathrm{GL}(n)$, see Proposition 4.1. We do not discuss here the case of unitary groups although it would be needed to construct the base change morphism $D_F[b^*H_E] \to D_E[H_E]$.

This paper was completed in early 2007, after I benefitted from discussions at the Eigenvarieties Semester at Harvard in late 2006 on this area which is new to me. Since then Loeffler [20] extended the theory, using techniques of Buzzard [3], to more general groups. Earlier Emerton [11] developed an alternative very general approach and proved all that we claim here. The introduction of [11] compares the various approaches. I hope that the explicit approach of this paper remains of interest, as is the base change relation which is here discussed. Thanks are due to the referee for extremely useful comments.

## 2. Local theory

Let $k$ be a commutative ring. Let $N$ be the group of unipotent upper triangular matrices in $\mathrm{GL}(n,k)$. The transpose $\overline{N} = {}^tN$ of $N$ is the group of unipotent lower triangular matrices in $\mathrm{GL}(n,k)$. Left and right matrix multiplication of $\mathrm{GL}(n,k)$ define two actions of $\mathrm{GL}(n,k)$ – by automophisms of $k$-algebras – on the algebra

$$R = k[\mathrm{GL}(n)] = k[\{X_{ij}; 1 \leqslant i,j \leqslant n\}, 1/\det(X_{ij})]$$

of algebraic functions on $\mathrm{GL}(n)$ over $k$. Thus the left action of $\mathrm{GL}(n,k)$ on $R$ is $(g_\ell \cdot f)(x) = f(g^{-1}x)$ ($g \in \mathrm{GL}(n,k)$, $f \in R$). If $f = X_{ij}$ and $x$ is $M = (X_{ij})$ in $\mathrm{M}(n,R)$, we have $g_\ell \cdot X_{ij} = (g^{-1}M)_{ij}$, or $g_\ell \cdot M = g^{-1}M$. The right action $(g_r \cdot f)(x) = f(xg)$ with $f = X_{ij}$ and $x = M$ becomes $g_r \cdot X_{ij} = (Mg)_{ij}$ or $g_r \cdot M = Mg$. These actions clearly commute.

Let ${}^{\overline{N}}R$ denote the sub-$k$-algebra of $R$ consisting of the elements of $R$ fixed by left multiplication by elements of $\overline{N}$. The $X_{1j}$ lie in ${}^{\overline{N}}R$. To construct other elements of ${}^{\overline{N}}R$ consider, for any commutative ring $A$, the representation $\mathrm{GL}(A^n) \to \mathrm{GL}(\bigwedge^i(A^n))$, $1 \leqslant i \leqslant n$. Let $e_1,\dots,e_n$ denote the standard basis of $A^n$. A basis of $\bigwedge^i(A^n)$ is given by $e_{j_1} \wedge e_{j_2} \wedge \dots \wedge e_{j_i}$ with $j_1 < j_2 < \dots < j_i$, ordered lexicographically on the $i$-tuples $j = (j_1,\dots,j_i)$. The choice of bases permits writing the map as $\rho_i : \mathrm{GL}(n,A) \to \mathrm{GL}\left(\binom{n}{i}, A\right)$. Then $\rho_i$ maps upper (resp. lower) unipotents of $\mathrm{GL}(n,A)$ to such elements of $\mathrm{GL}\left(\binom{n}{i}, A\right)$. We use this with $A = R$. Then $\rho_i(\overline{N})$ consists of lower triangular unipotent matrices in $\mathrm{GL}\left(\binom{n}{i}, A\right)$. Hence left multiplication by $\rho_i(\overline{N})$ fixes the top row of $\rho_i(M)$. Denote this top row by $(Y_{i,1},\dots,Y_{i,j},\dots,Y_{i,\binom{n}{i}})$. These $Y_{ij}$ are the determinants of the size $i$ minors in $M$ formed from the top $i$ rows and the columns $(j_1,\dots,j_i)$ if $j = (j_1,\dots,j_i)$, $1 \leqslant j \leqslant \binom{n}{i}$, in the lexicographic order. Note that $Y_{1j} = X_{1j}$, $Y_{n1} = \det M$.

The diagonal torus $T$ in $\mathrm{GL}(n,k)$ normalizes $\overline{N}$. Hence $T$ acts from the left on ${}^{\overline{N}}R$. The $n$-tuple $m = (m_1,\dots,m_n) \in \mathbb{Z}^n$ defines a weight (= character) $m : T \to k^\times$, $m(\mathrm{diag}(t_1,\dots,t_n)) = \prod_{1\leqslant i\leqslant n} t_i^{m_i}$. If $\delta_i = (0,\dots,0,1,0,\dots,0)$ with 1 at the $i$th place, $t \in T$, then $t_\ell \cdot X_{ij} = \delta_i^{-1}(t)X_{ij}$, $t_r \cdot X_{ij} = \delta_j(t)X_{ij}$. Hence $t_\ell \cdot Y_{ij} = (\prod_{1\leqslant a\leqslant i} \delta_a(t)^{-1})Y_{ij}$, $t_r \cdot Y_{ij} = (\prod_{1\leqslant a\leqslant i} \delta_{j_a}(t))Y_{ij}$. In particular $t_r \cdot Y_{i1} = (\prod_{1\leqslant a\leqslant i} \delta_a(t))Y_{i1}$.

An element $f \in R$ has *left weight* $m$ if $T$ acts on $k \cdot f$ from the left by $-m$, thus $t_\ell \cdot f = t^{-m} f$. It has *right weight* $m$ if $T$ acts on $k \cdot f$ from the right by $m$, thus $t_r \cdot f = t^m f$. Denote the $k$-vector space of left weight $m$ elements in $R$ by ${}_mR$. It is finite dimensional, and $R = \oplus_m {}_mR$. Since $T$ normalizes $\overline{N}$, we have ${}^{\overline{N}}R = \oplus_{m \in \mathbb{Z}^n} {}_m^{\overline{N}}R$. Since the right and left actions commute, each ${}_m^{\overline{N}}R$ is an algebraic representation of $\mathrm{GL}(n,k)$, of finite dimension. A weight $m = (m_1, \ldots, m_n)$ is called *positive*, and we write $m \geqslant 0$, if $m_i \geqslant m_{i+1}$ for all $i$ $(1 \leqslant i < n)$. Assume $k$ is a field of characteristic zero.

**Proposition 2.1.** *We have* ${}^{\overline{N}}R = \oplus_{m \geqslant 0} {}_m^{\overline{N}}R$. *For each* $m \geqslant 0$, ${}_m^{\overline{N}}R$ *is the irreducible algebraic representation of* $\mathrm{GL}(n,k)$ *of highest right weight* $m$. *It is generated over* $k$ *by the monomials in the* $Y_{ij}$ *of left weight* $m$. *The vector of right highest weight* $m$ *is the monomial in the* $Y_{i1}$.

**Proof.** By e.g. [16, Sec. 12.1.4], the space ${}^{\overline{N}}R$ is the direct sum of all irreducible representations of $\mathrm{GL}(n,k)$, each occurring with multiplicity one. The representation of highest weight $m \geqslant 0$ on the right is the left-weight $m$ subspace. This implies the first two assertions. The $k$-vector space generated by the monomials in the $Y_{ij}$ of left weight $m$ is nonempty by the expression $t_\ell \cdot Y_{ij} = \left( \prod_{1 \leqslant a \leqslant i} \delta_a(t)^{-1} \right) Y_{ij}$. It is stable under the right action of $\mathrm{GL}(n,k)$ since $\sum_j kY_{ij}$ is stable for each $i$. This space is all of ${}_m^{\overline{N}}R$ as ${}_m^{\overline{N}}R$ is irreducible. For this right action, $\left( \prod_{1 \leqslant i < n} Y_{i,1}^{m_i - m_{i+1}} \right) Y_{n,1}^{m_n}$ has weight $m$. The other monomials of left weight $m$ have lower weight by the description of $t_r \cdot Y_{ij}$. □

The $k$-algebra ${}^{\overline{N}}R = \oplus_{m \geqslant 0} {}_m^{\overline{N}}R$ is graded by the positive weights since ${}_m^{\overline{N}}R \cdot {}_u^{\overline{N}}R \subset {}_{m+u}^{\overline{N}}R$. Then ${}^{\overline{N}}R = \oplus_{b \in \mathbb{Z}} Y_{n,1}^b \cdot R'$. Here $R' = \oplus_{m'} {}_{m'}^{\overline{N}}R$; here $m' \geqslant 0, m' = (m_1, \cdots, m_{n-1}, 0), m_i \in \mathbb{Z}$. This $R'$ is a $k$-algebra of finite type, generated by the $Y_{ij}$ with $i < n$. It is integral since $R$ is. An alternative parametrization of the weights is given by replacing $m'$ by $\mu = (\mu_1, \cdots, \mu_{n-1}), \mu_i = m_i - m_{i+1}$ $(m_n = 0, 1 \leqslant i < n)$. Then $m' \geqslant 0$ means that $\mu_i \geqslant 0$ $(1 \leqslant i < n)$. In these coordinates $Y_{ij}$ has left weight $\delta_i = (0, \cdots, 0, 1, 0, \cdots, 0)$, thus $Y_{ij} \in {}_{\delta_i}R' = {}_{d_i}R, d_i = (1, \cdots, 1, 0, \cdots, 0), i$ times 1. We denote by ${}_\mu R'$ the subspace of $R'$ of elements with left weight $m'(\mu)$, that is weight $\mu$ in the new parametrization. We say that an element of $R'$ has weight $\mu$ if it lies in ${}_\mu R'$. Note that ${}_\mu R'$ is a right $\mathrm{GL}(n,k)$-module, and from now on we write $g(f)$ for $g_r \cdot f$.

Denote by $e_1, \cdots, e_n$ the standard basis of $k^n$, and by

$$Z_{ij} = e_{j_1} \wedge e_{j_2} \wedge \cdots \wedge e_{j_i}, \quad j = (j_1, j_2, \cdots, j_i), \quad 1 \leqslant j_b \leqslant n,$$

the basis of $\bigwedge^i(k^n)$ where the $j$ $(1 \leqslant j \leqslant \binom{n}{i})$ are ordered lexicographically. The map $\varphi : Z_{ij} \mapsto Y_{ij}$, $\bigwedge^i(k^n) \to {}_{\delta_i}R' = \sum_{1 \leqslant j \leqslant \binom{n}{i}} kY_{ij}$, is onto, and $\mathrm{GL}(n,k)$-equivariant. It extends to an equivariant surjection

$$\varphi : \mathrm{S} = \mathrm{Sym}\left( \bigoplus_{1 \leqslant i < n} \bigwedge^i(k^n) \right) = k[Z_{ij}] = \oplus_\mu \, {}_\mu S \to R' = \oplus_\mu \, {}_\mu R',$$
$$\mu \in \mathbb{Z}_{\geqslant 0}^{n-1}.$$

It is onto since the image of $\varphi$ contains all $Y_{ij}$ $(i < n)$. Denote by $J$ the *kernel of* $\varphi$; it is stable under the $\mathrm{GL}(n,k)$ action.

Fix a prime number $p$. We require working with a product situation. The index $r$ below would range over $1 \leqslant r \leqslant d$. Let $F_{p_r}$ be a finite field extension of $\mathbb{Q}_p$ (eventually we shall take $F_{p_r} = \mathbb{Q}_p$ for all $r$). Denote by $p_r$ also a generator of the maximal ideal $F_{p_r}^{00}$ of the ring of integers $F_{p_r}^0$ of $F_{p_r}$. Denote by $I_r$ or $\Gamma_0(p_r)$ the Iwahori subgroup of $\mathrm{GL}(n, F_{p_r}^0)$ consisting of the matrices which $\bmod\, p_r$ are upper triangular. For any sequence $a = a_{(r)} = (a_{r1}, \cdots, a_{rn})$ of non negative integers with $a_{r,i} \leqslant a_{r,i+1}$ $(1 \leqslant i < n)$, put $p_r^{a_{(r)}} = \mathrm{diag}(p_r^{a_{r1}}, \cdots, p_r^{a_{rn}})$ and consider $I_r p_r^a I_r \subset \mathrm{GL}(n, F_{p_r}) \cap M(n, F_{p_r}^0)$. Put $p_r^F$ for $p_r^{a_0}, a_0 = \mathrm{diag}(0, 1, \cdots, n-1)$. Put $\{a\}$ for the semigroup of all $a, p_r^{\{a\}}$ for the semigroup of all $p_r^a, I_r p_r^{\{a\}} I_r$ for the semigroup of all $I_r p_r^{a_r} I_r$. Note that $I_r p_r^a I_r \cap I_r p_r^{a'} I_r = \emptyset$ if $a \neq a'$ and $I_r p_r^a I_r \cdot I_r p_r^{a'} I_r = I_r p_r^{a+a'} I_r$.

Further we put $I_F = \prod_r I_r$; $a_F = (a_{(r)})$; $I_F p^{a_F} I_F = \prod_r I_r p_r^{a_{(r)}} I_r$; $p^F = \prod_r p_r^{a_0}$, and $I_F p^{\{a_F\}} I_F$ for the set of all $I_F p^{a_F} I_F$. We also write $I$ for any of the $I_r, F_p$ for any $F_{p_r}$, and $p$ for any $p_r, a$ for any of the $a_{(r)}$.

We continue with any factor, so that the index $r$ is now omitted.

**Lemma 2.2.** *For $\gamma \in I$ and any $i$ $(1 \leqslant i < n)$,*

$$\gamma(Z_{i,1}) \in F_p^{0\times} Z_{i,1} + p \sum_{j>1} F_p^0 Z_{i,j}, \quad 1 < j \leqslant \binom{n}{i},$$

*and*

$$\gamma(Z_{ij}) \in \sum_{j'} F_p^0 Z_{ij'} \quad \left(1 \leqslant j' \leqslant \binom{n}{i}\right).$$

*If* $j = (j_1, \cdots, j_i)$, $p^a(Z_{ij}) = p^{\sum_b a_{j_b}} Z_{ij}$, $1 \leqslant b \leqslant i$. *In particular* $p^a Z_{ij} \in p^{a_1+\cdots+a_i} F_p^0 Z_{ij}$ *and* $p^{a_0} Z_{ij} \in p^{i(i-1)/2} F_p^0 Z_{ij}$.

**Proof.** Recall that $Z_{ij} = e_{j_1} \wedge \cdots \wedge e_{j_i}$. The lemma follows from matrix multiplication. □

The ring $S = \mathrm{Sym}\big(\bigoplus_{1 \leqslant i < n} \bigwedge^i (F_p^n)\big) = \bigotimes_{1 \leqslant i < n} \mathrm{Sym}(\bigwedge^i (F_p^n))$ is the ring of multihomogeneous coordinates on the projective space $X = \prod \mathbb{P}_{F_p}^{\binom{n}{i}-1}$. Each $\mathbb{P}^{\binom{n}{i}-1}$ has homogeneous coordinates $(Z_{i,1} : \cdots : Z_{i,\binom{n}{i}})$. Recall that $J$ is the *kernel of the* $F_p$*-algebra morphism* $\varphi : S \to R'$. It is a multihomogeneous ideal. Let $\widetilde{J}$ be the coherent sheaf of ideals on $X$ associated to $J$. Let $Y$ denote the *closed subscheme of* $X$ *defined by* $\widetilde{J}$. Let $i : Y \hookrightarrow X$ be the underlying closed immersion. The group $\mathrm{GL}(n, F_p)$ acts naturally on $F_p^n, S, X, R'$ and $Y$ since $\varphi$ is $\mathrm{GL}(n, F_p)$-equivariant. The projections $\mathrm{pr}_i : X \to \mathbb{P}(\bigwedge^i (F_p^n))$ can be used to define the invertible sheaf $\mathcal{O}(\mu) = \mathrm{pr}_1^*(\mathcal{O}_{(\mu_1)}) \otimes \cdots \otimes \mathrm{pr}_{n-1}^*(\mathcal{O}_{(\mu_{n-1})})$ on $X$ and $\mathcal{O}_Y(\mu) = i^*\mathcal{O}(\mu)$ on $Y$. Recall that ${}_\mu R'$, the $\mu$-homogeneous part of $R'$, is the irreducible representation of $\mathrm{GL}(n, F_p)$ of highest weight $\mu$.

**Lemma 2.3.** *The canonical map* ${}_\mu R' \to H^0(Y, \mathcal{O}_Y(\mu))$ *is an isomorphism.*

**Proof.** For any $(n-1)$-tuple $s = (s_1, \cdots, s_{n-1})$ of integers $1 \leqslant s_b \leqslant \binom{n}{i}$, put $Z_s = \prod_{1 \leqslant i < n} Z_{i,s_i}$ and $X'_s = \{Z_s \neq 0\}$. The $X'_s$ define an open affine cover of $X$. Let $R'_{\varphi(Z_s)}$ be the localization of $R'$ at $\varphi(Z_s)$. It is graded by $\mathbb{Z}^{n-1}$. The canonical map (where $\mu = (\mu_1, \cdots, \mu_{n-1}) \in \mathbb{Z}^{n-1}, \mu_i \geqslant 0$)

$$\{x \in R'_{\varphi(Z_s)}; \deg(x) = \mu\} \to H^0(Y \cap X'_s, \mathcal{O}_Y(\mu))$$

is an $F_p$-linear isomorphism by construction. Since $R'$ is integral and $Y_{ij} = \varphi(Z_{ij})$ is nonzero, the fraction ring $R'_{\varphi(Z_s)}$ is a graded subring of the ring of fractions $R'_{\prod_{ij} Y_{i,j}}$. Each $Y_{ij}$ generates a prime ideal in $R'$, since the minor $Y_{ij}$ is irreducible in the factorial ring $R$, and $(Y_{ij}R) \cap R' = Y_{ij}R'$. The $Y_{ij}R'$

are pairwise distinct. Hence the intersection of the $\{x \in R'_{\varphi(Z_s)}; \deg(x) = \mu\}$ in $R'_{\prod_{ij} Y_{ij}}$ is ${}_{\mu}R'$, as asserted. □

Let $X'$ be the *open affine subset of* $X$ *defined by* $\prod_{1\leqslant i<n} Z_{i,1} \neq 0$. The coordinates $z_{ij} = Z_{ij}/Z_{i1}$ $\left(1 \leqslant j \leqslant \binom{n}{i}\right)$ identify $X'$ with $\prod_{1\leqslant i<n} \mathbb{A}_{F_p}^{\binom{n}{i}-1}$. The *ideal* $J' = H^0(X', \widetilde{J})$ *defines* $Y' = Y \cap X'$ in $X'$. Denote the image of $z_{ij}$ in $A(X')/J'A(X') = A(Y')$ by $y_{ij}$, where $A(X')$ denotes the *ring of regular functions on* $X'$ $(= H^0(X', \mathcal{O}_{X'}))$.

Given a scheme $Z$ of finite type over $F_p$, denote by $Z^{\mathrm{an}}$ the *associated k-rigid space*. Let $X_1$ denote the *affinoid subspace of* $X'^{\mathrm{an}} = \prod_{1\leqslant i<n} \mathbb{A}_{F_p}^{\binom{n}{i}-1,\mathrm{an}}$ *defined by* $|z_{ij}| \leqslant 1$. Let $A(X_1)$ be the affinoid algebra of rigid functions on the affinoid space $X_1$. The affinoid algebra $A(X_1) = F_p\langle z_{ij}\rangle$ has the sup norm, or Gauss norm $|\sum_{ijn} a_{ijn} z_{ij}^n| = \max_{i,j,n} |a_{ijn}|$, and ring $A(X_1)^0 = F_p^0\langle z_{ij}\rangle$ of integers (elements bounded by 1 in norm).

Put $Y_1 = X_1 \cap Y^{\mathrm{an}}$. Then $A(Y_1) = A(X_1)/J' \cdot A(X_1)$ has the quotient norm from $A(X_1)$. The point $e = (y_{ij} = 0)$ lies in $Y_1$. The action of $\mathrm{GL}(n, F_p)$ on $X^{\mathrm{an}}$ preserves $Y^{\mathrm{an}}$. Moreover,

**Lemma 2.4.** *The restriction of the action to the Iwahori subgroup* $I = \Gamma_0(p)$ *preserves* $X_1$ *and* $Y_1$*. The induced action on the affinoid algebras* $A(X_1)$ *and* $A(Y_1)$ *preserves* $A(X_1)^0$ *and* $A(Y_1)^0$.

**Proof.** The homogeneous coordinates of the $i$th factor $\mathbb{P}^{\binom{n}{i}-1}$ of $X$ are $(Z_{i,1} : \cdots : Z_{i,\binom{n}{i}})$. An element $\gamma \in I$ acts by

$$\gamma(Z_{ij}) = \sum_{1\leqslant s\leqslant\binom{n}{i}} a_{i,j,s} Z_{i,s}; \quad a_{i,j,s} \in F_p^0, \quad a_{i,1,s} \in pF_p^0$$

if $s > 1$ and $a_{i,1,1} \in F_p^{0\times}$. For $x$ in $X_1(\mathbb{C}_p) \subset X'\mathrm{an}(\mathbb{C}_p)$ we have $Z_{i,1}(x) \neq 0$. We may rescale to assume $Z_{i,1}(x) = 1$. Hence $|Z_{i,j}(x)| \leqslant 1$ (by definition of $X_1$) for all $j \leqslant \binom{n}{i}$. As $\gamma(Z_{i,1}) \equiv a_{i,1,1} Z_{i,1} \bmod \mathbf{p}F_p^0$, where $a_{i,1,1} \in F_p^{0\times}$, and $\gamma \in \mathrm{GL}(n, F_p)$ acts on a rational function $f$ on $X(\mathbb{C}_p)$ by $(\gamma \cdot f)(x) = f(x\gamma)$, we have $|Z_{i,1}(x\gamma)| = 1$. In particular it is nonzero, and the coordinates $(\gamma \cdot z_{ij}(x)) = (z_{ij}(x\gamma); 1 \leqslant j \leqslant \binom{n}{i})$ of $x\gamma$ are

$$\frac{\sum_{1\leqslant s\leqslant \binom{n}{i}} a_{i,j,s} z_{i,s}(x)}{\sum_{1\leqslant s\leqslant \binom{n}{i}} a_{i,1,s} z_{i,s}(x)} \qquad (z_{i1} = 1 \text{ and } a_{i,1,1} \in F_p^{0\times}).$$

The denominator is 1 in the absolute value, hence $x\gamma$ is in $X_1(\mathbb{C}_p)$, and $A(X_1)^0 = F_p^0\langle z_{ij}\rangle \subset A(X_1)$ is preserved under the induced action of $I$.

The assertion for $Y_1$ follows since $\mathrm{GL}(n, F_p)$ preserves $Y^{\mathrm{an}}$, and since $A(Y_1)$ has the quotient norm of $A(X_1)$. The set of values of the norm on $A(X_1)$ is discrete in $\mathbb{R}$; hence, $A(X_1)^0 \to A(Y_1)^0$ is onto. Hence the action of $I$ preserves $A(Y_1)^0$. □

In fact the claim of Lemma 2.4 holds for $p^a$ and $Ip^{\{a\}}I$ (they stabilize $X_1, A(X_1), A(X_1)^0$), since $p^a(z_{ij}) = p^{\sum_{1\leqslant b\leqslant i}(a_{j_b} - a_b)} z_{i,j} \in F_p^0 z_{ij}$.

**Lemma 2.5.** *The Banach spaces $A(X_1), A(Y_1)$ are orthonormalizable* (ONable) *over $F_p^0$.*

**Proof.** An orthonormal (ON) basis for $A(X_1) = F_p\langle z_{ij}\rangle$ over $F_p$ is given by 1 and the monomials in the $z_{i,j}$. An ideal such as $J'A(X_1)$, in an affinoid algebra, such as $A(X_1)$, is closed. Each of the $F_p$-Banach spaces $J'A(X_1)$ and $A(Y_1) = A(X_1)/J'A(X_1)$, with the norm induced from $A(X_1)$, satisfy the requirement $(N)$ of [21], since $|A(X_1)| = |F_p|$. Hence they are ON-able by [21]. Moreover, there is an isometric section to $A(X_1) \to A(Y_1)$, identifying $A(X_1)$ with $J'A(X_1) \oplus A(Y_1)$. □

**Lemma 2.6.** *The Iwahori subgroup $I$ acts linearly continuously on $A(X_1)$ and $A(Y_1)$ by operators of norm $\leqslant 1$. The $p^a$ are completely continuous on $A(X_1)$ and $A(Y_1)$ if $a$ is strictly increasing.*

**Proof.** Since $\gamma \in I$ acts on $A(X_1)$ preserving the unit ball $A(X_1)^0$, it acts by an endomorphism of norm $\leqslant 1$. The same applies to $A(Y_1)$, which is a quotient of $A(X_1)$. Each $p^a$ is diagonal in the basis of monomials in the $z_{ij}$. It acts by multiplication by a non negative power of $p$, hence has norm $\leqslant 1$. The norm is $< 1$ and $p^a$ is completely continuous precisely when $a$ is strictly increasing, using $p^a(z_{ij}) = p^{\sum_{1\leqslant b\leqslant i}(a_{j_b} - a_b)} z_{ij}$. □

**Corollary 2.7.** *For $\gamma_1, \gamma_2 \in I$ and increasing $a$ we have*

$$\gamma_1 p^a \gamma_2(z_{ij}) = \frac{\alpha_1 + \sum_{2 \leqslant s \leqslant \binom{n}{i}} \alpha_s z_{i,s}}{1 + p \sum_{2 \leqslant s \leqslant \binom{n}{i}} \beta_s z_{is}}, \qquad \alpha_s, \beta_s \in F_p^0.$$

*If $a$ is strictly increasing then $\alpha_s \in pF_p^0 = F_p^{00}$ for all $s > 1$.*

**Proof.** Here $\gamma_1$, $\gamma_2$ are upper triangular $\bmod p$, and $p^a$ acts as multiplication by $p^{\sum_{1 \leqslant b \leqslant i} a_b} v$, where $v$ is a diagonal matrix with coefficients of the form $p^\alpha, \alpha \geqslant 0$ in $\mathbb{Z}$. When $a$ is strictly increasing, $v$ is $\mathrm{diag}(1, 0, \cdots, 0) \bmod p$. Thus $p^{-\sum_{1 \leqslant b \leqslant i} a_b} \gamma_1 p^a \gamma_2$ acts on $Z_{ij}$ as an integral matrix which is upper triangular $\bmod p$ and whose first entry is a unit. When $a$ is strictly increasing, all entries except those in the first row, are zero $\bmod p$. Then $\gamma_1 p^a \gamma_2(z_{ij}) = p^{-\sum a_b} \gamma_1 p^a \gamma_2(Z_{ij}) / p^{-\sum a_b} \gamma_1 p^a \gamma_2(Z_{i1})$, as required. □

Given $\mu = (\mu_1, \cdots, \mu_{n-1}), \mu_i \in \mathbb{Z}_{\geqslant 0}$, the $\mathcal{O}_X$-module $\mathcal{O}(\mu)$ is locally free of rank one, generated by its global sections. These make the $F_p[\mathrm{GL}(n, F_p)]$-module ${}_\mu S$, by definition. Moreover, the restriction $\mathcal{O}'(\mu)$ of $\mathcal{O}(\mu)$ to $X'$ is free over $X'$ of rank one, generated by $e_\mu = \prod_{1 \leqslant i < n} Z_{i,1}^{\mu_i}$. The restriction to $X_1$ is free of rank one, denoted by $\mathcal{O}(\mu)_1$. We endow $\mathcal{O}(\mu)_1$ with the norm defined by $|e_\mu f| = |f|$. The map $\psi_\mu : A(X_1) \to \mathcal{O}(\mu)_1$, $\psi_\mu(f) = e_\mu f$, is an isometry, by definition.

The restriction $\mathcal{O}_Y(\mu) = i^* \mathcal{O}(\mu)$ of $\mathcal{O}(\mu)$ to $Y$ is an invertible sheaf whose module of global sections is the $F_p[\mathrm{GL}(n, F_p)]$-module ${}_\mu R'$ by Lemma 2.3. Its restriction to $Y_1$ is free of rank 1 over $A(Y_1)$, denoted $\mathcal{O}_Y(\mu)_1$, a basis consists of the section $e_{\mu,Y} = \prod_{1 \leqslant i < n} Y_{i,1}^{\mu_i}$. Define a norm on $\mathcal{O}_Y(\mu)_1$ by $|e_{\mu,Y} f| = |f|, f \in A(Y_1)$. The $\psi_\mu : A(Y_1) \to \mathcal{O}_Y(\mu)_1, f \mapsto e_{\mu,Y} f$, is an isometry.

Since for $\gamma \in I$ we have $\gamma(Z_{i1}) = Z_{i1} c_i(\gamma)$, where

$$c_i(\gamma) = a_{i11}(\gamma) + \sum_{1 < j \leqslant \binom{n}{i}} p a_{i1j}(\gamma) z_{ij}, \quad a_{i1j} \in F_p^0 \quad \text{and} \quad a_{i11} \in F_p^{0\times},$$

we see that $\gamma'(e_\mu f) = e_\mu \cdot \prod_{1 \leqslant i < n} c_i(\gamma')^{\mu_i} \cdot \gamma'(f)$ for any $\gamma' = \gamma_1 p^a \gamma_2, f \in A(X_1)$. Here $c_i$ are 1-cocycles $Ip^{\{a\}}I \to F_p\langle z_{ij}; 1 < j \leqslant \binom{n}{i} \rangle^\times \subset A(X_1)^\times$, thus $c_i(\gamma\gamma') = c_i(\gamma)\cdot\gamma(c_i(\gamma'))$. Their restrictions to $I$ are independent of $\mu$ and take values in $F_p^0\langle z_{ij}; 1 < j \leqslant \binom{n}{i}\rangle^\times \subset A(X_1)^{0\times}$. This gives an action of $Ip^{\{a\}}I$ on $\mathcal{O}(\mu)_1$, and using the projection $A(X_1) \to A(Y_1)$, replacing $z_{ij}$ by $y_{ij}$, also on $\mathcal{O}_Y(\mu)_1$.

Put $[\gamma]_\mu(f) = \left(\prod\limits_{1\leqslant i<n} c_i(\gamma)^{\mu_i}\right)\gamma(f)$, $\gamma \in Ip^{\{a\}}I$. Then $\psi_\mu : A(X_1) \to \mathcal{O}(\mu)_1$ is $Ip^{\{a\}}I$-equivariant where $A(X_1)$ is given the action $[\cdot]_\mu$. Namely, $\psi_\mu$ is an isomorphism of $F_p[I]$-modules. As $[I]_\mu$ preserves $J'A(X_1)$, the action $[\cdot]_\mu$ applies to the quotient $A(Y_1)$, and $\psi_\mu : A(Y_1) \to \mathcal{O}_Y(\mu)_1$ is still an isomorphism of $F_p[I]$-modules.

**Lemma 2.8.** *The natural maps* ${}_\mu S \to \mathcal{O}(\mu)_1$ *and* ${}_\mu R' \to \mathcal{O}_Y(\mu)_1$ *are injective for all* $\mu = (\mu_1, \cdots, \mu_{n-1}), \mu_i \geqslant 0$.

**Proof.** The first map is obtained from $F_p[z_{ij}] \hookrightarrow F_p\langle z_{ij}\rangle$. For the second note that ${}_\mu R'$ is an irreducible representation of $gl(n, F_p)$ over $F_p$. But $gl(n, F_p)$ is the Lie algebra also of the open subgroup $I$ of $\mathrm{GL}(n, F_p)$, so that ${}_\mu R'$ is an irreducible representation of $I$. The map ${}_\mu R' \to \mathcal{O}_Y(\mu)_1$ is $I$-equivariant, hence injective since it is nonzero: $e_{\mu,Y}$ is in its image.□

From now on we write $\mu' \in \mathbb{Z}_{\geqslant 0}^{n-1}$ for what was denoted by $\mu$ until now. Given $\mu = (\mu', \mu_n) \in \mathbb{Z}_{\geqslant 0}^{n-1} \times \mathbb{Z}$, write ${}_\mu S$, $\mathcal{O}(\mu)_1$, ${}_\mu R'$, for each of the $I$-modules ${}_{\mu'}S$, $\mathcal{O}(\mu')_1$, ${}_{\mu'}R'$, tensored by $\det^{\mu_n}$. Given $\mu_F = (\mu_{(r)}), \mu_{(r)} \in \mathbb{Z}_{\geqslant 0}^{n-1} \times \mathbb{Z}, 1 \leqslant r \leqslant d, 1 \leqslant i \leqslant n$, put ${}_{\mu_F}S = \otimes_r\, {}_{\mu_{(r)}}S, \mathcal{O}(\mu_F)_1 = \otimes_r \mathcal{O}(\mu_{(r)})_1$, etc. We shall consider below a totally real field $F$ of degree $d = [F : \mathbb{Q}]$ over $\mathbb{Q}$, and a prime $p$ which splits completely in $F$, into $p_r$. We shall need to consider the products of all objects associated with $\mu_F$, a $d$-tuple, with components indexed by $r$, $1 \leqslant r \leqslant d$.

Put $\mathbf{p} = 4$ if $p = 2, \mathbf{p} = p$ if $p \neq 2$. Denote by $I_1 = \Gamma_1(\mathbf{p})$ the tame group (see [14]), namely the group of matrices in $\mathrm{GL}(n, F_p^0)$ which $\mathrm{mod}\, \mathbf{p}$ are unipotent upper triangular. Then $I_1$ is a normal subgroup of $I$, $I/I_1 = (F_p^0/\mathbf{p})^{\times n}$. Put $\Delta^n = \mathrm{Hom}((F_p^0/\mathbf{p})^{\times n}, F_p^{0\times})$ and $\Delta = \Delta^1$. If $V$ is a representation of $F_p^0[I], \chi \in \Delta^n$, put $V_\chi$ or $V(\chi)$ for $V \otimes \chi$.

Recall that $p^a$ acts on ${}_\mu S$ by $p^a(e_\mu f) = e_\mu \nu_\mu(p^a)^{-1} \cdot p^a(f)$, where $\nu_\mu(p^a)^{-1} = p^{\sum_{1\leqslant i\leqslant n} \mu_i \sum_{1\leqslant b\leqslant i} a_b}$. Note that $i = n$ is included; it accounts for twisting with $\det^{\mu_n}$. This defines a character $\nu_\mu$ of the monoid $\{p^a; 0 \leqslant a_i \leqslant a_{i+1}\}$. This character is the restriction to $p^{\{a\}}$ of the inverse of the character of the representation of highest weight of ${}_\mu R'$. Then $\nu_\mu$ extends by 1 on $I$ to $Ip^{\{a\}}I$. Thus we can twist $\rho_\mu : Ip^{\{a\}}I \to \mathcal{O}(\mu)_1$

by $\nu_\mu$. This normalizes the action of $p^a$ so that $(\rho_\mu \otimes \nu_\mu)(p^a)(e_\mu f) = e_\mu p^a(f)$. Via $\psi_\mu : A(X_1) \to \mathcal{O}(\mu)_1$ we obtain an action $[\cdot]_\mu$ on $A(X_1)$. Put $c_i' = c_i \otimes \nu_\mu, c_i'$ equals $c_i$ on $I$. Thus for $\gamma = \gamma_1 p^a \gamma_2$; $\gamma_1, \gamma_2 \in I$; we have $[\gamma]_\mu(f) = \prod_{1 \leqslant i \leqslant n} c_i'(\gamma)^{\mu_i} \cdot \gamma(f)$; if $i < n, c_i'(\gamma) = \lambda + p \sum_{1 < j \leqslant \binom{n}{i}} d_j z_{ij}$ $(d_j \in F_p^0, \lambda \in F_p^{0\times})$; $c_n'(\gamma) = \det\gamma \in F_p^{0\times}$; $c_i'(p^a) = 1$ for all $i$ $(1 \leqslant i \leqslant n)$ (renormalization).

When $\mu = 0, [\cdot]_0$ coincides with the natural action of $Ip^{\{a\}}I$ on $A(X_1)$. In general $[\cdot]_\mu$ is the representation $[\cdot]_0$ twisted by the 1-cocycle $\prod_{1 \leqslant i \leqslant n} c_i'^{\mu_i}$. These actions preserve $J' \cdot A(X_1)$, hence transfer to the quotient $A(Y_1)$. Via $\psi_\mu$ they correspond to actions on $\mathcal{O}_Y(\mu)_1$.

**Lemma 2.9.** *Fix $\omega \in \mathbb{C}_p$ with $\omega^{p-1} = -p$ if $p > 2, \omega = 2$ if $p = 2$. Let $A$ be a commutative $k$-algebra, $f \in A$ with $|f| \leqslant |\mathbf{p}|, x \in A$ with $|x| < |\frac{\omega}{\mathbf{p}}|$. Then $(1+f)^x = 1 + \sum_{n \geqslant 1} \binom{x}{n} f^n$ converges in $A$, where $\binom{x}{n} = x(x-1)\cdots(x-n+1)/n!$.*

*One has: $(1+f)^x(1+f)^{x'} = (1+f)^{x+x'}$; $[(1+f)(1+f')]^x = (1+f)^x(1+f')^x$; $(1+f)^x = \exp(x \cdot \ln(1+f))$; $|(1+f)^x - 1| < p^{-\frac{1}{p-1}}$, in particular $|(1+f)^x| = 1$.*

**Proof.** If $|x| \leqslant 1, |x(x-1)\cdots(x-n+1)f^n/n!| \leqslant |\mathbf{p}/\omega|^n$ and the series converges since $|\mathbf{p}| < |\omega|$. If $|x| > 1$ then $|x(x-1)\cdots(x-n+1)f^n/n!| \leqslant |\mathbf{p}x/\omega|^n|\omega^n/n!|$, the series converges as $|\mathbf{p}x/\omega| < 1$. □

Recall that $F_p^{0\times} = \mu(F_p) \times (1 + \mathbf{p}F_p^0)$, the group $\mu(F_p)$ of roots of 1 in $F_p$ is isomorphic to $(F_p^0/\mathbf{p})^\times$. Let $\tau : (F_p^0/\mathbf{p})^\times \to F_p^{0\times}$ be the Teichmüller monomorphism. The cocycle $c_i' : Ip^{\{a\}}I \to F_p^0\langle z_{ij}\rangle^\times$ has its image, mod $\mathbf{p}$, in $(F_p^0/\mathbf{p})^\times$. Hence we put $\mathbf{c}_i = \tau(c_i' \bmod \mathbf{p})^{-1}c_i'$. Its image is now in $1 + \mathbf{p}A(X_1)^0$. We now assume that $F_p$ is unramified over $\mathbb{Q}_p$, so that $\mathbf{p}$ can be taken to be the rational number $p$ (or 4 if $p = 2$), as in the lemma.

The *space of weights* is $W^+(\mathbb{C}_p) = \mathrm{Hom}_{\mathrm{cts}}(\mathbb{Z}_p^{\times,n}, \mathbb{C}_p^\times)$. Thus each weight is a continuous character $\mathbb{Z}_p^{\times,n} \to \mathbb{C}_p^\times$. As $\mathbb{Z}_p^\times = (\mathbb{Z}/\mathbf{p})^\times \times (1 + \mathbf{p}\mathbb{Z}_p)$, we may write $W^+(\mathbb{C}_p)$ as the product of $\Delta^n = \mathrm{Hom}((\mathbb{Z}/\mathbf{p})^{\times,n}, \mathbb{Z}_p^\times)$ and $W(\mathbb{C}_p) = \mathrm{Hom}_{\mathrm{cts}}((1 + \mathbf{p}\mathbb{Z}_p)^n, \mathbb{C}_p^\times)$. Now for any normed space $A$, a continuous homomorphism $1 + \mathbf{p}\mathbb{Z}_p \to A^\times$ is given by $1 + \mathbf{p} \mapsto 1 + a$, $|a| < 1$, $a$ in $A$. Put $\nabla(A) = \{1 + a; a \in A, |a| < 1\}$. Then $\nabla(A) = \mathrm{Hom}(\mathrm{Sp}A, \nabla) = \mathrm{Hom}(X, \nabla)$ if $X = \mathrm{Sp}A$, and $W(\mathbb{C}_p)$ is $\nabla(\mathbb{C}_p)^n = \mathrm{Hom}(X^n, \nabla)$, $X = \mathrm{Sp}(\mathbb{C}_p)$. Note that $1 + \mathbf{p}$ is a topological generator of the multiplicative group $1 + \mathbf{p}\mathbb{Z}_p$, and we have an isomorphism $\mathbb{Z}_p \simeq 1 + \mathbf{p}\mathbb{Z}_p$, $z \mapsto (1 + \mathbf{p})^z$. Now a continuous homomorphism

$1+\mathbf{p}\mathbb{Z}_p \to \mathbb{C}_p^\times$ is given by $x : 1+\mathbf{p}u \mapsto (1+\mathbf{p}u)^x$ where the power series expansion for $(1+\mathbf{p}u)^x$ converges when $|x| < |\omega/\mathbf{p}| = p^{(p-2)/(p-1)}$ (or 2 when $p=2$; we often ignore this case below). Thus

$$W(\mathbb{C}_p) = \{(x_1,\dots,x_n) \in \mathbb{C}_p^n;\ |x_i| < p^{(p-2)/(p-1)},\ 1 \leqslant i \leqslant n\}.$$

This is the union, over all $q$ $(1 \leqslant q < p^{(p-2)/(p-1)})$ in $\mathbb{Q}$, of the affinoids

$$W_q(\mathbb{C}_p) = \{(x_1,\dots,x_n) \in \mathbb{C}_p^n;\ |x_i| \leqslant q\,(1 \leqslant i < n)\}.$$

Choose $u_q \in \mathbb{C}_p$ with $|u_q| = q$. Let $K_q$ be a finite field extension of $\mathbb{Q}_p(u_q)$. The associated affinoid algebra is

$$\begin{aligned} A(W_q) &= K_q\langle x_i/u_q;\ 1 \leqslant i \leqslant n\rangle \\ &= \Big\{\sum_{\mathbf{m}} \alpha_{\mathbf{m}} \mathbf{x}^{\mathbf{m}};\ \alpha_{\mathbf{m}} \in K_q,\ x_i \in \mathbb{C}_p,\ |x_i| \leqslant |u_q| = q\Big\}. \end{aligned}$$

If $F$ is a totally real extension of $\mathbb{Q}$ of degree $d = [F:\mathbb{Q}]$, and $E$ of $F$ of degree $\ell = [E:F]$, and $p$ splits completely in $E$, then we have

$$W_F(\mathbb{C}_p) = \mathrm{Hom}_{\mathrm{cts}}\Big((1+\mathbf{p}\mathcal{O}_{F,p})^n = \prod_r (1+\mathbf{p}\mathcal{O}_{p_r})^n, \mathbb{C}_p^\times\Big) = \cup_q W_{q,F}(\mathbb{C}_p),$$

$$W_{q,F}(\mathbb{C}_p) = \big\{(x_{ir});\ x_{ir} \in \mathbb{C}_p,\ |x_{ir}| \leqslant q,\ 1 \leqslant i \leqslant n,\ 1 \leqslant r \leqslant d\big\},$$

$$W_E(\mathbb{C}_p) = \mathrm{Hom}_{\mathrm{cts}}\Big((1+\mathbf{p}\mathcal{O}_{E,p})^n = \prod_{r,k} (1+\mathbf{p}\mathcal{O}_{p_{r,k}})^n, \mathbb{C}_p^\times\Big) = \cup_q W_{q,E}(\mathbb{C}_p),$$

$$W_{q,E}(\mathbb{C}_p) = \{(x_{irk});\ x_{irk} \in \mathbb{C}_p,\ |x_{irk}| \leqslant q,\ 1 \leqslant i \leqslant n,\ 1 \leqslant r \leqslant d,\ 1 \leqslant k \leqslant \ell\}.$$

Suppose $E$ is cyclic over $F$. Let $\sigma$ denote a generator of $\mathrm{Gal}(E/F)$. Then $\langle\sigma\rangle$ acts on $W_{q,E}(\mathbb{C}_p)$. We may label the $x_{i,r,k}$ $(1 \leqslant k \leqslant \ell = [E:F])$ so that $\sigma(x_{i,r,k}) = x_{i,r,k+1}$. Then the set of fixed points $W_{q,E}(\mathbb{C}_p)^\sigma$ of $\sigma$ in $W_{q,E}(\mathbb{C}_p)$ consists of the $(x_{i,r,k})$ with $x_{i,r,k} = x_{i,r,k+1}$ for all $i$, $r$, $k$. The norm map $E \twoheadrightarrow F$, $1+\mathbf{p}u_{irk} \mapsto \prod_k (1+\mathbf{p}u_{irk})$, maps the character $(x_{ir}) \in W_{q,F}(\mathbb{C}_p)$, to the character $(x_{irk}) \in W_{q,E}(\mathbb{C}_p)$ with $x_{irk} = x_{ir}$ for all $k$ $(1 \leqslant k \leqslant \ell)$, as

$$\begin{aligned} &(x_{ir}) \circ N :\ (1+\mathbf{p}u_{irk}; i,r,k) \mapsto (x_{ir}) : \Big(\prod_k (1+\mathbf{p}u_{irk}); i, r\Big) \\ &\mapsto \prod_i \prod_r \big[\prod_k (1+\mathbf{p}u_{irk})\big]^{x_{ir}}. \end{aligned}$$

Thus the image of $W_{q,F}(\mathbb{C}_p)$ in $W_{q,E}(\mathbb{C}_p)$ induced from the norm map is precisely the $\sigma$-invariant points $W_{q,E}(\mathbb{C}_p)^\sigma$.

We also have a map dual to $W_{q,F}(\mathbb{C}_p) \hookrightarrow W_{q,E}(\mathbb{C}_p)$, namely $A(W_{q,E}) = K_q\langle x_{irk}/u_q\rangle \twoheadrightarrow A(W_{q,F}) = K_q\langle x_{ir}/u_q\rangle$, which is the restriction: $x_{irk} \mapsto x_{ir}$ for all $k$. Thus the series $\sum_{\mathbf{m}} \alpha(\mathbf{m}) \prod_{i,r,k} x_{irk}^{m_{irk}}$, $\mathbf{m} = (m_{irk})$, is mapped to the series $\sum_{\mathbf{m}} \alpha(\mathbf{m}) \prod_{i,r} x_{ir}^{m_{ir}}$, $\mathbf{m} = (m_{ir})$, where $m_{ir}$ is $\sum_k m_{irk}$ and $\alpha((m_{ir}))$ is $\alpha((m_{irk}))$.

Fix $q_r \in \mathbb{Q}$ with $1 \leqslant q_r < |\omega/\mathbf{p}|$ $\big( = p^{\frac{p-2}{p-1}}$ if $p \neq 2, = 2$ if $p = 2\big)$ and $u_{q_r} \in \mathbb{C}_p$ with $|u_{q_r}| = q_r$. Let $K_{q_r}$ be a finite field extension of $\mathbb{Q}_p(u_{q_r})$. The affinoid ball of $q_r$-weights $W_{q_r} = \mathrm{Sp}\, K_{q_r}\langle x_1/u_{q_r}, \cdots, x_n/u_{q_r}\rangle$ has $\mathbb{C}_p$-points

$$W_{q_r}(\mathbb{C}_p) = \{x = (x_1, \cdots, x_n) \in \mathbb{C}_p^n; |x_i| \leqslant q_r \ (1 \leqslant i \leqslant n)\}.$$

Write $q_F = (q_r; 1 \leqslant r \leqslant d), 1 \leqslant q_r \leqslant |\omega/\mathbf{p}|$, and $W_{q_F}(\mathbb{C}_p) = \prod_r W_{q_r}(\mathbb{C}_p)$. Let $K_{q_F}$ be a finite field extension of $\mathbb{Q}_p(u_{q_r}; 1 \leqslant r \leqslant d)$.

*We shall write* $W_F$, $X_{1,F}$, *and* $Y_{1,F}$ *for* $W^d$, $X_1^d$, $Y_1^d$, where $d = [F : \mathbb{Q}]$, to indicate the dependence of these objects on $F$. Then $A(X_{1,F}) = A(X_1^d)$ $= \mathbb{Q}_p\langle z_{rij};\ 1 \leqslant r \leqslant d, 1 \leqslant i \leqslant n, 1 \leqslant j \leqslant \binom{n}{i}\rangle$ is $\otimes_{1\leqslant r\leqslant d} A(X_1)$.

Consider the affinoid algebras

$$A_{q_F}(X_{1,F}) = A(W_{q_F} \times_{K_{q_F}} X_{1,F})$$

$$= K_{q_F}\langle x_{ri}/u_{q_r}, z_{rij}; 1 \leqslant i \leqslant n,\ 1 \leqslant r \leqslant d, 1 \leqslant j \leqslant \tbinom{n}{i}\rangle$$

and $A_{q_F}(Y_{1,F}) = A_{q_F}(X_{1,F})/J' \cdot A_{q_F}(X_{1,F}) = A(W_{q_F} \times_{K_{q_F}} Y_{1,F})$ with the standard Gauss norm on $A_{q_F}(X_{1,F})$ and the quotient norm on $A_{q_F}(Y_{1,F})$.

Recall that $F$ is a totally real field of degree $d = [F : \mathbb{Q}]$ over $\mathbb{Q}$, and $p$ a rational prime which splits completely in $F$. Denote the $F$-primes dividing $p$ by $p_r$ $(1 \leqslant r \leqslant d)$. Then $F_{p_r} \simeq \mathbb{Q}_p$. Write $I_F$ for $\prod_r I_r$, where $I_r$ is the Iwahori $I$ in $\mathrm{GL}(n, F_{p_r}^0)$. Put $a_F = \{a_{(r)}\}$, where $a_{(r)} = (a_{ri}; 1 \leqslant i \leqslant n), a_{r_{i+1}} \geqslant a_{r_i} \geqslant 0$ in $\mathbb{Z}$, and $I_F p^{a_F} I_F$ for $\prod_r I_r p^{a_{(r)}} I_r$. We further write $I_F p^{\{a_F\}} I_F$ for the semigroup of all $I_F p^{a_F} I_F$.

We have a constant action of $I_F p^{\{a_F\}} I_F$ on $A(X_{1,F})$, and by extending scalars to $A(W_{q_F}) = K_{q_F}\langle x_{ri}/u_{q_r}; 1 \leqslant r \leqslant d, 1 \leqslant i \leqslant n\rangle$ on $A_{q_F}(X_{1,F})$. Denote it by $I_F p^{\{a_F\}} I_F : A_{q_F}(X_{1,F}) \to A_{q_F}(X_{1,F}), (\gamma, f) \mapsto \gamma(f)$.

A more interesting action is the twisted action. For $\gamma_r \in I p^{\{a_{(r)}\}} I$, $\mathbf{c}_i(\gamma_r)^{x_{ri}}$ exists in $A_{q_F}(X_{1,F}) = K_{q_F}\langle x_{ri}/u_{q_r}, z_{rij}\rangle$ since $|\mathbf{c}_i(\gamma_r) - 1| \leqslant |\mathbf{p}|$

and $|x_{ri}| \leqslant q_r < |\omega/\mathbf{p}|$. We then introduce

$$[\gamma_r](f) = \left( \prod_{1 \leqslant i \leqslant n} \mathbf{c}_i(\gamma_r)^{x_{ri}} \right) \gamma(f).$$

By definition, or normalization, of the $\mathbf{c}_i$, we have $[p^{a_{(r)}}]f = p^{a_{(r)}}(f)$, thus the $p^{a_{(r)}}$ act as constants (no twisting). Since the constant action preserves $J' \cdot A_{q_r}(X_{1,F})$, so does the twisted action $[\cdot]$. Hence the action of $Ip^{\{a\}}I$ descends to an action on $A_q(Y_1)$ and the action of $I_F p^{\{a_F\}} I_F$ descends to an action on $A_{q_F}(Y_{1,F})$. As usual, we write ONable for orthonormalizable.

**Proposition 2.10.** *The Banach algebras $A_q(X_1)$ and $A_q(Y_1)$ are ONable over $A(W_q)$, and $A_{q_F}(X_{1,F})$, $A_{q_F}(Y_{1,F})$ are ONable over $A(W_{q_F})$.*

**Proof.** The $F_p$-Banach modules $A(X_1)$ and $A(Y_1)$ are ONable, and $A_q(X_1)$, $A_q(Y_1)$ are obtained simply by extension of scalars to $A(W_q)$. □

Moreover, we have an isometric section to $A(X_1) \to A(Y_1)$, hence an isomorphism $A(X_1) = J' \cdot A(X_1) \oplus A(Y_1)$, each being ONable over $F_p$, hence we have an isomorphism $A_q(X_1) = J' \cdot A_q(X_1) \oplus A_q(Y_1)$, each term being $A(W_q)$-ONable.

**Proposition 2.11.** *The semigroup $I_F p^{\{a_F\}} I_F$ acts $A(W_{q_F})$-linearly and continuously via $[\cdot]$ on $A_{q_F}(X_{1,F})$ and $A_{q_F}(Y_{1,F})$, by operators of norm $\leqslant 1$.*

**Proof.** The constant action is $A(W_q)$-linear, and so is the twisted action $[\cdot]$ which differs from the constant action simply by multiplication by a factor independent of $q$. This action preserves the unit ball of $A_q(X_1)$. Indeed, if $f \in A_q(X_1)^0$, then $\gamma(f) \in A_q(X_1)^0$. We just need to see that $\mathbf{c}_i(\gamma)^{x_i}$ lies in $A_q(X_1)^0$. Put $\mathbf{c}_i(\gamma) = 1 + \mathbf{p} f_i$, $f_i \in A_q(X_1)^0$, so the claim follows from Lemma 2.9. By passage to the quotient we see that $\gamma$ preserves also $A_q(Y_1)^0$. The extension to $A_{q_F}(X_{1,F})$ and $A_{q_F}(Y_{1,F})$ is immediate. □

**Proposition 2.12.** *If $a_F$ is strictly increasing, $\gamma_1 \cdot p^{a_F} \cdot \gamma_2$ acts completely continuously.*

**Proof.** The operator $p^{a_F}$ is constant. It is obtained by extension of scalars to $A(W_{q_F})$ from the operator $p^{a_F}$ on $A(X_{1,F})$ and $A(Y_{1,F})$, and $p^{a_F}$ is completely continuous on $A(X_{1,F})$ and $A(Y_{1,F})$. □

**Proposition 2.13.** *If $q' > q$ then the canonical restriction $A_{q'}(X_1)_\chi \to A_q(X_1)_\chi$ commutes with the action of $I \cdot p^{\{a\}} \cdot I$ and induces an $I \cdot p^{\{a\}} \cdot I$-equivariant isomorphism $A_{q'}(X_1)_\chi \widehat{\otimes} A(W_q) \simeq A_q(X_1)_\chi$, and similarly with $X_1$ replaced by $Y_1$.*

**Proof.** This follows from the definitions. Recall that $V_\chi = V \otimes \chi$ if $V$ is an $F_p^0[I]$-module and $\chi : I/I_1 = (F_p^0/\mathbf{p})^{\times n} \to F_p^{0\times}$ extends by 1 on $p^{\{a\}}$. □

**Proposition 2.14.** *Given $\mu = (\mu_1, \cdots, \mu_n) \in \mathbb{Z}_{\geqslant 0}^{n-1} \times Z \subset W_q(K_q)$, put $\chi = (\tau^{\mu_1+\cdots+\mu_n}, \tau^{\mu_2+\cdots+\mu_n}, \cdots, \tau^{\mu_n}) \in \Delta^n$. Then the fiber of $A_q(X_1)_\chi$ at $\mu$, thus $x_i = \mu_i$, is naturally isomorphic to $\mathcal{O}(\mu)_1 \otimes_{F_p} K_q$ (the fiber of $A_q(Y_1)_\chi$ at $\mu$ is $\mathcal{O}_Y(\mu)_1 \otimes_{F_p} K_q$).*

**Proof.** The fiber at $\mu$ of $A_q(X_1)_\chi$ is $A_q(X_1) \otimes_{F_p} K_q$ with $I \cdot p^{\{a\}} \cdot I$ action $[\cdot]_\mu$, which is $\mathcal{O}(\mu)_1 \otimes_{F_p} K_q$, except for the modification by the 1-cocycles $\mathbf{c}_i$. The map $\psi_\mu$ gives an isomorphism of this fiber at $\mu$ with $\mathcal{O}(\mu)_1 \otimes_{F_p} K_q$ twisted by $\tau(\mathbf{c}_1)^{-\mu_1} \cdots \tau(\mathbf{c}_n)^{-\mu_n} = \tau_1^{-\mu_1}(\tau_1\tau_2)^{-\mu_2} \cdots (\tau_1 \cdots \tau_n)^{-\mu_n}$, where $\tau_i : (F_p^0/\mathbf{p}F_p^0)^{\times n} \to F_p^{0\times}$ is $(a_1, \cdots a_n) \mapsto \tau(a_i)$, as required. □

**Proposition 2.15.** *The map $I \cdot p^{\{a\}} \cdot I \times A_q(X_1) \to A_q(X_1), (\gamma, f) \mapsto [\gamma](f)$, is a representation of $I \cdot p^{\{a\}} \cdot I$, and the maps*
*$I \cdot p^{\{a\}} \cdot I \to A(W_q \times_{K_q} X_1)^\times, \gamma \mapsto \mathbf{c}_i(\gamma)^{x_i}$, are 1-cocycles.*

**Proof.** To verify that $[\cdot]$ is a representation, we need to show that $[\gamma\gamma'] = [\gamma][\gamma']$, for any $\gamma, \gamma'$ in $I \cdot p^{\{a\}} \cdot I$. These two $A(W_q)$-endomorphisms of $A_q(X_1)$ coincide at all fibers $\mathbb{Z}_{\geqslant 0}^n \subset W_q(\mathbb{C}_p)$, since $[\cdot]_\mu$ are all representations of $I \cdot p^{\{a\}} \cdot I$. Since $A_q(X_1)$ is ONable, it suffices to show that the matrix coefficients of $[\gamma\gamma']$ and $[\gamma][\gamma']$ in a fixed ON basis are equal. These coefficients are then functions in $A(W_q)$ which take the same values on $\mathbb{Z}_{\geqslant 0}^n$, hence on the Zariski dense, open subset $F_p^{0n}$. Hence they are equal. A similar argument applies to the 1-cocycles $\mathbf{c}_i^{x_i}$. □

Let $X$ be a reduced rigid space over $\mathbb{C}_p$. Let $B$ be an ONable (=orthonormalizable) Banach space over $\mathbb{C}_p$. The *Banach module over $X$ associated with $B$* is the sheaf of $\mathcal{O}_X^{\mathrm{an}}$-modules denoted $B_X = B \widehat{\otimes}_{\mathbb{C}_p} \mathcal{O}_X^{\mathrm{an}}$ which at each affinoid open subset $X'$ of $X$ takes the value

$$B_X(X') = (B \widehat{\otimes}_{\mathbb{C}_p} \mathcal{O}_X^{\mathrm{an}})(X') = B \widehat{\otimes}_{\mathbb{C}_p} (\mathcal{O}_X^{\mathrm{an}}(X')).$$

Here $\mathcal{O}_X^{\mathrm{an}}(X')$ is a Banach $\mathbb{C}_p$-algebra with the sup-norm, thus $B_X(X')$ can be viewed as a Banach $\mathcal{O}_X^{\mathrm{an}}(X')$-module.

If $\{b_i; i \geqslant 1\}$ is an ON basis of $B$, $\{b_i\widehat{\otimes}1; i \geqslant 1\}$ is an ON basis of the Banach $\mathcal{O}_X^{\mathrm{an}}(X')$-module $B_X(X')$. Clearly, $B_X$ is a presheaf for the weak $G$-topology on $X$. It follows from Tate's acyclicity theorem, as we have a global ON basis, that it is in fact a sheaf for the weak $G$-topology. Hence it extends uniquely to a sheaf in the strong $G$-topology on $X$, again denoted $B_X$.

If $X'$ is any open admissible subset of $X$, the topology on $B_X(X')$ is taken to be the weakest so that all maps $B_X(X') \to B_X(X'')$ be continuous, for all open affinoid subsets $X''$ of $X'$. This coincides with the underlying Banach space topology in the case that $X'$ is affinoid.

Again, $B_X = B\widehat{\otimes}_{\mathbb{C}_p}\mathcal{O}_X^{\mathrm{an}}$. Then $B \subset B_X(X)$. The sub-$\mathcal{O}_{\mathbb{C}_p}$-module of $B_X(X)$ consisting of the sections whose restriction to each open affinoid $X'$ has norm $\leqslant 1$ is a closed submodule denoted by $B_X(X)^0$. These sections of $B_X$ will be called *integral*. This applies in particular to $\mathcal{O}_X^{\mathrm{an}}(X) \supset \mathcal{O}_X^{\mathrm{an}}(X)^0$.

For each $\mu \in X(\mathbb{C}_p)$ the fiber of $B_X$ at $\mu$, which we denote by $B_{X,\mu}$ and also $B_X(\mu)$, is a Banach space over $\mathbb{C}_p$ defined by $B_X(X')\widehat{\otimes}_{\mathcal{O}_X^{\mathrm{an}}(X')}\mathbb{C}_p$ where $X'$ is any open affinoid in $X$ containing $\mu$ and $\mathcal{O}_X^{\mathrm{an}}(X') \to \mathbb{C}_p$ is the evaluation at $\mu$. Then $B_{X,\mu} \simeq B$.

A *morphism* $\varphi : \mathbb{B}_1 \to \mathbb{B}_2$ of Banach modules over $X$ is a morphism of $\mathcal{O}_X^{\mathrm{an}}$-modules whose restriction to any open affinoid $X' \subset X$ induces a continuous map $\varphi(X') : \mathbb{B}_1(X') \to \mathbb{B}_2(X')$ (it suffices to check this for an admissible cover of $X$). The morphism $\varphi$ is called *integral* if the $\varphi(X')$ have norm $\leqslant 1$. The morphism $\varphi$ defines an $\mathcal{O}_X^{\mathrm{an}}(X)$-linear continuous map $\mathbb{B}_1(X) \to \mathbb{B}_2(X)$, sending integral sections to integral sections if $\varphi$ is integral.

Let $K$ be a finite extension of $\mathbb{Q}_p$ in $\mathbb{C}_p$. A *$K$-structure* on a Banach space $B$ is an ONable $K$-Banach space $B^K$ with an isomorphism $B^K\widehat{\otimes}_K\mathbb{C}_p \simeq B$ . For an open affinoid $X'$ of $X, B_X^K(X') = B^K\widehat{\otimes}_K\mathcal{O}_X^{\mathrm{an}}(X')$ is a $K$-structure on $B_X(X') = B\widehat{\otimes}_{\mathbb{C}_p}\mathcal{O}_X^{\mathrm{an}}(X')$. If $B$ has a $K$-structure $B^K$, and $\widetilde{F}$ is a family of open affinoids $X' \subset X$, let $B_X(X, K, \widetilde{F})$ be the closed $K$-subspace of $B_X(X)$ consisting of the sections whose restriction to any $X' \in \widetilde{F}$ is $K$-rational.

Let $\mathbb{B}_1$ and $\mathbb{B}_2$ be ONable Banach modules over $X$ with $K$-rational structures $B_1^K$, $B_2^K$ wrt a family $\widetilde{F}$ of open affinoids, thus $B_i^K(X')$ is a $K$-structure on $\mathbb{B}_i(X')$. A morphism $\varphi : \mathbb{B}_1 \to \mathbb{B}_2$ is called *$K$-rational* if for each open affinoid $X'$ in $\widetilde{F}$, $\varphi$ maps $B_1^K(X') \subset \mathbb{B}_1(X')$ to $B_2^K(X') \subset \mathbb{B}_2(X')$.

The space of weights $W$ is the (increasing) union of the affinoid spaces $W_q$, $1 \leqslant q < |\omega/\mathbf{p}|(= p^{\frac{p-2}{p-1}}$, or $= 2$ if $p = 2)$. It is a rigid space over $\mathbb{C}_p$. We need the *space* $W_F = W^d$ *of multiweights* and the *products* $X_{1,F} = X_1 \times \cdots \times X_1$ and $Y_{1,F} = Y_1 \times \cdots \times Y_1$. Examples of Banach modules over $W_F$ include $A(X_{1,F};\chi_F, W_F) = A(X_{1,F})_{\chi_F} \widehat{\otimes}_{\mathbb{C}_p} \mathcal{O}^{\text{an}}_{W_F}$ and $A(Y_{1,F};\chi_F, W_F) = A(Y_{1,F})_{\chi_F} \widehat{\otimes}_{\mathbb{C}_p} \mathcal{O}^{\text{an}}_{W_F}$. By Proposition 2.11, $I_F \cdot p^{\{a_F\}} \cdot I_F$ acts by integral Banach endomorphisms on these spaces.

The Banach module $A(Y_{1,F};\chi_F, W_F)$, viewed as a representation of the semigroup $I_F p^{\{a_F\}} I_F$, is called the *analytic family of principal series representations* of $I_F$ of character $\chi_F$, for any $\chi_F \in \Delta_F^n$. Define $\tau_{\mu_F}$ : $I_F \to \prod_r (F^0_{p_r}/\mathbf{p}_r F^0_{p_r})^{\times n} \to F_p^{0\times n}$ by

$$\prod_r (\tau_r^{-\mu_{r1}-\cdots-\mu_{rn}}, \tau_r^{-\mu_{r2}-\cdots-\mu_{rn}}, \cdots, \tau_r^{-\mu_{rn}}), \quad \tau_r : (F^0_{p_r}/\mathbf{p}_r F^0_{p_r})^{\times} \to F^{0\times}_{p_r}.$$

By Proposition 2.14, for any $\mu_F = (\mu_{(r)}), \mu_{(r)} = (\mu_{r1}, \cdots, \mu_{rn})$ in $\mathbb{Z}^{n-1}_{\geqslant 0} \times \mathbb{Z}$, the fiber $A(X_{1,F};\chi_F, W_F, \mu_F)$ is isomorphic as an $F_p[I_F \cdot p^{\{a_F\}} \cdot I_F]$-module to $\mathcal{O}(\mu_F)_{1,\chi_F \tau_{\mu_F}}$, and $A(Y_{1,F};\chi_F, W_F, \mu_F)$ to $\mathcal{O}_Y(\mu_F)_{1,\chi_F\tau_{\mu_F}}$.

The ring $\mathcal{O}^{\text{an}}_{W_F}(W_F)$ is the product of $d$ copies of $\mathcal{O}^{\text{an}}_W(W)$.

**Lemma 2.16.** *The ring $\mathcal{O}^{\text{an}}_W(W)$ is canonically isomorphic to the ring of powers series which are convergent on all of $W(\mathbb{C}_p)$, with the weak topology given by the family $|f|_q = \sup_{x \in W_q(\mathbb{C}_p)} |f(x)|$ of norms. The subring $\mathcal{O}^{\text{an}}_W(W)^0$ is isomorphic to the subring of power series convergent on $W(\mathbb{C}_p)$ and bounded by 1 there. This is a local ring, complete for the topology defined by the closed ball $I_q$ of radius $\frac{1}{2}$ in $|\cdot|_q$ centered at 0. If $x'_i = \frac{\mathbf{p}}{\omega} x_i$ $(1 \leqslant i \leqslant n)$ then $\mathcal{O}^{\text{an}}_W(W)^0 = \mathbb{C}^0_p[[x'_1, \cdots, x'_n]]$. The topology here is defined by the ideal $(p, x'_1, \cdots, x'_n)$.*

*Given an ON basis $\{b_i; i \geqslant 1\}$ of $B$ there is a topological isomorphism $B_{W_F}(W_F)^0 \simeq C^0(\mathbb{Z}_{\geqslant 0}, \mathcal{O}^{\text{an}}_{W_F}(W_F)^0)$. For any topological ring $A$, $C^0(\mathbb{Z}_{\geqslant 0}, A)$ denotes the set of $f : \mathbb{Z}_{\geqslant 0} \to A$ with $\lim_{i\to\infty} f(i) = 0$, and $B_X = B\widehat{\otimes}_{\mathbb{C}_p} \mathcal{O}^{\text{an}}_X$.*

The closed ball of radius $|\omega/\mathbf{p}|$ is defined over $L = \mathbb{Q}_p(\omega) = \mathbb{Q}_p(\mu_p)$. In the coordinates $x'_i = \frac{\mathbf{p}}{\omega} x_i$ over $L$ $(\mathbf{p}/\omega \in L^0)$, $W$ is isomorphic to the open ball with center 0 and radius 1 over $L$. Let $B_{W_F} = B\widehat{\otimes}_{\mathbb{C}_p}\mathcal{O}^{\text{an}}_{W_F}$ be a Banach module over $W_F$, and $B^L$ an $L$-structure of $B$ over $L$. As an $L$-structure on $W_F$ we take the open affinoid $W_{1,F} = W_1^d$ which is defined over $\mathbb{Q}_p$. Put $\Lambda = \mathcal{O}^{\text{an}}_{W_F}(W_F, L) \cap \mathcal{O}^{\text{an}}_{W_F}(W_F)^0$. Let $B_\Lambda = B(W_F, L) \cap B(W_F)^0$ be the module of $\Lambda$-adic sections of a Banach module $B$ over $W_F$.

**Lemma 2.17.** *The isomorphism*

$$\mathcal{O}^{\mathrm{an}}_{W_F}(W_F)^0 \simeq \mathbb{C}_p^0\big[\big[x'_{ri}; 1 \leqslant r \leqslant d, 1 \leqslant i \leqslant n\big]\big]$$

*induces a topological isomorphism* $\Lambda \simeq L^0[[x'_{ri}; 1 \leqslant r \leqslant d, 1 \leqslant i \leqslant n]]$ (= *complete local ring*). *The isomorphism*

$$B_{W_F}(W_F)^0 \simeq C^0(\mathbb{Z}_{\geqslant 0}, \mathcal{O}^{\mathrm{an}}_{W_F}(W_F)^0)$$

*induces an isomorphism of topological* $\Lambda$*-modules* $B_\Lambda \simeq C^0(\mathbb{Z}_{\geqslant 0}, \Lambda)$, *where* $B_\Lambda$ *is an ONable Banach space over* $L$.

*In particular, the maximal ideal of* $\Lambda$ *is* $m = (p, x'_{ri};\ 1 \leqslant r \leqslant d, 1 \leqslant i \leqslant n)$, *and* $B_\Lambda$ *is complete and separated in the* $m$*-adic topology.*

Let us also recall the notion of systems of *ONB-modules* (orthonormalizable Banach modules), introduced in [9, Sec. 4.3]. Fix a complete nondiscrete nonarchimedean field $K$.

A *system* of Banach $K$*-spaces* is a sequence $\mathbb{B} = \{i_n : B_n \to B_{n+1}; n \geqslant 1\}$ of compact $K$-linear maps between $K$-ONB spaces $B_n$. Put $B^+ = \varinjlim B_n$.

Let $W$ be a separated and reduced rigid $K$-space. A *sheaf* of ONB-modules over $W$ is a sheaf $C$ over $W$ such that for each open affinoid $W'$ in $W$, $C(W')$ has a structure of an ONB $A(W')$-module such that for any open affinoid $W'' \subset W'$ the canonical map $C(W')\widehat{\otimes}_{A(W')}A(W'') \to C(W'')$ is an isomorphism of Banach $A(W'')$-modules.

A *system* of Banach *modules* over $W$ is a sequence $\mathbb{B} = \{i_n : B_n \to B_{n+1}; n \geqslant 1\}$ where $B_n$ is a sheaf of ONB-modules over $W$ and $i_n$ is a morphism of sheaves such that $i_n(W') : B_n(W') \to B_{n+1}(W')$ is an $A(W')$-linear compact map for every open affinoid $W'$ in $W$. Write $B(W', n)$ for $B_n(W')$ and $\mathbb{B} = \{B(W', n); W', n \geqslant 1\}$. Put $B(W')^+$ for the $A(W')$-module $\varinjlim B(W', n)$.

If $\mathbb{B} = \{i_n : B_n \to B_{n+1}\}$ is a system of Banach spaces, the *associated* (to $\mathbb{B}$) *system* of Banach modules over $W$ is $\mathbb{B} = \{B(W', n) = A(W')\widehat{\otimes}_K B_n\}$, where $i_n(W') : B(W', n) \to B(W', n+1)$ is $\mathrm{id}_{A(W')} \otimes i_n$, the compact $A(W')$-linear map obtained by exension of scalars from $i_n$.

Let $\mathbb{B} = \{B(W', n)\}$ be a system of Banach modules over $W$. Fix $x \in W$ or $x \in W(L)$ where $L/k_x$ is finite ($k_x$ is the residue field at $x \in W$ of the ring of rigid analytic functions $A(W)$ on $W$). Denote by $\mathbb{B}_x$ the system of Banach $k_x$- (or $L$-)spaces $B_{x,n} = B(W', n)\widehat{\otimes}_{A(W')}k_x$ (or $= B(W', n)\widehat{\otimes}_{A(W')}L$) where $W'$ is any open affinoid in $W$ with $x \in W'$, and $i_{x,n} = i_n(W') \otimes \mathrm{id} : B_{x,n} \to B_{x,n+1}$ is the induced compact map. In applications, the $i_{x,n}$ are often injective.

An *endomorphism* $\beta = (\beta(W',n))$ of $\mathbb{B}$ consists of a continuous $A(W')$-linear endomorphism $\beta(W',n)$ of $B(W',n)$ for each open affinoid $W'$ in $W$ and $n$ large enough, depending on $W'$, such that the $\beta(W',n)$ commute with the maps $i_n(W') : B(W',n) \to B(W',n+1)$ and with open affinoid base change maps $B(W',n) \to B(W'',n)$ whenever the latter maps are defined. Identify $\beta$ and $\beta'$ if $\beta(W',n) = \beta'(W',n)$ for every $W'$ and $n(\gg n(W'))$. The set $\mathrm{End}(\mathbb{B})$ of these endomorphisms is a $K$-algebra.

Let $H$ be a $K$-algebra. A *representation* of $H$ on $\mathbb{B}$ is a morphism $\rho : H \to \mathrm{End}\mathbb{B}$ of $K$-algebras. In this case we say: $\mathbb{B}$ is a *system* of Banach *$H$-modules* on $W$.

Denote by $\mathrm{Comp}(\mathbb{B})$ the ideal in $\mathrm{End}(\mathbb{B})$ of the $\beta = (\beta(W',n))$ such that for every open affinoid $W'$ in $W$ and $n \gg n(W')$ there exists a continuous $A(W')$-linear endomorphism $T(W',n) : B(W',n+1) \to B(W',n)$ with $\beta(W',n) = T(W',n) \circ i_n(W')$ and $\beta(W',n+1) = i_n(W') \circ T(W',n)$. In particular $\beta(W',n)$ is compact for $n$ large enough, and one has the Fredholm series for the orthonormalizable $B(W',n)$. By [11, Proposition A2.3], $\det[I - t\beta(W',n)|B(W',n)] \in 1 + tA(W')\{\{t\}\}$ is independent of $n(\gg n(W'))$. The construction of Fredholm series commutes with open affinoid base change by [11, Lemma A2.5]; hence, with $B(W',n)$, by definition of $\mathbb{B}$. Since $W$ is separated, all these series are obtained by restriction from a unique Fredholm series $\mathrm{Fred}_{\mathbb{B}}(\beta)(t,\mu) \in 1 + tA(W)\{\{t\}\}$.

Let $\beta$ be a compact endomorphism of an ONable $K$-Banach space $B$. The characteristic series $p_\beta(t) = \det(I - t\beta)$ of $\beta$ decomposes as $\prod_{i \geqslant 0} p_i(t)^{n_i}$ where the $p_i(t) \in 1 + tK[t]$ are irreducible distinct polynomials in $K[t]$ with $p_i \to 1$ as $i \to \infty$ in the sup-norm of the coefficients on $K[t]$. By [20], $B$ decomposes as the topological direct sum of $\ker\beta$ and the finite dimensional spaces $B(p_i) = \ker(p_i^F(\beta)^{n_i})$, $p^*(t) = t^d p(1/t)$ if $\deg p = d$, and $\beta$ has characteristic polynomial $p_i^*(t)^{n_i}$ on $B(p_i)$.

Let $H$ be a $K$-algebra. Let $\rho : H \to \mathrm{End}_K B$ be a representation such that $\beta \in Z(\rho(H))$, the center of $\rho(H)$. Then $\rho(H)$ stabilizes the $B(p_i)$. Denote by $B(p_i)^{ss}$ the semisimplification of the $\rho(H)$-module $B(p_i)$, and $X_\beta(B)$ for the $\rho(H)$-module $\oplus_{i \geqslant 0} B(p_i)^{ss}$, viewed as a direct sum, with multiplicites, of irreducible $\rho(H)$-modules. Write $|X_\beta(B)|$ for the set of equivalence classes. Note that $X_\beta(B) = X_{\beta'}(B)$ if $\beta', \beta \in Z(\rho(H))$ are compact endomorphisms of $B$ which commute with $\rho(H)$ and $\ker \beta' = \ker \beta$.

Let $W$ be a reduced, separated, rigid $K$-space. Let $\mathbb{B}$ be a system of ONB-modules on $W$. Let $\rho : H \to \mathrm{End}_K(\mathbb{B})$ be a representation of a commutative $K$-algebra $H$. Suppose $\beta$ in $H$ has $\rho(\beta)$ in $\mathrm{Comp}(\mathbb{B})$. Let $p_{\rho(\beta)}(\mu, t) = \mathrm{Fred}_{\mathbb{B}}(\rho(\beta))(\mu, t) \in 1 + tA(W)\{\{t\}\}$ be the Fredholm series associated with $\mathbb{B}$ and $\rho(\beta)$, namely with $(\det[I - t\rho(\beta)|B(W', n)]; W', n)$.

The *spectral variety* $Z_{\rho(\beta)} \subset W \times \mathbb{A}^{1,\mathrm{an}}$ is defined by $p_{\rho(\beta)}(\mu, t) = 0$. There are the natural projections $(\mathrm{pr}_W, \mathrm{pr}_{\mathbb{A}}) : Z_{\rho(\beta)} \to W \times \mathbb{A}^{1,\mathrm{an}}$. This $Z_{\rho(\beta)}$ has a canonical admissible covering $\mathcal{C} = \mathcal{C}(\mathrm{Fred}_{\mathbb{B}}(\rho(\beta)))$ consisting of open affinoids $Y \subset Z_{\rho(\beta)}$ finite and flat over their image $\mathrm{pr}_W(Y)$, and open closed in $\mathrm{pr}_W^{-1}(\mathrm{pr}_W(Y))$ (see [10, Proposition A5.8]; [3, Sec. 4] for general $W/K$). Using this covering $\mathcal{C}$, [9; Sec. 7] constructed the eigenvariety $D_{\rho(\beta)}$ associated with $(\mathbb{B}, \rho, H, \beta)$, a finite morphism $\pi : D_{\rho(\beta)} \to Z_{\rho(\beta)}$, a ring homomorphism $a : H \to A(D_{\rho(\beta)})$, and a commutative diagram

$$\begin{array}{ccccc} & & D_{\rho(\beta)} & & \\ & \swarrow^{k} & \downarrow{\scriptstyle \pi} & \searrow^{\alpha(\beta)^{-1}} & \\ W & \xleftarrow{\mathrm{pr}_W} & Z_{\rho(\beta)} & \xrightarrow{\mathrm{pr}_{\mathbb{A}}} & \mathbb{A}^{1,\mathrm{an}} \end{array}$$

Given systems $\mathbb{B}_i$ $(i = 1, 2)$ and the associated $(\pi_i : D_i \to Z_i, a_i : H \to A(D_i))$ with $a_i(\beta)^{-1} = \mathrm{pr}_{\mathbb{A}} \circ \pi_i$, by a *morphism* $(\pi_1, a_1) \to (\pi_2, a_2)$ we mean a pair $(\varphi_D : D_1 \to D_2, \varphi_Z : Z_1 \to Z_2)$ of morphisms which make a commutative square with $\pi_1$ and $\pi_2$, such that $a_2(h) \circ \varphi_D = a_1(h)$ for all $h$ in $H$. Given a pair $(\pi, a)$ define its *reduction* $(\pi^{\mathrm{red}}, a^{\mathrm{red}})$ to be the compositions

$$\pi^{\mathrm{red}} : D^{\mathrm{red}} \overset{\mathrm{can}}{\hookrightarrow} D \xrightarrow{\pi} Z, \quad a^{\mathrm{red}} : H \xrightarrow{a} A(D) \to A(D)/\mathrm{Nil\ rad}A(D).$$

A morphism $(\varphi_D, \varphi_Z)$ is called *closed immersion* if both $\varphi_D$ and $\varphi_Z$ are.

A subset $W''$ of a rigid $K$-space $W$ is called *Zariski dense* if any *analytic subset* (see [2, Sec. 9.5.2]) of $W$ containing $W''$ is equal to $W$. If in addition for every $x \in W''$ and every open affinoid neighborhood $W'$ of $x$ in $W$, the intersection $W'' \cap W'$ is Zariski dense in every irreducible component of $W'$ which contains $x$, we say that $W''$ is *Zariski very dense* in $W$.

A *classical structure over* $W''$ of a system of Banach modules $\mathbb{B}$ with an action $\rho$ of a $K$-algebra $H$ is the data of a finite dimensional Banach $k_x$-subspace $B_x^{\mathrm{cl}}$ of $B_x^+ = \varinjlim B_{x,n}$ stable under the action of $H$, for each $x$ in

$W''$. An eigenvector of an endomorphism $\beta$ of a Banach space $B$ has *slope* $a \in \mathbb{R}$ if $a = \mathrm{ord}(\lambda)$, where $\lambda$ is the associated eigenvalue. Denote by $B_x^{\leqslant a}$ (resp. $B_x^a$) the finite dimensional subspace of $B_x^+$ spanned by the vectors on which $\beta$ has slope at most $a$ (resp. $= a$). Put $W_a'' = \{x \in W''; B_x^{\leqslant a} \subset B_x^{\mathrm{cl}}\}$. Each of $B_x^{\leqslant a}$, $B_x^a$, $W_a''$ depends on $\beta$.

A *control* statement on a rigid $K$-space $W$ with a classical structure $\{B_x^{\mathrm{cl}}\}$ is the following

$(CL:\beta)$ For every $a \in \mathbb{R}$ and open affinoid $W'$ in $W$, the intersections $W' \cap W''$ and $W_a'' \cap W'$ have the same Zariski closure in $W'$.

We use the following Théorème 1 of Chenevier [6, p. 180]. Let $W$ be a reduced, separated, relatively factorial rigid analytic space over $K$.

**Theorem 2.18.** *Suppose $\mathbb{B}_1$, $\mathbb{B}_2$ are systems of ONable Banach $H$-modules on $W$ with actions $\rho_i : H \to \mathrm{End}_K \mathbb{B}_i$ and $\beta \in H$ with $\rho_i(\beta)$ in $\mathrm{Comp}\mathbb{B}_i$, and with classical structures on a Zariski very dense subset $W''$ of $W$, which satisfy $(CL:\rho_i(\beta))$. If $\det[I - t\rho_1(h\beta)|B_{1,x}^{\mathrm{cl}}]$ divides $\det[I - t\rho_2(h\beta)|B_{2,x}^{\mathrm{cl}}]$ in $k_x[t]$ for every $h$ in $H$ and $x$ in $W''$, then*

(1) $\mathrm{Fred}_{\mathbb{B}_1}(h\beta)$ *divides* $\mathrm{Fred}_{\mathbb{B}_2}(h\beta)$ *in* $A(W)\{\{t\}\}$;

(2) *there exists a canonical closed immersion* $(\varphi_D : D_{1\,\rho(\beta)} \to D_{2\,\rho(\beta)}$, $\varphi_Z : Z_{1\,\rho(\beta)} \to Z_{2\,\rho(\beta)})$ *of* $(\pi_1 : D_{1\,\rho(\beta)} \to Z_{1\,\rho(\beta)}, a_1 : H \to A(D_{1\,\rho(\beta)}))$ *into* $(\pi_2 : D_{2\,\rho(\beta)} \to Z_{2\,\rho(\beta)}, a_2 : H \to A(D_{2\,\rho(\beta)}))$;

(3) *for every $x$ in $W$ one has $X_\beta(\mathbb{B}_{1,x}) \subset X_\beta(\mathbb{B}_{2,x})$.*

## 3. Global theory

In this section, we introduce various spaces of automorphic forms and families of $p$-adic automorphic forms, Hecke operators and spectral and eigenvarieties for them. We shall be interested not only in the theory over the ground field $F$, but also in a relative situation where $E/F$ is a cyclic extension of prime degree $\ell$ prime to $2n$.

Let $F$ be a totally real number field of degree $d$ over $\mathbb{Q}$. Let $F'$ be a totally imaginary quadratic extension of $F$. Let $D$ be a central simple algebra over $F'$ of rank $n^2$ with $F$-involution $*$ whose restriction to $F'$ is the nontrivial automorphism of $F'/F$. Consider the $F$-group $G_F$ which associates to any $F$-algebra $R$ the group $G_F(R) = \{g \in D \otimes_F R;\ gg^* = 1\}$. At a place $v$ of $F$ where $D$ is unramified, namely split, we have $G_F(F_v) = \mathrm{GL}(n, F_v)$ if $v$ splits in $F'$, or $G_F(F_v)$ is the unitary group $\mathrm{U}(n, F_v'/F_v)$ if $v$ stays prime in $F'$. At each infinite place $v$ of $F$ the group $G_F(F_v)$

is a unitary group of signature $(p_v, q_v)$. Our assumption on $G_F$ is that $G_F(F_\infty) = \prod_r G_F(F_{\infty_r})$, where the product is over all archimedean places $\infty_r$ of $F$, is compact, namely that the signature of $*$ on $G_F(F_{\infty_r})$ is $(n, 0)$ or $(0, n)$, for each $r$ $(1 \leqslant r \leqslant d = [F : \mathbb{Q}])$. The center of $G_F(F)$ is $F'^1 = \ker N_{F'/F}$, the group of $x$ in $F'$ whose norm to $F$ is 1. We also fix a prime $p$ of $\mathbb{Q}$ which is split completely in $F'$. Denote the $F$-primes over $p$ by $p_r$ $(1 \leqslant r \leqslant d)$, and also uniformizers will be denoted by $p_r$. Then $G_F(F_{p_r}) = \mathrm{GL}(n, F_{p_r})$ for each $F$-place $p_r$ over $p$.

Fix a compact open subgroup $U = U^p \times U_p$ of $G_F(\mathbb{A}_f)$, where $\mathbb{A}_f = \mathbb{A}_{F,f}$ denotes the ring of adèles of $F$ without real components. We assume $U^p$ is an open compact subgroup in $G_F(\mathbb{A}_f^p)$, where $\mathbb{A}_f^p$ means no components also at the places of $F$ above $p$, and $U_p = \prod_r U_{p_r}$, product over the places $p_r$ of $F$ over $p$, $U_{p_r}$ open compact subgroup of $G_F(F_{p_r})$. The double coset $X_F(U) = G_F(F)\backslash G_F(\mathbb{A}_f)/U$ is finite, represented by $x_1, \dots, x_h$ in $G_F(\mathbb{A}_f)$. We take $U$ to be small enough so that $G_F(F) \times U$ acts freely on $G_F(\mathbb{A}_f)$, thus $\gamma x_i u = x_i$ has only the trivial solution, namely $x_i^{-1} G_F(F) x_i \cap U$ is $\{1\}$ for each $i$. The assumption that $G_F(F_\infty)$ is compact implies also that $G_F(F)$ is discrete in $G_F(\mathbb{A}_f)$.

We fix a field isomorphism $\overline{\mathbb{Q}}_p \simeq \mathbb{C}$. Given $\mu = (\mu_1, \dots, \mu_n)$ in $\mathbb{Z}_{\geqslant 0}^{n-1} \times \mathbb{Z}$, ${}_\mu R'(k)$ denotes the algebraic representation of $\mathrm{GL}(n, k)$ of highest weight $\mu$ constructed above. Here $k$ is any field of characteristic zero, and the construction is compatible with extension of fields. Thus ${}_\mu R'(k) \subset {}_\mu R'(\mathbb{C})$ for any finite extension $k$ of $\mathbb{Q}_p$ as well as ${}_\mu R'(\mathbb{R}) \subset {}_\mu R'(\mathbb{C})$. Namely ${}_\mu R'(\mathbb{C})$ is a representation of both subgroups $G_F(\mathbb{R})$ and $G_F(k)$ of $G_F(\mathbb{C}) = \mathrm{GL}(n, \mathbb{C})$, as well as of $G_F(\mathbb{C})$.

The group $G_F(F) \times G_F(F_\infty) \times U$ acts linearly on the vector space of functions $\phi : G_F(\mathbb{A}) \to \mathbb{C}$ by $((\gamma, x_\infty, u)\phi)(g) = \phi(\gamma^{-1} g x_\infty u)$.

As suggested above, we consider also an analogous situation where $E$ is a totally real cyclic extension of $F$ of degree $\ell$ not dividing $2n$. To simplify matters, we consider only $E$ such that each finite place $v$ of $F$ where $D$ ramifies, splits in $E$. Fix a generator $\sigma$ of $\mathrm{Gal}(E/F)$. Put $E' = EF'$, and define $G_E$ using the central simple algebra $D_E = D_F \otimes_{F'} E'$ over $E'$. Inspired by the theory of base change for automorphic representations of $G_E(\mathbb{A}_E)$, discussed briefly in the next section, we consider here in parallel to the theory over $F$, also the effect of twisting by $\sigma$ the (completely analogous) theory over $E$. To specify that objects are defined with respect to $E$, we use an index $E$, for example $\phi_E$. We take the open compact subgroup $U_E$ to be $\sigma$-invariant. Then $\sigma$ acts on $\phi_E : G_E(\mathbb{A}_E) \to \mathbb{C}$ by $(\sigma\phi_E)(g) = \phi_E(\sigma g)$.

Let $\mu_F = (\mu_{(r)}; 1 \leqslant r \leqslant d)$, $d = [F : \mathbb{Q}]$, be a $d$-tuple of $n$-tuples $\mu_{(r)} = (\mu_{r1}, \dots, \mu_{rn}) \in \mathbb{Z}_{\geqslant 0}^{n-1} \times \mathbb{Z}$. Then ${}_{\mu_F}R'(\mathbb{C}) = \otimes_r {}_{\mu_{(r)}}R'(\mathbb{C})$ is a representation of $\prod_{1\leqslant r\leqslant d} G_F(\mathbb{C})$, in particular of $\prod_r G_F(F_{\infty_r})$, where $\infty_r$ are the real places of $F$ (and $F_{\infty_r} = \mathbb{R}$ for each $r$). Thus $F \otimes_{\mathbb{Q}} \mathbb{R}$, denoted $F_\infty$, is the product of the $F_{\infty_r} = \mathbb{R}$ over all $r$ $(1 \leqslant r \leqslant d)$, and we put $G_F(F_\infty) = \prod_r G_F(F_{\infty_r}) \subset \prod_r G_F(\mathbb{C})$.

The *space* ${}_{\mu_F}R'(G_F, U, \mathbb{C})$ of *automorphic forms on* $G_F(\mathbb{A})$ with *weight* $\mu_F$ and *level* $U$ is the $\mathbb{C}$-vector space spanned by the complex valued functions $\phi'$ on $X(U) = G_F(F)\backslash G_F(\mathbb{A})/U$ which generate under $G_F(F_\infty)$ the representation ${}_{\mu_F}R'(\mathbb{C})^*$ dual to ${}_{\mu_F}R'(\mathbb{C})$.

In other words, the vectors in ${}_{\mu_F}R'(G_F, U, \mathbb{C})$ are functions $g \mapsto \phi'(g)\varphi$, $\phi'(g) \in \mathbb{C}$, $\varphi \in {}_{\mu_F}R'(\mathbb{C})^*$, with $x_\infty \cdot \phi'(g)\varphi = \phi'(gx_\infty)\varphi = \phi'(g)x_\infty\varphi$. Writing $\phi'(g, \varphi)$ for $\phi'(g)\varphi$, the last equality can be expressed as $\phi'(gx_\infty, \varphi) = \phi'(g, x_\infty\varphi)$. Note that $\varphi \mapsto \phi'(g, \varphi) = \phi'(g)\varphi$ is linear in $\varphi \in {}_{\mu_F}R'(\mathbb{C})^*$.

Over $E$ we write $\mu_E = (\mu_{F,i}; 0 \leqslant i < \ell)$, and $\sigma$ acts by permutation: $\sigma(\mu_E) = (\mu_{F,i+1})$, where the index is taken mod $\ell$. In particular $\varphi_E \in {}_{\mu_E}R'(\mathbb{C})^* = \otimes_i {}_{\mu_{F,i}}R'(\mathbb{C})^*$ is generated by pure tensors $\otimes_i \varphi_{\mu_{F,i}}$, $\varphi_{\mu_{F,i}} \in {}_{\mu_{F,i}}R'(\mathbb{C})^*$. This pure tensor is mapped by $\sigma$ to $\otimes_i \varphi_{\mu_{F,i+1}}$. Then $\sigma$ maps $g \mapsto \phi'_E(g)\varphi_E$ to $\sigma \cdot \phi'_E(g)\varphi_E = \phi'_E(\sigma g)\sigma\varphi_E$, and for $x_{E,\infty} = (x_{F,\infty,i})$ we have

$$\sigma \cdot x_{E,\infty} \cdot \phi'_E(g)\varphi_E = \phi'_E(\sigma(gx_{E,\infty}))\sigma\varphi_E = \phi'_E(\sigma g)\sigma(x_{E,\infty}\varphi_E).$$

To have a model over smaller fields, consider the space ${}_{\mu_F}R(G_F, U, \mathbb{C})$ of functions $\phi : G_F(F)\backslash G_F(\mathbb{A}_{F,f}) \to {}_{\mu_F}R'(\mathbb{C})$ such that $\phi(gu) = u_p^{-1}\phi(g)$ for all $u$ in $U$ and $g$ in $G_F(F)\backslash G_F(\mathbb{A}_{F,f})$. Here we used the fixed isomorphism $\overline{\mathbb{Q}}_p \simeq \mathbb{C}$, thus $\mathbb{Q}_p \subset \mathbb{C}$, and we are assuming that $p$ splits completely in $F$, thus $u_p = (u_{p_r}; 1 \leqslant r \leqslant d)$ is the $p$-component of $u$. Each $u_{p_r}$ lies in the open compact subgroup $U_{p_r}$ of $G_F(F_{p_r}) = \mathrm{GL}(n, \mathbb{Q}_p)$.

Over $E$, $\sigma$ acts by $\sigma \cdot \phi_E(g) = \sigma(\phi_E(\sigma^{-1}g))$, thus

$$(\sigma \cdot \phi_E)(gu) = \sigma(\phi_E(\sigma^{-1}(gu))) = \sigma(\sigma^{-1}u_p^{-1}\phi_E(\sigma^{-1}g)) = u_p^{-1}(\sigma \cdot \phi_E)(g)$$

so that $\sigma\phi_E$ lies again in ${}_{\mu_E}R'(\mathbb{C})$.

The spaces ${}_{\mu_F}R(G_F, U, \mathbb{C})$ and ${}_{\mu_F}R'(G_F, U, \mathbb{C})$ are isomorphic. Inverse isomorphisms are given by: $\phi \mapsto \phi'$ is $\phi'(g, \varphi) = \varphi(g_\infty^{-1}g_p\phi(g_f))$, where $g = (g_\infty, g_f) \in G_F(F_\infty) \times G_F(\mathbb{A}_{F,f})$. In the other direction: $\phi' \mapsto \phi$ is defined as follows. Given $g_f \in G_F(\mathbb{A}_{F,f})$, the linear form $\varphi \mapsto \phi'(g_f, \varphi)$

on ${}_{\mu_F}R'(\mathbb{C})^*$ is represented by a unique vector $g_p\phi(g_f)$ in ${}_{\mu_F}R'(\mathbb{C})$. By construction, $\varphi(g_p\phi(g_f))$ is $\phi'(g_f,\varphi)$, and thus

$$\varphi(g_pu_p\phi(g_fu)) = \phi'(g_fu,\varphi) = \phi'(g_f,\varphi) = \varphi(g_p\phi(g_f))$$

for all $\varphi$, thus $u_p\phi(g_fu) = \phi(g_f)$ for all $u$ in $U$.

These spaces are finite dimensional over $\mathbb{C}$ via the isomorphism

$${}_{\mu_F}R(G_F,U,\mathbb{C}) \to {}_{\mu_F}R'(\mathbb{C})^h, \quad \phi \mapsto (\phi(x_1),\dots,\phi(x_h)).$$

We assume also that $p$ splits completely in $E$, and check that the isomorphism $\phi_E \to \phi'_E$, ${}_{\mu_E}R(G_E,U_E,\mathbb{C}) \to {}_{\mu_E}R'(G_E,U_E,\mathbb{C})$ is $\sigma$-invariant: $\phi'_E(\sigma g)\sigma\varphi_E = \sigma\varphi_E(\sigma g_\infty^{-1}\sigma g_p\phi_E(\sigma g_f))$.

For any finite extension $k$ of $\mathbb{Q}_p$ the space ${}_{\mu_F}R(G_F,U,k)$ can be defined to be the space of all functions

$$\phi : G_F(F)\backslash G_F(\mathbb{A}_{F,f}) \to {}_{\mu_F}R'(k)$$

such that $\phi(gu) = u_p^{-1}\phi(g)$ for all $g \in G_F(F)\backslash G_F(\mathbb{A}_{F,f})$ and $u$ in $U$. In particular ${}_{\mu_F}R(G_F,U,k)$ is a $k$-structure of ${}_{\mu_F}R(G_F,U,\mathbb{C})$, corresponding to ${}_{\mu_F}R'(k)^h \subset {}_{\mu_F}R'(\mathbb{C})^h$.

Over $E$, as usual, $\sigma$ permutes the coordinates of ${}_{\mu_E}R'$ and sends $g$ to $\sigma g$.

This construction generalizes as follows. For any ring $A$ and $A[U_p]$-module $V$ let $V(G_F,U)$ be the space of functions $\phi : G_F(F)\backslash G_F(\mathbb{A}_{F,f}) \to V$, such that $\phi(gu) = u_p^{-1}\phi(g)$, $u \in U$, $g \in G_F(F)\backslash G_F(\mathbb{A}_{F,f})$. It is isomorphic to $V^h$ by $\phi \mapsto (\phi(x_1),\dots,\phi(x_h))$. The functor $V \mapsto V(G_F,U)$ is exact, ($A[U_p]$-modules) $\to$ ($A$-modules). It commutes with extension of scalars in $A$. If $V$ is a Banach $A$-module, $V(G_F,U)$ has the norm $|\phi| = \max_{1\leqslant i\leqslant h}|\phi(x_i)|$ obtained from that of $V$. It is isomorphic to $V^h$ as a normed $A$-module.

If $\langle\sigma\rangle$ acts on $V$, put $(\sigma\cdot\phi_E)(g) = \sigma(\phi_E(\sigma^{-1}g))$.

If $\mathbb{B}$ is a (Banach) $\mathcal{O}_X^{\mathrm{an}}[U_p]$-module on a rigid space $X$, so is $\mathbb{B}(G_F,U) = \{X' \mapsto \mathbb{B}(X')(G_F,U)\}$. We write $\mathbb{B}(G_F,U)(X')$ for $\mathbb{B}(X')(G_F,U)$.

Denote by $U_1(p)_F$ any subgroup $U$ with $U_{p_r} = \Gamma_1(\mathbf{p}_r)$ for each $F$-prime $p_r$ over $p$, where $\mathbf{p}_r$ denotes the generator of the maximal ideal $F_{p_r}^{00}$ in the ring $F_{p_r}^0$ of integers in $F_{p_r}$ unless $F_{p_r} = \mathbb{Q}_2$ in which case $\mathbf{p}_r$ is 4. The group $\Gamma_1(\mathbf{p}_r)$ consists of the matrices in $\mathrm{GL}(n,F_{p_r}^0)$ which are congruent to a unipotent upper triangular matrix mod $\mathbf{p}_r$. Also we put $U_0(p)_F$ for

$U_1(p)_F I_F$, where $I_F = \prod_r I_{p_r}$ and $I_{p_r}$ is the Iwahori upper triangular subgroup in $\mathrm{GL}(n, F^0_{p_r})$ when $F_{p_r}$ is not $\mathbb{Q}_2$, and if it is, $I_{p_r}$ is the group of matrices which are upper triangular mod 4.

Given an $A[\prod_r I_{p_r}]$-module $V$, $I_F = \prod_r I_{p_r}$ acts on $V(G_F, U_1(p)_F)$ by $(\gamma\cdot\phi)(g) = \gamma\phi(g\gamma)$. The action of the subgroup $\prod_r \Gamma_1(\mathbf{p}_r)$ is trivial. Hence there is an action of $\prod_r(\mathbb{Z}/\mathbf{p}_r)^{\times n}$ on $V(G_F, U_1(p)_F)$. If $A$ contains the $(p-1)$th roots of 1 and $\frac{1}{p-1}$ (or $\frac{1}{2}$ when $p=2$) we have diagonalization

$$V(G_F, U_1(p)_F) = \oplus_{\chi_F \in \Delta_F^n} V(G_F, U_0(p)_F)(\chi_F).$$

Here $\Delta_F^n = \mathrm{Hom}(\prod_r(\mathbb{Z}/\mathbf{p}_r)^{\times n}, \mathbb{Z}_p^\times)$ and $V(G_F, U_0(p)_F)(\chi_F)$ is the space of all $\phi$ in $V(G_F, U_1(p)_F)$ with $\phi(gu) = \chi_F(u)u^{-1}\phi(g)$ for all $u$ in $I_F = \prod_r I_{p_r}$. It is also denoted by $V_{\chi_F}(G_F, U_0(p)_F)$.

Thus ${}_{\mu_F}R'(G_F, U_1(p)_F) = \oplus_{\chi_F\in\Delta_F^n}\, {}_{\mu_F}R'_{\chi_F}(G_F, U_0(p)_F)$. The space of *automorphic forms over* $\mathbb{Q}_p$ of *weight* $\mu_F$, *level* $U_0(p)_F$ and *character* $\chi_F$ at $p$ is denoted by ${}_{\mu_F}R'_{\chi_F}(G_F, U_0(p)_F)$.

If $\langle\sigma\rangle$ acts on $V_E$, put $\sigma\cdot\phi_E(g) = \sigma(\phi_E(\sigma^{-1}g))$. At this stage the extension to the semidirect product with $\langle\sigma\rangle$ is clear, and will be mentioned only when used.

To simplify the notations we write $A_\#$ for $A(G_F, U_0(p)_F)$ from now on.

The Banach module over $W_F$ of *p-adic automorphic forms* on the $F$-group $G_F$ of level $U_0(p)_F$ and type $\chi_F \in \Delta_F^n$ at $p$ is the Banach module $A(Y_{1,F}; \chi_F, W_F)_\# : W' \mapsto A(Y_{1,F})_{\chi_F} \widehat{\otimes}_{\mathbb{C}_p} \mathcal{O}^{\mathrm{an}}_{W_F}(W')_\#$. It is a sheaf which associates to each open affinoid $W'$ in $W_F$ the Banach space of $p$-adic automorphic forms of type $(G_F, U_0(p)_F, \chi_F)$ over $W'$. For any weight $\mu_F \in W_F(\mathbb{C}_p)$ denote by $A(Y_{1,F}; \chi_F, W_F, \mu_F)_\#$ the fiber at $\mu_F$ of the sheaf $A(Y_{1,F}; \chi_F, W_F)_\#$. It is equal to $\mathcal{O}_{W_F}(\mu_F)_{1,\chi_F\tau_{\mu_F},\#}$. Define *p-adic automorphic forms of type* $(G_F, U_0(p)_F, \chi_F)$ and *weight* $\mu_F$ to be the elements in the $\mathbb{C}_p$-Banach space $A(Y_{1,F}; \chi_F, W_F, \mu_F)_\#$.

As usual, $\mu_F = (\mu_{(r)})$, $\mu_{(r)} = (\mu_{r1}, \dots, \mu_{rn}) \in \mathbb{Z}^{n-1}_{\geqslant 0}\times\mathbb{Z}$ $(1\leqslant r\leqslant d)$. Then $A(Y_{1,F}; \chi_F, W_F, \mu_F)_\#$ contains the $\mathbb{C}_p$-subspaces ${}_{\mu_F}R'_{\chi_F\tau_{\mu_F},\#}$, where $\tau_{\mu_F} = (\tau_{\mu_{(r)}})$ and $\tau_{\mu_{(r)}} = (\tau^{-\mu_{r1}-\dots-\mu_{rn}}, \tau^{-\mu_{r2}-\dots-\mu_{rn}}, \dots, \tau^{-\mu_{rn}})$ is a character in $\Delta^n$, thus $\tau_{\mu_F}$ is in $\Delta_F^n$. We then define the space $A(Y_{1,F}; \chi_F, W_F, \mu_F)^{\mathrm{cl}}_\#$ of *classical* $p$-adic automorphic forms of weight $\mu_F$ on $G$ over $F$ with level $U_0(p)_F$ and character $\chi_F$ to be the space ${}_{\mu_F}R'_{\chi_F\tau_{\mu_F},\#}$. In particular a classical $p$-adic automorphic form of weight $\mu_F$ and character $\chi_F$ at $p$ is a $p$-adic automorphic form of weight $\mu_F$ but of character $\chi_F\tau_{\mu_F}$.

Recall that $\Lambda = \Lambda_F = \mathcal{O}^{\mathrm{an}}_{W_F}(W_F, L) \cap \mathcal{O}^{\mathrm{an}}_{W_F}(W_F)^0$. For a Banach module $B_{W_F} = B \widehat{\otimes}_{\mathbb{C}_p} \mathcal{O}^{\mathrm{an}}_{W_F}$ over $W_F$, where $B$ is ONable over $\mathbb{C}_p$ with $L$-structure $B'$, the subscript $\Lambda$ means $B_{W_F,\Lambda} = B_{W_F}(W_F, L) \cap B_{W_F}(W_F)^0$. Then we have the $\Lambda$-module $A(Y_{1,F}; \chi_F, W_F, \Lambda)_\#$ $(= A(Y_{1,F}; \chi_F, W_F)_{\Lambda,\#})$ of *$\Lambda$-adic automorphic forms* of type $(G, U_0(p)_F, \chi_F)$. The Banach modules $A(Y_{1,F}; \chi_F, W_F)_\#$ and $A(X_{1,F}; \chi_F, W_F)_\#$ are ONable over $W_F$, associated with $A(Y_{1,F})_\#$ and $A(X_{1,F})_\#$. The $\Lambda$-modules $A(Y_{1,F}; \chi_F, W_F, \Lambda)_\#$ and $A(X_{1,F}; \chi_F, W_F, \Lambda)_\#$ are isomorphic to $C^0(\mathbb{N}, \Lambda)$. They are flat, complete and separated since $C^0(\mathbb{N}, \Lambda) \simeq \Lambda\langle t\rangle$ as a topological $\Lambda$-module, and $\Lambda\langle t\rangle$ is flat over $\Lambda$ as $\Lambda$ is a complete local Noetherian ring.

Let $\zeta \in G_F(\mathbb{A}_f)$ have component $\zeta_p = (\zeta_{p_r})$ with $\zeta_r = \zeta_{p_r} \in I_r p_r^{a(r)} I_r$, $a_{(r)} = (a_{r1} \leqslant a_{r2} \leqslant \cdots \leqslant a_{rn})$ for each $r$ $(1 \leqslant r \leqslant d)$. Then $U_0(p)_F \zeta U_0(p)_F = \coprod_{1 \leqslant \alpha \leqslant A(\zeta)} \zeta_\alpha U_0(p)_F$ for some $\zeta_\alpha \in U_0(p)_F \zeta U_0(p)_F$, $1 \leqslant \alpha \leqslant A(\zeta)$. Under the right action of $I_F$, we have the decomposition

$$_{\mu_F}R'(F, G_F, U_1(p), \mathbb{C}) = \oplus_{\chi_F \in \Delta_F^n}\, {}_{\mu_F}R'(F, G_F, U_0(p)_F, \mathbb{C})(\chi_F).$$

Fix a character $\chi_F : I_F = \prod_r I_r \to \prod_r (F^0_{p_r}/\mathbf{p}_r)^{\times n} \to \mathbb{C}^\times$. Extend $\chi_F$ trivially by 1 to $p^{\{a_F\}}$.

Define the *Hecke operator* $T(\zeta)$ on $_{\mu_F}R'(F, G_F, U_0(p)_F, \mathbb{C})(\chi_F)$ by

$$(T(\zeta)\phi)(x) = \sum_{1 \leqslant \alpha \leqslant A(\zeta)} {\chi_F}^{-1}(\zeta_{\alpha,p})\phi(x\zeta_\alpha), \qquad x \in G_F(F)\backslash G_F(\mathbb{A}).$$

The operator $T(\zeta)$ is independent of the choice of the $\zeta_\alpha$. The action transferred to $_{\mu_F}R(F, G_F, U_1(p), \mathbb{C})(\chi_F)$ descends to the space $_{\mu_F}R(F, G_F, U_0(p)_F)(\chi_F)$, which has a $\mathbb{Q}_p$-structure, by the formula

$$(T(\zeta)\phi)(x) = \sum_{1 \leqslant \alpha \leqslant A(\zeta)} (\zeta_\alpha)_p \cdot \phi(x\zeta_\alpha), \qquad x \in G_F(F)\backslash G_F(\mathbb{A}_f).$$

Note that the scalars $\chi_F(\zeta_\alpha)$ appear in the last formula through the action of $I_F$ on $_{\mu_F}R'_{\chi_F}(\mathbb{Q}_p) = {}_{\mu_F}R'(\mathbb{Q}_p) \otimes \chi_F$.

Let now $V$ be any $A$-module with $A$-linear action of $I_F \cdot p^{\{a_F\}} \cdot I_F$. Then the last formula for $T(\zeta)$, for $\zeta$ as above, defines an $A$-linear operator on $V_\#$. This defines the Hecke operators in particular on $A(Y_{1,F}; \chi_F, W_F)_\#$ and $A(X_{1,F}; \chi_F, W_F)_\#$.

Here is a useful factorization of the Hecke operators $T(\zeta)$ on the space $V_\#$. Recall the isomorphism $\iota : V_\# \to V^h$, $\phi \mapsto (\phi(x_i))$, $1 \leqslant i \leqslant h = \#X(U_0(p)_F)$.

**Lemma 3.1.** *For $\zeta \in G_F(\mathbb{A}_f)$ with $\zeta_p \in I_F \cdot p^{a_F} \cdot I_F$, $T(\zeta)$ is equal to $\sum_{1\leqslant\alpha\leqslant hA(\zeta)} T_\alpha \cdot \sigma_\alpha$. Each $\sigma_\alpha$ is – via $\iota$ – a permutation of the $h$ coordinates followed by a projection to one of them. Each $T_\alpha$ is a multiplication by a diagonal matrix in $U \cdot p^{a_F} \cdot U$, $U = U_0(p)_F$.*

**Proof.** Recall the decomposition $U\zeta U = \coprod_{1\leqslant\alpha\leqslant A(\alpha)} \zeta_\alpha U$, and $G_F(\mathbb{A}_f) = \coprod_{1\leqslant s\leqslant h} G_F(F)x_s U$, $U = U_0(p)_F$. Then $x_s\zeta_\alpha = \gamma_\alpha(s)x_{\delta_\alpha(s)}u_\alpha(s)$ where $\delta_\alpha$ is a permutation of $\{1,\dots,h\}$, $\gamma_\alpha(s) \in G_F(F)$ and $u_\alpha(s) \in U$. Let $\sigma_{\alpha,s}$ be the map $(v_1,\dots,v_h) \mapsto (0,\dots,0,v_{\delta_\alpha(s)},0,\dots,0)$ where $v_{\delta_\alpha(s)}$ is at the $s$th place. Let $T_{\alpha,s}$ be the operator of multiplication of $(\iota : V_\# \to)\ V^h$ by $\mathrm{diag}((\zeta_\alpha u_\alpha(s)^{-1})_p; 1 \leqslant s \leqslant h)$. Then we claim: $\iota T(\zeta)\iota^{-1} = \sum_{\alpha,s} T_{\alpha,s}\sigma_{\alpha,s}$. Indeed, given $\phi \in V_\#$, $\iota(\phi) = (\phi(x_1),\dots,\phi(x_h))$,

$$\iota T(\zeta)\iota^{-1}(\iota(\phi)) = ((T(\zeta)\phi)(x_1),\dots,(T(\zeta)\phi)(x_h)).$$

By definition of $T(\zeta)$ we have $(T(\zeta)\phi)(x_s) = \sum_{1\leqslant\alpha\leqslant A(\alpha)} (\zeta_\alpha)_p \cdot \phi(x_s\zeta_\alpha)$. But $\phi(x_s\zeta_\alpha) = \phi(\gamma_\alpha(s)x_{\delta_\alpha(s)}u_\alpha(s)) = (u^{-1}_{\alpha(s)})_p \cdot \phi(x_{\delta_\alpha(s)})$, as asserted. □

Let $\mathbb{H}_\Lambda$ denote the $\Lambda$-Hecke algebra generated over $\Lambda$ by the double cosets $U_0(p)_F g U_0(p)_F$, $g \in G_F(\mathbb{A}_f)$, or alternatively by the constant measures on $G_F(\mathbb{A}_f)$ supported on these sets, with volume 1. Let $H_\Lambda$ be the $\Lambda$-subalgebra of $\mathbb{H}_\Lambda$ generated by the subalgebra of the Iwahori double cosets $I_{p_r} \cdot p_r^{a_{(r)}} \cdot I_{p_r}$ $(a_{(r)} > 0)$ at each prime $p_r$ of $F$ over $p$, and by a commutative subalgebra of the Hecke algebra of $G_F(\mathbb{A}_f^p)$ with respect to $U_0(p)_F^p$. Then $H_\Lambda$ is commutative.

**Proposition 3.2.** *The commutative $\Lambda$-algebra $H_\Lambda$ acts on each of the spaces $A(Y_{1,F};\chi_F,W_F)_\#$ and $A(X_{1,F};\chi_F,W_F)_\#$ by integral and rational endomorphisms, which consequently preserve*

$$A(Y_{1,F};\chi_F,W_F,\Lambda)_\# = A(Y_{1,F};\chi_F,W_F,L)_\# \cap A(Y_{1,F};\chi_F,W_F)^0_\#$$

*and*

$$A(X_{1,F};\chi_F,W_F,\Lambda)_\# = A(X_{1,F};\chi_F,W_F,L)_\# \cap A(X_{1,F};\chi_F,W_F)^0_\#.$$

**Proof.** Any $I_{p_r}p_r^{a_{(r)}}I_{p_r}$ acts by integral automorphisms on the Banach modules $A(Y_{1,F};\chi_F,W_F)$ and $A(X_{1,F};\chi_F,W_F)$ as it acts on $A(W_{q_F} \times_{K_{q_F}}$

$X_{1,F})$ and $A(W_{q_F} \times_{K_{q_F}} Y_{1,F})$. It acts rationally since it preserves $A(W_1^F \times_{\mathbb{Q}_p} X_{1,F})$ and $A(W_1^F \times_{\mathbb{Q}_p} Y_{1,F})$. As $V_\# = V^h$ and $T(\zeta) = \sum_\alpha T_\alpha \cdot \sigma_\alpha$, the proposition follows. □

Denote by $H_{\chi_F,\Lambda}$ the closure of the image of $H_\Lambda$ in the space $E_{\chi_F}$ of $u \in \mathrm{End} A(Y_{1,F}; \chi_F, W_F)_\#$ which are integral and rational. Put on End the sup norm. Put on $E_{\chi_F}$ the weakest topology such that all restrictions $E_{\chi_F} \to \mathrm{End}^{\mathrm{cont}}_{A(W_F)}(A(Y_{1,F}; \chi_F, W_F)_\#(W'))$ are continuous.

Using the isomorphism $A(Y_{1,F}; \chi_F, W_F, \Lambda)_\# \simeq C^0(\mathbb{N}, \Lambda)$ of topological $\Lambda$-modules, $E_{\chi_F}$ is topologically isomorphic to the $\Lambda$-module $C^0(\mathbb{N}, \Lambda)^{\mathbb{N}}$ with its $m$-adic topology, where $m$ denotes the maximal ideal of $\Lambda$. It is complete and separated.

**Proposition 3.3.** *The commutative $\Lambda$-algebra $H_{\chi_F,\Lambda}$ is profinite (in particular compact) and semilocal. The map $\Lambda \hookrightarrow H_{\chi_F,\Lambda}$ is continuous.*

**Proof.** Since $H_{\chi_F,\Lambda}$ is the closure of the image of $H_\Lambda$ in $E_{\chi_F}$, its topology is generated by the $(m^\ell E_{\chi_F}) \cap H_{\chi_F,\Lambda}$. Hence $H_{\chi_F,\Lambda}$ is complete and separated, and the restriction to $\Lambda$ is the natural $m$-adic topology on $\Lambda$ since $(m^\ell E_{\chi_F}) \cap \Lambda = m^\ell \Lambda$.

To show that $H_{\chi_F,\Lambda}$ is profinite, it suffices to show that the quotients $H_{\chi_F,\Lambda}/(m^\ell E_{\chi_F} \cap H_{\chi_F,\Lambda})$ are finite. Since $\Lambda/m^\ell\Lambda$ is finite, it suffices to show that $I_{p_r} \cdot p^{\{a_{(r)}\}} \cdot I_{p_r}$ acts on $A(Y_{1,F}; \chi_F, W_F, \Lambda)/m^\ell$ by only finitely many endomorphisms. It suffices to see this on $A(X_{1,F}; \chi_F, W_F, \Lambda)/m^\ell$. This follows from the fact that $1 + p_r^\ell M_n(F^0_{p_r})$ is a normal subgroup of finite index in $I_{p_r}$ which acts trivially, and that $p_r^{\{a_{(r)}\}}$ acts via a finite number of distinct automorphisms mod $m^\ell$.

In a commutative ring $A$ complete and separated for a topology given by a decreasing family of ideals $I_1 \supset I_2 \supset \dots$ with $I_r I_s \subset I_{r+s}$ we have $I_1 \subset \mathrm{Rad}(A)$. In particular $A$ is semilocal iff $A/I_1$ is. In our case $A = H_{\chi_F,\Lambda}$, $I_\ell = m^\ell E_{\chi_F} \cap H_{\chi_F,\Lambda}$, and $A/I_1$ is finite (hence semilocal). □

It follows that $H_{\chi_F,\Lambda}$ is a product of local proArtinian algebras, with finite residue fields of characteristic $p$, thus $H_{\chi_F,\Lambda} = \prod_{1 \leqslant \alpha \leqslant A} e_{\chi_F,\alpha} H_{\chi_F,\Lambda}$ for idempotents $e_{\chi_F,\alpha}$. Denote by $\psi_{\chi_F,\alpha} : H_{\chi_F,\Lambda} \to \overline{\mathbb{F}}_p$ $(1 \leqslant \alpha \leqslant A)$ the residual characters.

**Proposition 3.4.** *The set $\{\psi_{\chi_F,\alpha}\,(1 \leqslant \alpha \leqslant A)\}$ is equal to the set of reductions mod $m_{\mathbb{C}_p^0}$ of the characters $H_{\chi_F,\Lambda} \to \mathbb{C}_p^0$ in $A(Y_{1,F}; \chi_F, W_F, \mu_F)^{\mathrm{cl}}_\#$.*

**Proof.** Consider the $\overline{\mathbb{F}}_p[I_F \cdot p^{\{a_F\}} \cdot I_F]$-modules

$$A(Y_{1,F}; \chi_F, W_F, \mu_F)^0 \otimes_{\mathbb{C}_p^0} \overline{\mathbb{F}}_p$$

and

$$A(X_{1,F}; \chi_F, W_F, \mu_F)^0 \otimes_{\mathbb{C}_p^0} \overline{\mathbb{F}}_p).$$

As $\mu_F$ varies over $W_F(\mathbb{C}_p)$, these modules are isomorphic to each other as the $\mathbf{c}_i^{x_i}$ are trivial mod $m_{\mathbb{C}_p^0}$ (by end of Lemma 2.9) and the $p^{a_F}$ are constant. Let $\overline{A}(Y_{1,F}; \chi_F)$ and $\overline{A}(X_{1,F}; \chi_F)$ be the common $\overline{\mathbb{F}}_p[I_F \cdot p^{\{a_F\}} \cdot I_F]$-modules. Consider the composition of the $\mathbb{Z}_p[I_F \cdot p^{\{a_F\}} \cdot I_F]$-equivariant natural maps

$$_{\mu_F}V^0_{\chi_F\tau_{\mu_F}} \hookrightarrow A(X_{1,F}; \chi_F, W_F, \mu_F)^0 \twoheadrightarrow \overline{A}(X_{1,F}; \chi_F).$$

It factorizes via $_{\mu_F}R'^0_{\chi_F\tau_{\mu_F}} \otimes_{\mathbb{Z}_p} \overline{\mathbb{F}}_p$. We claim that the union of the images of the map displayed above, as $\mu_F = (\mu_{(r)})$, $\mu_{(r)} \in \mathbb{Z}^{n-1}_{\geqslant 0} \times \mathbb{Z}$, varies, covers $\overline{A}(Y_{1,F}; \chi_F)$. Indeed $\overline{A}(X_{1,F}; \chi_F) \to \overline{A}(Y_{1,F}; \chi_F)$ is onto, and the union of the images of the natural maps $_{\mu_F}V_{\chi_F\tau_{\mu_F}} \otimes_{\mathbb{Z}_p} \overline{\mathbb{F}}_p \to \overline{A}(X_{1,F}; \chi_F)$ is the whole of $\overline{A}(X_{1,F}; \chi_F)$.

Note that the $\mathbb{F}_p$-vector space $A(Y_{1,F}; \chi_F, W_F, \Lambda)/mA(Y_{1,F}; \chi_F, W_F, \Lambda)$ is an $\mathbb{F}_p$-structure of $A(Y_{1,F}; \chi_F, 0_F) \otimes_{\mathbb{C}_p^0} \overline{\mathbb{F}}_p = \overline{A}(Y_{1,F}; \chi_F)$ which is stable under $I_F \cdot p^{\{a_F\}} \cdot I_F$.

The functor $V \mapsto V_\#$ is exact and commutes with extension of scalars. Hence $\overline{A}(Y_{1,F}; \chi_F)_\#$ is an $\overline{\mathbb{F}}_p[I_F \cdot p^{\{a_F\}} \cdot I_F]$-module isomorphic to $A(Y_{1,F}; \chi_F, W_F, \Lambda)_\# \otimes_\Lambda \overline{\mathbb{F}}_p$. Moreover, it is the union of the images of the natural maps from $A(Y_{1,F}; \chi_F, W_F, \mu_F)^{\mathrm{cl},0}_\#$.

By Deligne–Serre [10, Lemma 6.11], these last spaces are obtained by extension of scalars to $\mathbb{C}_p^0$ from the free, finite type $L^0$-module $_{\mu_F}V_{\chi_F\tau_{\mu_F},\#} \otimes_{\mathbb{Z}_p} L^0$, which is stable under $H_{\chi_F,\Lambda}$. □

**Proposition 3.5.** *The $T(\zeta)$, $\zeta \in G_F(\mathbb{A}_f)$ with $\zeta_p \in I_F \cdot p^{\{a_F\}} \cdot I_F$, are rational and integral endomorphisms of*

$$A(Y_{1,F}; \chi_F, W_F)_\# \quad \text{and} \quad A(X_{1,F}; \chi_F, W_F)_\#.$$

*If each $a_{(r)}$ is strictly increasing, $T(\zeta)$ is completely continuous over any open affinoid of $W_F$.*

**Proof.** By Lemma 3.1, if $\zeta_{p_r} \in I_r \cdot p_r^{a_{(r)}} \cdot I_r$ and $a_{(r)}$ is strictly increasing, $T(\zeta)$ is a finite sum of compositions of completely continuous diagonal operators and operators of norm $\leqslant 1$. □

**Definition 3.6.** *When $a_F = (a_{(r)})$, $a_{(r)} = (a_{r1} \leqslant \cdots \leqslant a_{rn})$ is increasing and positive, let $T(p^{a_F}) \in H_{\chi_F,\Lambda}$ be the Hecke operator $T(\zeta)$ on $V_\#$ where $\zeta \in G_F(\mathbb{A}_f)$ is the identity everywhere except at the places $p_r$ of $F$ over $p$, where it is $p^{a_F} = (p^{a_{(r)}})$. Put $T(p^F)$ for $p^F = p^{a_F}$ with $a_{(r)} = (0, 1, 2, \ldots, n-1)$ for all $r$. The $T(p^{a_F})$ pairwise commute.*

From the general Fredholm theory of [8, Sec. A.2], for each $a_F$ one has

**Theorem 3.7.** *There exists unique entire series $P^{a_F}_{\chi_F}(\mu_F, t)$, $Q^{a_F}_{\chi_F}(\mu_F, t)$ in $A(W_F)\{\{t\}\}$ such that for all $\mu_F = (\mu_{(r)})$, $\mu_{(r)} \in \mathbb{Z}^{n-1}_{\geqslant 0} \times \mathbb{Z}$,*

$$P^{a_F}_{\chi_F}(\mu_F, t) = \det[I - tT(p^{a_F})|A(Y_{1,F}; \chi_F, W_F, \mu_F)_\#],$$

$$Q^{a_F}_{\chi_F}(\mu_F, t) = \det[I - tT(p^{a_F})|A(X_{1,F}; \chi_F, W_F, \mu_F)_\#].$$

As usual, if $X$ is a rigid space, $A(X)$ denotes the ring of rigid functions on $X$, and $A(X)\{\{t\}\}$ denotes the space of *entire* functions on $X$, namely $A(X \times \mathbb{A}^{1,\mathrm{an}})$. If $A$ is a local ring with maximal ideal $m_A$, a power series $F(t) = \sum_{i \geqslant 0} a_i t^i$ in $A[[t]]$ is called *entire* if $\max\{n \geqslant 0;\, a_i \in m_A^n\}/i \to \infty$ as $i \to \infty$. The ring of these is denoted by $A\{\{t\}\}$, and such a series is called *Fredholm* if $F(0) = 1$. Thus, when $X$ is affine, $P \in A(X)\{\{t\}\}$ is a power series $P(x,t) = \sum_{i \geqslant 0} a_i(x) t^i$ with $|a_i| q^i \to 0$ as $i \to \infty$ for all $q > 0$ in $\mathbb{R}$. Here $|.|$ is the sup norm on $A(X)$. The functions $P^{a_F}_{\chi_F}(\mu_F, t)$ and $Q^{a_F}_{\chi_F}(\mu_F, t)$ are Fredholm series on $W_F$, namely they lie in $1 + tA(W_F)\{\{t\}\}$. We shall use the case when $A$ is $\Lambda_F = L^0[[x_{ri};\, 1 \leqslant i \leqslant n,\, 1 \leqslant r \leqslant d]]$. A Fredholm series on $\Lambda$ gives by restriction a Fredholm series on $X = W_F$. Here $A(W_F)$ is $\mathcal{O}^{\mathrm{an}}_{W_F}(W_F)$, and $\Lambda_F$ is $\mathcal{O}^{\mathrm{an}}_{W_F}(W_F, L) \cap \mathcal{O}^{\mathrm{an}}_{W_F}(W_F)^0$.

A stronger form is given by

**Proposition 3.8.** *Let $a_F = (a_{(r)})$, $a_{(r)}$ strictly increasing for each $r$, $v \in H_{\chi_F,\Lambda_F} \cdot p^{a_F} \cdot H_{\chi_F,\Lambda_F}$. Then there exists a unique Fredholm series over $\Lambda_F$ whose evaluation at each $\mu_F = (\mu_{(r)})$, $\mu_{(r)} \in \mathbb{Z}^{n-1}_{\geqslant 0} \times \mathbb{Z}$, coincides with $\det[I - tv|A(Y_{1,F}; \chi_F, W_F, \mu_F)_\#]$. In particular the $P^{a_F}_{\chi_F}(\mu_F, t)$ are in $1 + t\Lambda_F\{\{t\}\}$.*

**Proof.** The assumption on $v$ (namely on $a_F$) implies that its restriction to the ONable Banach $A(W_{q_F})$-module $A_{q_F}(Y_{1,F}; \chi_F)_\#$ is completely continuous. Then we get an entire series $P_{\chi_F, v, q_F}(t) = \det[I - tv|A_{q_F}(Y_{1,F}; \chi_F)_\#]$. Compatibility with extension of scalars gives a power series $P_{\chi_F,v}(t) \in 1 + t\mathcal{O}^{\mathrm{an}}_{W_F}(W_F)\{\{t\}\}$, since as each $q_r$ grows in $[1, |\omega/\mathbf{p}|) \cap p^{\mathbb{Q}}$, the $W_{q_F} \times \mathbb{A}^{1,\mathrm{an}}$ provide an admissible cover of $W_F \times \mathbb{A}^{1,\mathrm{an}}$. Since $H_{\chi_F,\Lambda_F}$ acts by rational and integral endomorphisms on $A(Y_{1,F};\chi_F,W_F)_\#$, the coefficients of $P_{\chi_F,v}(t)$ are in fact in $\Lambda_F$. Since the Fredholm determinant is compatible with evaluation at $\mu_F \in W_F(\mathbb{C}_p)$, the uniqueness claim of the proposition follows since $(\mathbb{Z}^{n-1}_{\geqslant 0} \times \mathbb{Z})^d$ is Zariski dense in $\mathrm{Spec}(\Lambda_F)$. □

The same proof establishes also a twisted analogue. Let $a_E$ be strictly increasing, $v \in H_{\chi_E,\Lambda_E} \cdot p^{a_E} \cdot H_{\chi_E,\Lambda_E}$. Then there exists a unique Fredholm series over $\Lambda_E$ whose evaluation at each $\mu_E = (\mu_{(r),i})$, $0 \leqslant i < \ell$, $\mu_{(r),i} \in \mathbb{Z}^{n-1}_{\geqslant 0} \times \mathbb{Z}$, coincides with $\det[I - tv\sigma | A(Y_{1,E};\chi_E,W_E,\mu_E)_\#]$. In particular $P^{a_E\sigma}_{\chi_E}(\mu_E,t) = \det[I - tT(p^{a_E})\sigma | A(Y_{1,E};\chi_E,W_E,\mu_E)_\#]$ lies in $1 + t\Lambda_E\{\{t\}\}$.

We now explain a strategy to construct a base change morphism of eigenvarieties associated with $G_F$ and $G_E$. We shall use Theorem 2.18. For that, we need to show that $\det[I - th_E p^{a_E}\sigma | A(Y_{1E};\chi_E,W_E,\mu_E)^{\mathrm{cl}}_\#]$ is divisible by $\det[I - th_F p^{a_F} | A(Y_{1F};\chi_F,W_F,\mu_F)^{\mathrm{cl}}_\#]$ for suitably related $\mu$, $\chi$, $h$, where e.g. $h_E p^{a_E}$ indicates the double coset with respect to $U_0(p)_E$ in $H_E$ represented by $h_E p^{a_E}$.

By [21, Cor. 3, p. 77], $\det(I - tu) = \exp[-\sum_{m=1}^{\infty} \frac{1}{m}\,\mathrm{tr}(u^m)t^m]$.

The operator $\sigma^m$ maps $\phi_E$ in $A(Y_{1E};\chi_E,W_E,\mu_E)_\#$ to $\sigma^m \cdot \phi_E$ in the space $A(Y_{1E};\chi_E,W_E,\sigma^m\mu_E)_\#$. Then the trace of $(h_E p^{a_E}\sigma)^m$ is zero unless $\sigma^m\mu_E = \mu_E$. We then restrict the sheaf $A(Y_{1E};\chi_E,W_E)_\#$ to $W_F = W_E^\sigma$.

We wish to claim that $\det[I - tb^*(h_E p^{a_E}) | A(Y_{1F};\chi_F,W_F,\mu_F)^{\mathrm{cl}}_\#]$ divides $\det[I - th_E p^{a_E}\sigma | A(Y_{1E};\chi_E,W_E,\mu_E)^{\mathrm{cl}}_\#]$ where $\mu_E$ is $\mu_F$ and $\chi_F = \chi_E \circ N_{E/F}$. Both systems here are parametrized by $\mu_F$ in $W_F$, and so we have the Hecke algebra $H_E$ and two systems of Banach $H_E$-modules on $W = W_F$. We shall show below that both systems have classical structures on a Zariski very dense subset $W''$ of $W$, which satisfy the control conditions (CL) with $\beta = p^E$ and $p^F$. One system is $\mathbb{B}_1 = A(Y_{1F};\chi_F,W_F)_\#$ with $\chi_F = \chi_E \circ N_{E/F}$. The other is $\mathbb{B}_2 = A(Y_{1E};\chi_E,W_E)_\#$ restricted to $W_F$, with $H_E$ acting on $\mathbb{B}_1$ via $b^* : H_E \to H_F$. The map $b^*$ is the Hecke algebra homomorphism dual to the base change L-group homomorphism $b$ to be discussed in the next section. The map $b$ also defines the lifting of unramified representations from $G_F(F_v)$ to $G_E(E_v)$, $\pi_{F,v} \mapsto b(\pi_{F,v})$.

To prove the divisibility of the determinants, we need to show the equality of the traces, thus that $\mathrm{tr}\{\pi_F\}(b^*(h_E p^{a_E})^m) = \mathrm{tr}\{b(\pi_F)\}((h_E p^{a_E}\sigma)^m)$ for each integer $m \geqslant 0$. On $\{\pi_E\}$ which is not $\sigma$-invariant, the trace, twisted, is zero. If $\ell$ divides $m$, $\sigma^m$ is trivial, and we get extra terms. For this reason we deduce divisibility but not equality of $\det[I - tb^*(h_E p^{a_E})]$ and $\det[I - th_E p^{a_E}\sigma]$. If the packet $\{\pi_E\}$ is $\sigma$-invariant, at each place $v$ where $h_{E,v}$ is spherical we have $\mathrm{tr}\{\pi_{E,v}\}(h_{E,v}\sigma) = \mathrm{tr}\{\pi_{E,v}\}(h_{E,v})$, and only unramified $\pi_{E,v}$ have nonzero trace at places $v$ where $h_{E,v}$ is spherical.

Note the general fact that if $\pi = \otimes_v \pi_v$ is an admissible representation of an adèle group and $f = \otimes_v f_v$ is a test function, then $\mathrm{tr}\,\pi(f) = \prod_v \mathrm{tr}\,\pi_v(f_v)$. Further, if $f_v$ is $K_v$-biinvariant, where $K_v$ is a compact open subgroup, then $\mathrm{tr}\,\pi_v(f_v) = \mathrm{tr}\,\pi_v^{K_v}(f_v)$.

Let us clarify what is the commutative Hecke algebra $H_F$ and the analogous algebra $H_E$, and what is the map $b^* : h_E \mapsto h_F = b^*(h_E)$. The Hecke algebras are generated (as modules) by products $\otimes_v h_{F,v}$, $\otimes_v h_{E,v}$ over all finite places. For a place $v$ outside a finite set $S$ which contains the places of $F$ which ramify in $E$, and the places over $p$, the component $h_{F,v}$ ranges over the convolution algebra $H_{F,v}$ of compactly supported mod center functions which are biinvariant under the standard compact subgroup $G_F(R_{F,v})$ in $G_F(F_v)$. The fundamental lemma asserts in this case that the dual $b^* : H_{E,v} \to H_{F,v}$ to the base change homomorphism $b : {}^LG_F \to {}^LG_E$ has the property that $h_{E,v}$ and $b^*(h_{E,v}) = h_{F,v}$ have matching orbital integrals.

At the places $p_r$ of $F$ over $p$, the component is Iwahori biinvariant, supported on a double coset $I_{p_r} p^{a_0} I_{p_r}$, where $a_0$ is an increasing $n$-tuple. Such functions on $G_E(E_v)$ and $G_F(F_v)$ have matching orbital integrals. Thus at the place $p$ we have $\mathrm{tr}\{\pi_{F,p_r}\}(I_{F_r} p^{a_0} I_{F_r}) = \mathrm{tr}\{b(\pi_{F,p_r})\}(I_{E_r} p^{a_{E,0}} I_{E_r} \cdot \sigma)$.

At the places of $S$ outside $p$ we also take functions with matching orbital integrals. At $F$-places $v$ which split in $E$ we have $f_F = f_1 * \cdots * f_\ell$ if $f_E = (f_1, \ldots, f_\ell)$ and $\mathrm{tr}\,\pi_E(f_E \times \sigma)$ is zero unless $\pi_E = \pi_1 \times \cdots \times \pi_\ell$ on $G_E(E_v) = G_F(F_v) \times \cdots \times G_F(F_v)$ has $\pi_1 = \cdots = \pi_\ell$ in which case it is equal to $\mathrm{tr}\,\pi(f_F)$. If $v$ stays prime in $E$, we use matching functions $f_{F,v}$ and $f_{E,v}$ which are measures of volume 1 on congruence subgroups $U_v = (1 + \boldsymbol{\pi}_v^k D_F(R'_v)) \cap G_F(F_v)$ and the analogous group for $E$.

Thus $H_F$ is the span of products which are spherical functions for all $v$ outside $S$, Iwahori biinvariant over $p$, and a constant measure concentrated on a congruence subgroup at the remaining places $v$ in $S - \{p\}$. In particular, the component of the open compact subgroup $U_0(p)_F$ is the standard compact subgroup $G_F(R_{F,v})$ for all finite $v \notin S$, the Iwahori subgroup $I_{p_r}$ at each place $p_r$ of $F$ over $p$, and the congruence subgroup at each place $v \neq p$ in $S$.

For the equality of the trace formulae we are using (1) the base change lifting which asserts that there is a bijection from the set of packets of automorphic representations of $G_F(\mathbb{A}_F)$ to the set of $\sigma$-invariant representations of $G_E(\mathbb{A}_E)$, (2) that $A(Y_{1F};\chi_F,W_F,\mu_F)^{\mathrm{cl}}_{\#}$ is the space of automorphic forms on $G_F(\mathbb{A}_F)$ whose component at $p$ is the Iwahori-invariant $p$-adic principal series representation ${}_{\mu_F}R'_{\chi_F\tau_{\mu_F}} = {}_{\mu_F}R' \otimes \chi_F\tau_{\mu_F}$, viewed as an $F_p^0[I]$-module.

More details on the base change lifting of automorphic representations are given in the next section. In the rest of this section we discuss the control condition (CL) which is satisfied by our systems.

**Definition 3.9.** *Let $P_{\chi_F}$ be $P^F_{\chi_F}$, thus $a_F = (a_{(r)})$, $a_{(r)} = p_r^{(0,1,\dots,n-1)}$ for all $r$, and $Q_{\chi_F} = Q^F_{\chi_F}$. Let $B$ be an ONable Banach space over $\mathbb{C}_p$. Let $\beta \in \mathrm{End}_{\mathbb{C}_p} B$ be a completely continuous endomorphism of $B$. For any $q \in \mathbb{Q}$ denote by $B(\beta,q)$ the maximal subspace of $B$ stable under $\beta$ on which all eigenvalues of $\beta$ have valuation $q$. Then $\dim_{\mathbb{C}_p} B(\beta,q)$ is finite since $\beta$ is completely continuous. A nonzero vector in $B$ is said to have a* finite slope for *$\beta$ if it lies in the (direct) sum over $q \in \mathbb{Q}$ of $B(\beta,q)$.*

Let $f \in A(Y_{1,F};\chi_F,W_F,\mu_F)_{\#}$ be a form of finite slope for $T(p^{a_F})$, $a_F = (a_{(r)})$, each $a_{(r)}$ is strictly increasing, where $p^{a_F}$ is viewed as an element of $G_F(\mathbb{A}_{F,f})$ with trivial component outside $p$. Then we claim it has finite slope for all $T(p^{a'_F})$ with $a'_{(r)}$ strictly increasing for all $r$, $a'_F = (a'_{(r)})$. Indeed, each $T(p_r^{a_{(r)}})$ is a monomial in $n+1$ operators $T(p_r^{(0,\dots,0,1,\dots,1)})$, each of which occurs nontrivially. The finite dimensional subspace of $A(Y_{1,F};\chi_F,W_F,\mu_F)_{\#}$ on which $T(p^{a_F})$ has a given slope $q$, is stable under each $T(p^{a'_F})$, as these commute with $T(p^{a_F})$. But $T(p^{a_F})$ is invertible on this subspace. Hence so are the $T(p_r^{(0,\dots,0,1,\dots,1)})$, and also all $T(p^{a'_F})$ with strictly increasing $a'_F$. In conclusion, a vector has finite slope for some $T(p^{a_F})$, strictly increasing $a_F$, iff it is so for any other $T(p^{a'_F})$, strictly increasing $a'_F$.

**Proposition 3.10.** *If $p > 2$, every classical automorphic form of type $(G_F, U_0(p), \chi_F)$ has finite slope.*

**Proof.** It suffices to observe that by Bernstein's presentation [17, 14] of the Iwahori algebra, if $V$ is any admissible irreducible complex representation of $\mathrm{GL}(n,F_{p_r})$ containing a nonzero subspace on which the

Iwahori subgroup $I_F$ acts via a one-dimensional representation, the operators $I_F \cdot p^{a_F} \cdot I_F$ are invertible, hence their eigenvalues are nonzero, and have finite slope. $\square$

Consider $\mu_F = (\mu_{(r)})$, $\mu_{(r)} = (\mu_{r1}, \ldots, \mu_{rn}) \in \mathbb{Z}_{\geqslant 0}^{n-1} \times \mathbb{Z}$, $a_F = (a_{(r)})$, $a_{(r)} = (a_{r1} < a_{r2} < \cdots < a_{rn})$, $r$ ranges over the $F$-places $p_r$ over $p$.

**Proposition 3.11.** *For $\alpha_r \in \mathbb{Q}$, $\alpha_r \leqslant \min_{1\leqslant i<n}(a_{r,i+1} - a_{r,i})(\mu_{r,i} + 1)$, we have*

$$A(Y_{1,F}; \chi_F, W_F, \mu_F)^{\mathrm{cl}}_{\#}(T(p_r^{a_{(r)}}), \alpha_r; 1 \leqslant r \leqslant d)$$
$$= A(Y_{1,F}; \chi_F, W_F, \mu_F)_{\#}(T(p_r^{a_{(r)}}), \alpha_r; 1 \leqslant r \leqslant d).$$

*If $a_F = (a_{(r)})$, $a_{(r)} = (0, 1, 2, \ldots, n-1)$, the condition is*

$$\alpha_r \leqslant \min_{1\leqslant i<n}(\mu_{ri}) + 1.$$

**Proof.** Define $A(X_{1,F}; \chi_F, W_F, \mu_F)^{\mathrm{cl}}_{\#} = {}_{\mu_F}V_{\chi_F\tau_{\mu_F},\#}$ (for $Y_{1,F}$ replace $V$ by $R'$), and

$$A(Y_{1,F}; \chi_F, W_F, \mu_F)^{\mathrm{ncl}}_{\#} = A(Y_{1,F}; \chi_F, W_F, \mu_F)_{\#}/A(Y_{1,F}; \chi_F, W_F, \mu_F)^{\mathrm{cl}}_{\#}.$$

Define $A(X_{1,F}; \ldots)^{\mathrm{ncl}}_{\#}$ analogously. There is a canonical surjective Hecke equivariant map

$$A(X_{1,F}; \chi_F, W_F, \mu_F)^{\mathrm{ncl}}_{\#} \to A(Y_{1,F}; \chi_F, W_F, \mu_F)^{\mathrm{ncl}}_{\#}.$$

For $a_{(r)}$ as in the claim, consider the action on $A(X_{1,F}; \chi_F, W_F, \mu_F)_{\#}$ of $\prod_r [p_r^{a_{(r)}}]_{\mu_F}$. On a monomial $\prod_{r,i} z_{r,i,j}^{\mu'_{r,i}}$, it acts by multiplication with $\prod_r p_r^{\mu'_{ri}\sum_k (a_{r,j_k} - a_{r,k})}$. The monomial for which this exponent is minimal is $\prod_{r,i} z_{r,i,2}^{\mu'^{r}_{r,i}}$. The exponent of $p_r$ for this monomial is $\sum_i \mu'_{r,i}(a_{r,i+1} - a_{r,i})$.

The action of $T(\prod_r p_r^{a_{(r)}})$ on $A(X_{1,F}; \chi_F, W_F, \mu_F)_{\#}$ decomposes in the form $\sum_{1\leqslant\alpha\leqslant A} [T_\alpha]_{\mu_F} \cdot \sigma_\alpha$ where $\sigma_\alpha$ are operators of norm 1 and the $T_\alpha$ in $I_F \cdot p^{a_F} \cdot I_F$ act diagonally via $\iota$, on $A(X_{1,F}; \chi_F, W_F, \mu_F)^h$.

We claim that $T(\prod_r p_r^{a_{(r)}})/\prod_r p_r^{\min_{1\leqslant i<n}(\mu_{r,i}+1)(a_{r,i+1}-a_{r,i})}$ is integral on $A(X_{1,F}; \chi_F, W_F, \mu_F)^{\mathrm{ncl}}_{\#}$. It suffices to see this for each $T_\alpha$. Since $I_F$

acts by automorphisms of norm $\leqslant 1$, it suffices to check this for $[p_r^{a(r)}]_{\mu_F}/p_r^{\min_{1\leqslant i<n}(\mu_{r,i}+1)(a_{r,i+1}-a_{r,i})}$.

Now $A(X_{1,F};\chi_F,W_F,\mu_F)^{\mathrm{ncl}}_{\#}$ is isometrically isomorphic, together with the action of $[p_r^{a(r)}]_{\mu_F}$, to the subspace of (for $h$ see the 3rd paragraph of this section)

$$A(X_{1,F};\chi_F,W_F,\mu_F)_{\#}\simeq A(X_{1,F};\chi_F,W_F,\mu_F)^h$$

generated by the vectors whose coordinates are the monomials of weight $\mu'_{(r)}=(\mu'_{r1},\dots,\mu'_{r,n-1})$ such that there exists $i<n$ with $\mu'_{r,i}>\mu_{r,i}$. Hence

$$[p_r^{a(r)}]_{\mu_{(r)}}/p_r^{\min_{1\leqslant i<n}(\mu_{r,i}+1)(a_{r,i+1}-a_{r,i})}$$

is integral on $A(X_{1,F};\chi_F,W_F,\mu_F)^{\mathrm{ncl}}_{\#}$. Thus

$$|[p_r^{a(r)}]_{\mu_F}/p^{\min_{1\leqslant i<n}(\mu_{r,i}+1)(a_{r,i+1}-a_{r,i})}|\leqslant 1$$

and $|T(p_r^{a(r)})_{\mu_F}/p_r^{\min_{1\leqslant i<n}(\mu_{r,i}+1)(a_{r,i+1}-a_{r,i})}|\leqslant 1$.

Passing to the quotient the same result holds for $T(p_r^{a(r)})$ on $A(Y_{1,F};\chi_F,W_F,\mu_F)^{\mathrm{ncl}}_{\#}$. In particular the eigenvalues of $T(p_r^{a(r)})$ have $p_r$-valuation $\geqslant \min_{1\leqslant i<n}(\mu_{r,i}+1)(a_{r,i+1}-a_{r,i})$. Consequently, if $T(p_r^{a(r)})$ has an eigenvector with eigenvalue of $p_r$-valuation $<\min_{1\leqslant i<n}(\mu_{r,i}+1)(a_{r,i+1}-a_{r,i})$, its image in $A(Y_{1,F};\chi_F,W_F,\mu_F)^{\mathrm{ncl}}_{\#}$ is zero. □

We continue with the construction of the eigenvariety.

**Proposition 3.12.** *Let $A$ be a reduced affinoid algebra, $\beta$ a completely continuous endomorphism of an ONable Banach $A$-module $N$ with characteristic series* $\det(I-t\beta)=Q(t)S(t)$, $S\in 1+tA\{\{t\}\}$, $Q\in 1+tA[t]$, *such that the leading coefficient of the polynomial $Q$ is invertible, and $Q$ and $S$ coprime in $A\{\{t\}\}$. Then $N$ decomposes as a direct sum $N=N'\oplus N''$ of two closed $A$-submodules $N'$ and $N''$ stable under $\beta$ such that* (1) *$N'$ is locally free of rank* $\deg Q$ *and* $\det(I-t\beta|N')=Q(t)$; (2) *if $Q^*(t)=t^{\deg Q}Q(1/t)$, $Q^*(\beta)|N''$ is invertible.*

If $W'$ is an open affinoid in $W_F$, the Banach $A(W')$-module

$$A(Y_{1,F};\chi_F,W')_{\#}=(A(Y_{1,F};\chi_F)\widehat{\otimes}_{\mathbb{C}_p}A(W'))_{\#}$$

is ONable over $A(W')=\mathcal{O}^{\mathrm{an}}_{W_F}(W')$. The operator $T(p^F)$ restricted to this module has characteristic series in $A(W')\{\{t\}\}$, denoted $P_{\chi_F}(W';t)$. If

$P_{\chi_F}(W';t) = Q(t)S(t)$ in $A(W')\{\{t\}\}$, $Q(t) \in 1 + tA(W')[t]$ has invertible leading coefficient, $(Q,S) = 1$, Proposition 3.12 applies since $A(W')$ is a reduced affinoid algebra, hence semisimple, to give a decomposition

$$A(Y_{1,F};\chi_F,W')_{\#} = A(Y_{1,F};\chi_F,W')'_{\#} \oplus A(Y_{1,F};\chi_F,W')''_{\#}$$

into submodules stable under $T(p^F)$, such that

(1) $A(Y_{1,F};\chi_F,W')'_{\#}$ is locally free $A(W')$-module of rank $\deg Q$, on which $T(p^F)$ has characteristic polynomial $Q^*(t)$;

(2) $A(Y_{1,F};\chi_F,W')'_{\#}$ is $\ker Q^*(T(p^F))$ and $Q^*(T(p^F))$ is invertible on $A(Y_{1,F};\chi_F,W')''_{\#}$;

(3) the commutator of $T(p^F)$ in $\mathrm{End}_{A(W')}[A(Y_{1,F};\chi_F,W')_{\#}]$ stabilizes $A(Y_{1,F};\chi_F,W')'_{\#}$.

Thus we take an open affinoid $W' \subset W_F$ and decomposition $P_{\chi_F} = QS$ in Proposition 3.12, $A = A(W')$ and $N = [A(Y_{1,F};\chi_F)\widehat{\otimes}_{\mathbb{C}_p}A(W')]_{\#}$, and $N' = N'(Q,S)$ the summand provided by Proposition 3.12. Let $H(W') = H(W';Q,S)$ denote the image of the algebra $H_{\chi_F,\Lambda} \otimes_\Lambda A(W')$ in $\mathrm{End}_{A(W')}N'$. The $A(W')$-module $\mathrm{End}_{A(W')}N'$ equiped with the norm map is complete and of finite type over the noetherian Banach algebra $A(W')$. Consequently its sub $A(W')$-modules are closed of finite type over $A(W')$.

In particular, $H(W') \subset \mathrm{End}_{A(W')}N'$ is a commutative $A(W')$-algebra of finite rank over $A(W')$, hence an affinoid algebra, closed, and the image of the continuous map $H_{\chi_F,\Lambda} \to \mathrm{End}_{A(W')}N'$ lies in $H(W')^0 = H(W') \cap (\mathrm{End}_{A(W')}N)^0$.

Define the *local eigenvariety* to be $D(W') = \mathrm{Sp}H(W')$. It is an affinoid depending on $W'$ and $Q$. From $A(W') \to H(W') = A(W')\widehat{\otimes}_{\mathbb{C}_p}H_{\chi_F,\Lambda}$ we get a finite morphism $\kappa : D(W') \to W' \subset W_F$. Let $Z(W')$ be the *spectral variety*, thus the rigid hypersurface in $W' \times \mathbb{A}^{1,\mathrm{an}}$ defined by $Q(\mu_F,t) = 0$. Denote by $\mathrm{pr}_{W'} : Z(W') \to W'$ and $\mathrm{pr}_{\mathbb{A}} : Z(W') \to \mathbb{A}^{1,\mathrm{an}}$ the natural projections. Since $Q(T(p^F)) = 0$, from $A(W') \to H(W')$ we get a map

$$A(Z(W')) = A(W')[t]/Q^*(t) \to A(D(W')) = H(W'), \quad t \mapsto T(p^F),$$

hence a dual map $D(W') \to Z(W')$.

We shall now consider a point $x$ in $D(W')(\mathbb{C}_p)$. Denote its image in $Z(W')(\mathbb{C}_p) \subset W'(\mathbb{C}_p) \times \mathbb{C}_p$ by $(\mu_F, \lambda^{-1})$. We shall use the following.

**Lemma 3.13.** *Let $A$ be a commutative ring, $N$ a finite type projective module over $A$, $H$ a commutative subalgebra of $\mathrm{End}_A N$ over $A$, $I$ an ideal of $A$. Then the kernel of the natural map $H/IH \to \mathrm{End}_{A/I}(N/IN)$ is nilpotent.*

Applying this with $A = A(W')$, $N = A(Y_{1,F}; \chi_F, W')'_{\#}$, $H = H(W') = \mathrm{Im}[H_{\chi_F,\Lambda} \otimes_\Lambda A(W') \to \mathrm{End}_{A(W')}N]$, $I = x$, we note that $A(W')/x = \mathbb{C}_p$, $N/I = A(Y_{1,F}; \chi_F, W_F, \mu_F)'_{\#}(T(p^F); \lambda)$ and so

$$\mathrm{Im}[H_{\chi_F,\Lambda} \otimes_\Lambda \mathbb{C}_p \to \mathrm{End}_{\mathbb{C}_p}[A(Y_{1,F}; \chi_F, W', \mu_F)^N_{\#}(T(p^F); \lambda)]]$$

has the same $\mathbb{C}_p$-points as $H/x \simeq \mathbb{C}_p$. Thus a choice of $x \in D(W')(\mathbb{C}_p)$ whose image in $Z(W')(\mathbb{C}_p)$ is $(\mu_F, \lambda^{-1})$ is the same as a choice of a nonzero subspace $S$ of $A(Y_{1,F}; \chi_F, W', \mu_F)'_{\#}(T(p^F); \lambda)_x$ and a $\mathbb{C}_p$-valued character $\beta$ of $H_{\chi_F,\Lambda}$ such that $T\xi = \beta(T)\xi$ for each $T$ in $H_{\chi_F,\Lambda}$ and $\xi$ in $S$. Namely it is a choice of an action of the Hecke algebra $H_{\chi_F,\Lambda}$ by scalars on a subspace of $p$-adic modular forms of weight $\mu_F$ and $T(p^F)$-eigenvalue $\lambda$. Thus we conclude the following.

**Theorem 3.14.** *There is a bijection between the points $x \in D(W')(\mathbb{C}_p)$ whose image in $Z(W')(\mathbb{C}_p)$ is $(\mu_F, \lambda^{-1})$ and the set of subspaces of $A(Y_{1,F}; \chi_F, W', \mu_F)'_{\#}(T(p^F); \lambda)$ on which $H_{\chi_F,\Lambda}$ acts by scalars.*

Each such subspace is a multiple of an irreducible $H_{\chi_F,\Lambda}$-module.

The local eigenvarieties $D(W') = \mathrm{Sp}H(W')$ are affinoids that can be patched together to define an admissible covering of a non quasi-compact rigid space $D(W_F)$ over $W_F$, called the *eigenvariety*. The construction, following Coleman, Mazur [9], and Buzzard [3], Chenevier [5] in general, uses the admissible covering which we define next.

Let $P(\mu, t) \in 1 + tA(W)\{\{t\}\}$ ($\mu \in W$) be a Fredholm series on a reduced rigid space $W$. The associated Fredholm hypersurface $Z_P(W) = \mathrm{Sp}A(W)[t]/(P(\mu, t))$ is the closed rigid analytic subspace of $W \times \mathbb{A}^{1,\mathrm{an}}$ defined by $P(\mu, t) = 0$. It is called the *spectral variety*. The series $P$ and the space $Z_P$ are called $\Lambda$-adic if $P \in 1 + t\Lambda\{\{t\}\}$.

The projection $\mathrm{pr}_W : Z_P \hookrightarrow (W \times \mathbb{A}^{1,\mathrm{an}} \twoheadrightarrow) W$ is flat by [5, (5.4.3)]: if $A$ is an affinoid algebra, $f \in 1 + tA\langle t\rangle$, then $A\langle t\rangle/(f(t))$ is flat over $A$.

Consider the set $\mathcal{C}$ of open affinoids $Y$ on $Z_P$ such that (i) $\mathrm{pr}_W(Y)$ is an open affinoid in $W$, (ii) $Y$ is a rigid connected component of $\mathrm{pr}_W^{-1}(\mathrm{pr}_W(Y))$, and (iii) $\mathrm{pr}_W|Y : Y \to \mathrm{pr}_W(Y)$ is finite.

**Proposition 3.15.** *The family $\mathcal{C}$ makes an admissible covering of $Z_P$.*

**Proof.** See Buzzard [3], developing Coleman [7]. □

**Proposition 3.16.** *Any member $Y$ of $\mathcal{C}$ defines, and is defined by, the following data: open affinoid $V$ of $W_F$, and factorization*

$$P(\mu,t) = Q(\mu,t)S(\mu,t) \in 1 + tA(V)\{\{t\}\}, \text{ where } Q(\mu,t) \in 1 + tA(V)[t]$$

*has invertible dominant coefficient, and $(Q,S) = 1$ in $A(V)\{\{t\}\}$.*

**Proof.** Given $Y \in \mathcal{C}$, put $V = \mathrm{pr}_W(Y)$ and $Z_V = Z_P \cap (V \times \mathbb{A}^{1,\mathrm{an}})$. The affinoid $Y$ is an open in $Z_V$ by $(ii)$, hence it is flat over $V$. By $(ii)$, $(iii)$ it is a closed subset in $Z_V$ which is finite and flat over $V$. In other words, $A(Y)$ is locally free over $A(V)$, generated over $A(V)$ by the image of $t$. Then $t$ acts by multiplication on $A(Y)$. Its characteristic polynomial, $Q \in A(V)[t]$, has constant coefficient 1. Then the surjection $A(V)[t]/(Q) \to A(Y)$ is an isomorphism since both $A(V)$-modules are projective and of the same rank over $A(V)$. Hence $Q(t)$ divides $P(t)$. Consequently the constant coefficient of $Q(t)$ is invertible, and $(Q,P/Q) = 1$ in $A(V)\{\{t\}\}$. As $P(0) = 1$ we may assume $Q(0) = 1$. This determines $Q$ uniquely.

Conversely, given $V$, $Q$, $S$ as in the proposition, define $Y$ to be the closed affinoid in $Z_V$ defined by $Q = 0$. This $Y$ is open and closed in $Z_V$ as a Bezout relation between $S$ and $Q$ provides an idempotent defining $Y$, hence $Y$ is an open affinoid. Hence it is flat over $V$, finite since $Q(t) = 0$ in $A(Y)$ by definition of $Y$. □

## 4. Automorphic representations

We shall now sketch briefly the mostly conjectural theory of automorphic representations for the unitary groups $G_F$ and $G_E$, which are associated with rank $n$ central simple algebras over $F'$ and $E'$ and an involution. Here $E/F$ is a cyclic extension of totally real number fields of odd prime degree, and $F'/F$, $E'/E$ are totally imaginary quadratic extensions with $E' = F'E$. Automorphic representations of $G_F$ are irreducible submodules of $L^2(G_F(F)\backslash G_F(\mathbb{A}_F))$ under right translation by $G_F(\mathbb{A}_F)$. Here $L^2$ means square integrable complex valued functions. The group $G_F$ is anisotropic, namely the homogeneous space $G_F(F)\backslash G_F(\mathbb{A}_F)$ is compact, since $G_F$ is anisotropic at least at one local place, and it is so at each of the archimedean places $\infty_r$ $(1 \leqslant r \leqslant d = [F:\mathbb{Q}])$ of $F$. Namely, $G_F(F_{\infty_r})$ are compact. Hence the analytic derivation of the trace formula for $G_F$ is simple. Moreover one sees that it is also stable, namely the geometric side of this trace formula, which consists of orbital integrals (of elliptic elements only, as $G_F$ is anisotropic) depends only on the stable conjugacy classes of the rational elements ($\gamma \in G_F(F)$ is called *elliptic* if it lies in an anisotropic torus, and a *stable* conjugacy class is the intersection with $G_F(F)$ of a conjugacy class in $G_F(\overline{F})$, $\overline{F}$ denotes an algebraic closure of $F$).

Basic results on the automorphic representations of $G_F$ can be obtained by comparison of its trace formula with the twisted — by the involution defining $G_F(F)$ in $\mathrm{GL}(n, F')$ — trace formula for $\mathrm{GL}(n, F')$, or more precisely the multiplicative group of the central over $F'$ simple algebra with involution used to define $G_F$. This latter group has rigidity theorem (two automorphic representations which are equivalent locally at almost all places are equivalent) and multiplicity one theorem, by the Deligne-Kazhdan comparison with $\mathrm{GL}(n, F')$. For this base change comparison, transfer of orbital integrals of test functions, and of spherical functions, is known due to works by Waldspurger, Kottwitz and Clozel. Using this base change comparison, one can conclude that the automorphic representations of $G_F$ are stable, namely there is a partition of $L^2(G_F(F)\backslash G_F(\mathbb{A}_F))$ into packets, which are products of local packets. Each local packet is a singleton at any place $v$ of $F$ which splits in $F'$, where $G_F(F_v)$ is $\mathrm{GL}(n, F_v)$ or an inner form of it. It is uniquely defined by its lift to $G_F(F'_v)$ when $v$ remains prime in $F'$, thus $F'_v = F' \otimes_F F_v$ is a field. There are no common irreducible representations in two different local packets. Here one deals first with square integrable (also named discrete series) packets, then induces to get all packets.

A study of the inner structure of the packets requires much work in the stabilization of the trace formula and transfer of orbital integrals to all twisted endoscopic groups of $\mathrm{GL}(n)$ over $F'$ and the endoscopic groups of the quasisplit form of $G_F$. This is so far done only for $n = 3$ in [12]. But we do not use here this far more elaborate study. We assume multiplicity one theorem for $G_F$, namely that equivalent irreducible constituents of $L^2(G_F(F)\backslash G_F(\mathbb{A}_F))$ are equal. Most – but not all – of this, is proven in [12] when $n = 3$.

Next we outline the theory of base change for automorphic representations from $G_F$ to $G_E$, where $E/F$ is a cyclic extension of prime degree $\ell$ of totally real fields, $E \cap F' = F$ and $E' = EF'$, and the central simple algebra $D_E$ over $E'$ defining $G_E$ is $D_F \otimes_{F'} E'$. The involution on $D_E$ is such that $G_E$ is anisotropic at each archimedean place $\infty_{rk}$ of $E$. The theory of base change from $G_F$ to $G_E$ would be based on a comparison of the trace formula of $G_F$ with the twisted by $\sigma$, a generator of the Galois group of $E$ over $F$, trace formula of $G_E$. Both groups $G_F$ and $G_E$ are anisotropic, hence there are no analytic difficulties in the derivation of their trace formulae, which are stable (and twisted stable in the case of $G_E$, twisted by $\sigma$). Comparison of the geometric sides of these formulae is possible since transfer of orbital integrals for base change is known by works of Kottwitz, Clozel, and Waldspurger.

Note also that the norm map, which relates the twisted stable conjugacy classes in $G_E(E)$ with the stable conjugacy classes in $G_F(F)$, is onto. Further, the norm map is surjective from the center of $G_E(\mathbb{A}_E)$ (mod the rational points) to the center (mod rationals) of $G_F(\mathbb{A}_F)$. Namely the norm

$$N_{E'/F'} : \{z/\overline{z};\, z \in \mathbb{A}_{E'}^{\times}/E'^{\times}\} \to \{x/\overline{x};\, x \in \mathbb{A}_{F'}^{\times}/F'^{\times}\}$$

is onto, when $[E : F]$ is odd. Here $z \mapsto \overline{z}$ indicates conjugation in $E'/E$ and $x \mapsto \overline{x}$ in $F'/F$. For a proof see [4, Appendix B]. In particular the map $N_{E'/F'}\mathbb{A}_{E'}^{\times}/N_{E'/F'}E'^{\times} \to \{x/\overline{x};\, x \in \mathbb{A}_{F'}^{\times}/F'^{\times}\}$, $x \mapsto x/\overline{x}$, is onto.

Note that in the theory of base change for $\mathrm{GL}(n)$ the image $N_{E/F}C_E$ of the idèle class group $C_E = \mathbb{A}_E^{\times}/E^{\times}$, which is the center of $\mathrm{GL}(n, \mathbb{A}_E)$ modulo the rational points of the center of $\mathrm{GL}(n)$ over $E$, under the norm map from $E$ to $F$, has index $\ell = [E : F]$ in the analogous object over $F$. The implication of this nonsurjectivity is that the theory of base change for $\mathrm{GL}(n)$ relates orbits $\{\pi \otimes \chi^i;\, 0 \leqslant i < \ell\}$, usually of length $[E : F]$, of cuspidal representations $\pi$ of $\mathrm{GL}(n, \mathbb{A}_F)$, with $\sigma$-invariant $\sigma$-cuspidal representations $\pi_E$ of $\mathrm{GL}(n, \mathbb{A}_E)$. Thus this base change map has fibers of size $\ell$ when $\pi \otimes \chi$ is not equivalent to $\pi$. When $\pi \otimes \chi \simeq \pi$ its base change lift is not cuspidal, but rather induced, yet $\sigma$-cuspidal (not induced from a $\sigma$-invariant representation).

Returning to our unitary groups $G_F$ and $G_E$, a comparison of the $\sigma$-twisted trace formula for $G_E$ with the trace formula for $G_F$ would show (only the case of $n = 3$ is currently available, see [4]) that there is a bijection between the set of packets of automorphic representations of $G_F$ with the set of such packets of $G_E$ which contain a $\sigma$-invariant representation. Note that since $G_F$, $G_E$ are anisotropic, namely they do not have proper parabolic subgroups over their base fields $F$, $E$, their automorphic representations are all cuspidal. A global packet $P$ is a restricted product of local packets, $P_v$, that is, $P$ consists of all $\otimes_v \pi_v$, $\pi_v \in P_v$ for all $v$, and $\pi_v$ is the unique unramified member $\pi_v^0$ of $P_v$ for almost all $v$. In the case of $G_F$ and $G_E$, each member of a packet is automorphic, and the base change map relates only those packets which contain a $\sigma$-invariant (irreducible) $\pi_E$. It has not yet been shown that when $(\ell, n) = 1$ the packet of such $\pi_E \simeq {}^{\sigma}\pi_E$ consists only of $\sigma$-invariant representations, except for

$n = 3$, where this is proven in [4], where it is also shown that this result does not always hold for $\ell = n = 3$.

The rigidity theorem asserts that a packet of automorphic representations of $G_F$ or $G_E$ is determined by almost all of its components. Fix a finite set $S$ of places of $F$ containing the archimedean ones, those above $p$, and those which ramify in $E$ or $F'$ or where the involution is ramified. To simplify matters assume that $D$ ramifies only at places $v$ of $F$ which split in $E$.

To define the base change lifting, it then suffices to define it locally. At a place $v$ of $F$ which splits in $E$, the map is the diagonal $\pi_v \mapsto \pi_v \times \cdots \times \pi_v$, as $G_E(E_v) = G_F(F_v) \times \cdots \times G_F(F_v)$, $\ell = [E : F]$ factors in the product. This includes the archimedean places, which are all real, and the places above $p$. At $v \notin S$ which stays prime in $E$, the dual group homomorphism

$$^L G_F = \widehat{G}_F \rtimes W_F \to {}^L G_E = (\mathrm{R}_{E/F} G_E)^\wedge \rtimes W_F$$

underlying the base change lifting, via the Satake transform defines a map of unramified representations $\pi_v \mapsto \pi_{E,v}$. Here the connected component $\widehat{G}_F$ is $\mathrm{GL}(n, \mathbb{C})$. The connected component $(\mathrm{R}_{E/F} G_E)^\wedge$ of the dual group of $G_E$, viewed as an $F$-group via the restriction of scalars $\mathrm{R}_{E/F} G_E$ from $E$ to $F$, is $\mathrm{GL}(n, \mathbb{C}) \times \cdots \times \mathrm{GL}(n, \mathbb{C})$ ($\ell$ times). The Weil group $W_F$ acts through its quotient $\mathrm{Gal}(F'/F)$ on $\widehat{G}_F$. On $(\mathrm{R}_{E/F} G_E)^\wedge$ it acts through its quotient $\mathrm{Gal}(E'/F)$. This group is generated by $\sigma$, a generator of $\mathrm{Gal}(E/F)$, which permutes the factors in $(\mathrm{R}_{E/F} G_E)^\wedge$, and $\tau$, the generator of $\mathrm{Gal}(F'/F)$, which acts on each factor $\widehat{G}_F = \mathrm{GL}(n, \mathbb{C})$. The conjugacy classes in the complex groups $^L G_F$ and $^L G_E$ are represented by diagonal elements in the connected components. Thus the base change map, restricted to $\widehat{T}_F \times \tau \to \widehat{T}_E \times \sigma\tau$, $\widehat{T}_F$ is the diagonal subgroup in $\widehat{G}_F$ and $\widehat{T}_E = \widehat{T}_F \times \cdots \times \widehat{T}_F$ is in $(\mathrm{R}_{E/F} G_E)^\wedge$, defines a lifting of unramified representations, as an unramified $\pi_v$ is parametrized by (the Weyl group orbit of) $t \times \tau$ in $\widehat{T}_F \times \tau$, and an unramified $\pi_{E_v}$ by $t_E \times \sigma\tau$ in $\widehat{T}_E \times \sigma\tau$. Both Galois elements are simply the Frobenius in the unramified case. The base change global lifting theorem asserts that given an automorphic representation $\pi_F$ of $G_F(\mathbb{A}_F)$ there exists an automorphic representation $\pi_E$ of $G_E(\mathbb{A}_E)$ with $\pi_{E,v} = b^*(\pi_{F,v})$ for all finite $v \notin S$.

Having defined the base change lifting we consider only $\pi$ whose components outside $S$ are unramified, namely invariant under the hyperspecial maximal compact subgroups $G_F(R_v)$ and $G_E(R_{E,v})$, where $R_v$ is the ring of integers of $F_v$ and $R_{E,v}$ of $E_v$. At the places $v$ of $F$ which split in

$E$, if $\pi_v$ has a nonzero vector invariant under a compact open subgroup $U_v$ of $G_F(F_v)$, then $\pi_{E,v} = \pi_v \times \cdots \times \pi_v$ has a nonzero vector invariant under $U_v \times \cdots \times U_v$. At the remaining finite set of finite places in $S$ of $F$ which stay prime in $E$, we assume the following statement of control of ramification of admissible representations under base change.

**Admissible control statement.** If the admissible irreducible representation $\pi_v$ of $G_F(F_v)$ ($= \mathrm{GL}(n, F_v)$ or the quasisplit $\mathrm{U}(n, F'_v/F_v)$) has a nonzero vector invariant under $\mathrm{U}(k)$ ($= (I + \boldsymbol{\pi}_v^k \mathrm{M}(n, R'_v)) \cap G_F(F_v)$), $k > 0$, then its base change lift $\pi_{E_v}$ to $G_E(E_v)$ ($= \mathrm{GL}(n, E_v)$ or $\mathrm{U}(n, E'_v/E_v)$) has a nonzero vector invariant under $\mathrm{U}_E(k_E)$ ($= (I + \boldsymbol{\pi}_{E_v}^{k_E} \mathrm{M}(n, R'_{E_v})) \cap G_E(E_v)$), where $k_E = k$ if $E_v/F_v$ is unramified and $k_E = k[E_v : F_v]$ if $E_v/F_v$ is ramified.

Here $\mathrm{M}(n)$ denotes the ring of $n \times n$ matrices, $R'_v$ is the ring of integers in $F'_v$, and $R'_{E,v}$ in $E'_v$. The Statement follows if the characteristic functions $f_{\mathrm{U}(k)}$ of $\mathrm{U}(k)$ and $f_{\mathrm{U}_E(k_E)}$ of $\mathrm{U}_E(k_E)$ have matching stable orbital integrals. Then $\mathrm{tr}\{\pi_v\}(f_{\mathrm{U}(k)}) = \mathrm{tr}\{\pi_{E_v}\}(f_{\mathrm{U}_E(k_E)} \times \sigma)$ for corresponding packets $\{\pi_v\}$ and $\{\pi_{E_v}\}$. When $F'_v = F_v \oplus F_v$ the packet is a singleton, and the lifting is the usual base change lifting for $\mathrm{GL}(n)$ from $F_v$ to $E_v$. Following is a proof of the matching statement for $G = \mathrm{GL}(n)$. The case of the unitary group is not yet done. Note that a proof of a similar statement in an analogous case is given in [15, Sec. 10].

**Proposition 4.1.** *When $G = \mathrm{GL}(n)$, the characteristic functions $f_{\mathrm{U}(k)}$ of $\mathrm{U}(k) = I + \boldsymbol{\pi}^k M(n, R)$ in $G(F)$ and $f_{\mathrm{U}_E(k_E)}$ of $\mathrm{U}_E(k_E) = I + \boldsymbol{\pi}_E^{k_E} M(n, R_E)$ in $G(E)$ have matching stable orbital integrals. Here $E/F$ is a cyclic extension of local fields, $R$ is the ring of integers of $F$, $R_E$ of $E$, $k_E$ is $k$ if $E/F$ is unramified, or $k[E : F]$ if not.*

**Proof.** We have $\boldsymbol{\pi}_E^{f(E/F)} = \boldsymbol{\pi}$ where $f(E/F)$ is the ramification index of $E/F$. Put $f = f_{\mathrm{U}(k)}$, and $\phi = f_{\mathrm{U}_E(k_E)}$. Then $\mathrm{U}_E(kf(E/F)) \cap G(F) = \mathrm{U}(k)$. We consider the twisted orbital integral

$$\int_{G_\delta^\sigma(E)\backslash G(E)} \phi(g^{-1}\delta\sigma(g))dg.$$

Here $G_\delta^\sigma(E) = \{g \in G(E);\, g^{-1}\delta\sigma(g) = \delta\}$. If the integral is nonzero, replace $\delta$ by a $\sigma$-conjugate to assume $\delta = I + \boldsymbol{\pi}^k \varepsilon$. We denote by $\varepsilon$ (and variants) an element of $M(n, R_E)$. Put $N\delta = \delta\sigma(\delta)\ldots\sigma^{\ell-1}(\delta)$, $\langle\sigma\rangle =$

$\mathrm{Gal}(E/F)$, $\ell = [E:F] \neq p$. Then $\sigma(N\delta) = \delta^{-1} \cdot N\delta \cdot \delta$. So the $G(R_E)$-conjugacy class of $N\delta$ is $\sigma$-invariant, namely defined over $F$, so we can replace $\delta$ by $g^{-1}\delta\sigma(g)$ for some $g \in G(R_E)$ to have that $N\delta \in G(R)$, in fact in $U(k)$.

It is well-known that if $Nx$ and $Ny$ are conjugate then $x$ is $\sigma$-conjugate to $y$ ($x$, $y \in G(E)$). Indeed, we may assume that $u = Nx$ lies in $G(F)$ and $Nx = Ny$. Let $G_u$ be the centralizer of $u$ in $G$. As $x^{-1} \cdot Nx \cdot x = \sigma(Nx) = Nx$, $x$ lies in $G_u(E)$, and $\sigma_x(g) = x\sigma(g)x^{-1}$ defines an automorphism of $G_u(E)$ of order $\ell$. It defines a twisted form ${}_xG_u$ of $G_u$, with ${}_xG_u(F) = G_x^\sigma(E)$. As ${}_xG_u$ is the multiplicative group of the twisted form ${}_xM(n)_u$ of the centralizer $M(n)_u$ of $u$ in $M(n)$, it follows from [22, Chap. X, Sec. 1, Ex. 2] that $H^1(\mathrm{Gal}(E/F), {}_xG_u(E)) = \{1\}$. Now put

$$c_\tau = y\sigma(y)\dots\sigma^{r-1}(y)\sigma^{r-1}(x)^{-1}\dots\sigma(x)^{-1}x^{-1},$$

where $\tau = \sigma^r$. Then $c_\sigma x\sigma(c_\tau)x^{-1} = c_{\sigma\tau}$, namely $\tau \mapsto c_\tau$ is a cocycle of $\mathrm{Gal}(E/F)$ with values in ${}_xG_u(E)$. Hence there is an $h \in G_u(E)$ with $yx^{-1} = c_\sigma = h^{-1}x\sigma(h)x^{-1}$. Thus $y = h^{-1}x\sigma(h)$ is $\sigma$-conjugate to $x$.

Consequently, as $(N\delta)^{1/\ell} \in \mathrm{U}(k)$, we may assume (replacing $\delta$ by a $\sigma$-conjugate) that $\delta \in \mathrm{U}(k)$. If $g \in G_\delta^\sigma(E)$ then $g^{-1}\delta\sigma(g) = \delta$, so $g^{-1}\delta^\ell g = \delta^\ell$, hence $g^{-1}\delta g = \delta$ on taking the $\ell$th root, so $g \in G(F)$, and $G_\delta^\sigma(E)$ equals $G_{N\delta}(F)$, the centralizer of $N\delta$ in $G(F)$.

Now suppose $\phi(g^{-1}\delta\sigma(g)) \neq 0$. Then $g^{-1}\delta\sigma(g) = \delta'$, where $\delta' = I + \boldsymbol{\pi}^k\varepsilon'$, $\varepsilon' \in M(n, R_E)$. Put $\|g\| = \max\{|g_{i,j}|;\ 1 \leqslant i,j \leqslant n\}$. We assume as we may that $|\boldsymbol{\pi}| < \|g\| \leqslant 1$. Note that $g = g_F + \frac{1}{\ell}\sum_{1\leqslant i<\ell}(g - \sigma^i g)$, where $g_F = \frac{1}{\ell}\sum_{0\leqslant i<\ell}\sigma^i g$ is in $G(F)$. To estimate the remainder, note that $g^{-1}\delta\sigma(g) = \delta'$ implies $\sigma(g)^{-1}\delta\sigma^2(g) = \sigma(\delta')$, $\dots$, $\sigma^{i-1}(g)^{-1}\delta\sigma^i(g) = \sigma^{i-1}(\delta')$. Multiplying we get $g^{-1}\delta^i\sigma^i(g) = \delta'\sigma(\delta')\dots\sigma^{i-1}(\delta')$. Denote the right side here by $N_i(\delta')$. Then $\delta^i\sigma^i(g) = gN_i(\delta')$, where $\delta^i = I + \boldsymbol{\pi}^k\varepsilon_i$, $N_i(\delta') = I + \boldsymbol{\pi}^k\varepsilon_i'$. Then $\sigma^i(g) - g = \boldsymbol{\pi}^k(g\varepsilon_i' - \varepsilon_i\sigma^i(g))$. Hence $\|\sigma^i g - g\| \leqslant |\boldsymbol{\pi}|^k$. So $g = g_F(I + \boldsymbol{\pi}^k\varepsilon'')$, hence $\phi(g^{-1}\delta\sigma(g)) = \phi(g_F^{-1}\delta g_F) = f(g_F^{-1}\delta^\ell g_F)$ if $N\delta = \delta^\ell$ is a regular element, so our integral becomes

$$\int_{G_{N\delta}(F)\backslash G(F)} f(g^{-1}N\delta g)dg$$

if the measures are normalized suitably. This completes the proof of the matching of the orbital integrals of $f$ and $\phi$. □

Consider now an open compact subgroup $U_0(p)_F$ which is the product over $v$ of the maximal compact $G_F(R_v)$ for all $v \notin S$, the Iwahori $I_{p_r} \subset \mathrm{GL}(n, R_{p_r})$ at each $F$-prime $p_r$ over $p$, and the $U_v$ specified as above at the other finite $v \in S$. An automorphic representation $\pi$ of $G_F$ with a nonzero vector invariant under $U_0(p)_F$ and with a rational weight $\mu_F$ is the same as a system of Hecke eigenvalues, namely a subspace of $A(Y_{1,F}; \chi_F, W_F, \mu_F)_\#$, on which the Hecke algebra $H_{\chi_F,\Lambda}$ acts by scalars. The action factors through a finite dimensional subspace as the dimension of the space of automorphic forms with fixed ramification at all finite places, and fixed infinitessimal character, or weight, is finite.

The points $x \in D_F(\mathbb{C}_p)$ with image $\mu_F$ in $W_F$ parametrize the isomorphism classes of systems of eigenvalues of $H_{\chi_F,\Lambda}$ acting on

$$A(Y_{1,F}; \chi_F, W_F, \mu_F)_\#.$$

Namely each such $x \in D_F(\mathbb{C}_p)$ with $\kappa(x) = \mu_F$, is an automorphic representation with weight $\mu_F$ and a $U_0(p)_F$-fixed nonzero vector such that the eigenvalue of the Hecke operators $T(p^F)$ is $\lambda$, where $\pi_F(x) = (\mu_F, \lambda^{-1})$ in $Z_F(\mathbb{C}_p)$. Note that the *classical weights* $\mu_F = (\mu_{(r)})$, $\mu_{(r)} \in \mathbb{Z}^{n-1}_{\geqslant 0} \times \mathbb{Z}$ for all $r$ $(1 \leqslant r \leqslant d = [F:\mathbb{Q}])$ make a Zariski dense subset in the weight space $W_F$.

A point $x \in D_F(\mathbb{C}_p)$ will be called *classical* if $\mu_F$ (here $\pi_F(x)$ is $(\mu_F, \lambda^{-1})$) is classical, and there is a classical automorphic form $\phi_x \neq 0$ in $A(Y_{1,F}; \chi_F, W_F, \mu_F)^{\mathrm{cl}}_\#$ with $T(\phi_x) = a(T)(x)\phi_x$ for all $T$ in $H_{\chi_F,\Lambda}$ where

$$a : H_{\chi_F,\Lambda} \to \mathcal{O}^{\mathrm{an}}_{D_F}(D_F)^0$$

is the continuous ring homomorphism of Theorem 1.2.

Recall that a subset $W''$ of a rigid $K$-space $W$ is called *Zariski dense* if any *analytic subset* (see [2, Sec. 9.5.2]) of $W$ containing $W''$ is equal to $W$. If in addition for every $x \in W''$ and every open affinoid neighborhood $W'$ of $x$ in $W$, the intersection $W'' \cap W'$ is Zariski dense in every irreducible component of $W'$ which contains $x$, $W''$ is called *Zariski very dense* in $W$.

**Proposition 4.2.** *The classical points in $D_F(\mathbb{C}_p)$ are Zariski very dense in $D_F$.*

**Proof.** Let $\mu_F$ be a classical weight in $W_F(\mathbb{C}_p)$, and $\phi_x$ an automorphic form with $\pi_F(x) = (\mu_F, \lambda^{-1})$. Choose a small enough neighborhood of $x$ in $D_F(\mathbb{C}_p)$ so that it is isomorphic to its image in $W_F(\mathbb{C}_p)$, and such that the slope is constant on the neighborhood of $x$ (the valuation is locally constant and the projection $D_F(\mathbb{C}_p) \to W_F(\mathbb{C}_p)$ is continuous). Thus the open affinoid neighborhood of $\mu_F$ contains (by [2, 7.2.5]) a ball of center $\mu_F$ of radius sufficiently small, thus in particular $\mu_F + p^N(\mathbb{Z}_{\geqslant 0}^{n-1} \times \mathbb{Z})^d$ for a sufficiently large $N$. The weights $\widetilde{\mu}_F$ in this set satisfying the conditions of Proposition 3.11 are Zariski dense in the neighborhood of $\mu_F$ in $W_F(\mathbb{C}_p)$, and the result of Proposition 3.11 is that given any rational choice ($\alpha_r \in \mathbb{Q}$) of eigenvalues of $T(p_r^{a(r)})$, for large enough $N$ the space of automorphic forms $A(Y_{1,F}; \chi_F, W_F, \mu_F)_\#$ with $T(p_r^{a(r)})$-eigenvalues $\alpha_r$ for each $r$ ($1 \leqslant r \leqslant d$) consists of classical automorphic forms. Since the map from the neighborhood of $x$ in $D_F(\mathbb{C}_p)$ to the neighborhood of $\mu_F$ in $W_F(\mathbb{C}_p)$ is an isomorphism, we obtain a Zariski dense set of classical points in any neighborhood of $x$ in $D_F(\mathbb{C}_p)$, as required. □

The base change lifting would assert that there exists a bijection from the set of classical points in $D_F(\mathbb{C}_p)$ onto the set of classical $\sigma$-invariant points in $D_E(\mathbb{C}_p)$, where $E/F$ is a cyclic extension of odd prime degree not dividing $n$ of totally real fields, and $\sigma$ is a generator of the Galois group of $E/F$. Since the classical points are Zariski dense in $D_F$ and $D_E$, we should conclude from Theorem 2.18, by continuity and density, the existence of a morphism from $D_F$ to the $\sigma$-invariant part of $D_E$.

The base change map $\pi_F \to \pi_E$ would send automorphic representations $\pi_F$ with a nonzero vector fixed under the open compact subgroup $U_0(p)_F$ to such $\pi_E$ with a group $U_0(p)_E$ in such a way that outside the finite set $S$ of finite $F$-places, the $v \notin S$ component of $U_0(p)_F$ is the maximal compact subgroup $G_F(R_v)$ and for $v \notin S_E$ (= places of $E$ over $S$) the component of $U_0(p)_E$ is the analogous $G_E(R_{E,v})$, at the finite split places $v \in S$ which stay prime in $E$. I do not know to relate $U_0(p)_{E,v}$ to $U_0(p)_{F,v}$, so the eigenvariety $D_F$ will be taken to be the union, namely direct image, of the $D_F = D_F(U_0(p)_F)$ where $U_0(p)_F$ is fixed as above at the places outside $S$ and those that split in $E$, but the component at $v \in S$ that stays prime in $E$ varies. Of course $D_E$ is analogously defined, and the base change morphism $D_F \to D_E$ would be defined for eigenvarieties $D_F = \operatorname{liminj} D_F(U_0(p)_F)$, $D_E = \operatorname{liminj} D_E(U_0(p)_E)$ as just specified.

It will be interesting to construct the base change morphism $D_F \to D_E$ geometrically, and conclude from it the base change lifting of automorphic representations: first the $p$-adic ones, and then of classical ones, for those $p$-adic automorphic forms with slope controlled by the weight, when it is classical. However, this I do not know to do. The reason is that the

Eigenvariety is defined to be the spectrum not of the Hecke algebra, but rather the image of the Hecke algebra in the ring of endomorphisms of the space of $p$-adic automorphic forms. To relate the eigenvarieties directly geometrically it appears that the spaces of automorphic forms need to be related by base change, although such a relation does not exist: it is only the Hecke algebras or more precisely their actions, which are related by base change, namely the relation provided by base change lifting of automorphic representations relates systems of Hecke eigenvalues, not automorphic forms themselves.

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